mathematics

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A collection of seminorms linking the $\mathcal{A}$-numerical radius and the operator $\mathcal{A}$-seminorm

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Abstract: We investigate a novel operator seminorm, $\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$, for an $\mathcal{A}$-bounded operator Q, where $\mathcal{A}$ is a positive operator on a complex Hilbert space $(\mathscr{K},\langle\cdot,\cdot\rangle)$. This seminorm is defined using a continuous increasing and bijective function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ and an interpolational path $\mathfrak{m}_\lambda$ of the symmetric mean $\mathfrak{m}$. Specifically, $\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}=\sup\Big\{f^{-1}\Big(f(|\langle Qy,y\rangle_\mathcal{A}|)\mathfrak{m}_\lambda f(\|Qy\|_\mathcal{A})\Big):y\in\mathscr{K},\|y\|_\mathcal{A}=1\Big\}$, where f^{-1} represents the reciprocal function of f, and $\langle\cdot,\cdot\rangle_\mathcal{A}$ and $\|\cdot\|_\mathcal{A}$ denote the semi-inner product and seminorm, respectively, induced by $\mathcal{A}$ on $\mathscr{K}$. We explore various bounds and relationships associated with this new concept, establishing connections with existing literature.

Keywords: operator seminorm; $\mathcal{A}$-bounded operator; $\mathcal{A}$-numerical radius; symmetric mean; bounds

MSC: 47B65; 47B37; 46C05; 47L05

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Citation: Aljawi, S.; Feki, K.; Taki, Z. A collection of seminorms linking the $\mathcal{A}$-numerical radius and the operator $\mathcal{A}$-seminorm. *Mathematics* **2024**, *12*, 1122. https://doi.org/10.3390/math12071122

Academic Editor: Enrique Alfonso Sanchez Pérez

Received: 9 March 2024
Revised: 2 April 2024
Accepted: 4 April 2024
Published: 8 April 2024

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1. Introduction and Preliminaries

Let $\mathscr{K}$ be a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$, and its associated norm $\|\cdot\|$. We denote by $\mathrm{BL}(\mathscr{K})$ the C^*-algebra of all bounded operators acting on $\mathscr{K}$. Let I represent the identity operator on $\mathscr{K}$. The symbols Q^*, $\mathrm{Ran}(Q)$, $\|Q\|$, and $w(Q)$ stand for the adjoint, range, operator norm, and numerical radius of $Q\in\mathrm{BL}(\mathscr{K})$, respectively.

Let us recall that an operator $\mathcal{A}$ belonging to the space $\mathrm{BL}(\mathscr{K})$ is said to be positive, denoted as $\mathcal{A}\geq 0$, if it satisfies the condition $\langle\mathcal{A}u,u\rangle\geq 0$ for all $u\in\mathscr{K}$. In such cases, we denote the square root of $\mathcal{A}$ as $\sqrt{\mathcal{A}}$.

In what follows, we consider $\mathcal{A}\geq 0$. The positive semi-definite sesquilinear form on $\mathscr{K}$ induced by $\mathcal{A}$, denoted as $\langle\cdot,\cdot\rangle_\mathcal{A}$, is defined as follows:

$$\langle y,z\rangle_\mathcal{A}:=\langle\mathcal{A}y,z\rangle\quad(y,z\in\mathscr{K}).$$

We define the seminorm induced by $\langle\cdot,\cdot\rangle_\mathcal{A}$ as $\|\cdot\|_\mathcal{A}$, where $\|y\|_\mathcal{A}^2=\langle y,y\rangle_\mathcal{A}$ for every $y\in\mathscr{K}$. It is well-known that the space $\Big(\mathscr{K},\|\cdot\|_\mathcal{A}\Big)$ is normed (resp. complete) if and only if $\mathcal{A}$ is one-to-one (resp. $\mathrm{Ran}(\mathcal{A})$ is closed in $\mathscr{K}$). Furthermore, applying the Cauchy-Schwarz inequality yields

$$|\langle y,z\rangle_\mathcal{A}|\leq\|y\|_\mathcal{A}\|z\|_\mathcal{A}\quad(y,z\in\mathscr{K}).\tag{1}$$

Before proceeding further, it is worth revisiting some key definitions and results.

Definition 1. *Let $Q \in BL(\mathscr{K})$. An operator $R \in BL(\mathscr{K})$ is referred to as the $\mathcal{A}$-adjoint of the operator Q if it satisfies the following equality:*

$$\langle Qu, v\rangle_{\mathcal{A}} = \langle u, Rv\rangle_{\mathcal{A}} \quad (u, v \in \mathscr{K}).$$

We denote the set of all operators in $BL(\mathscr{K})$ that have $\mathcal{A}$-adjoints as $BL_{\mathcal{A}}(\mathscr{K})$.

It is evident that R is an $\mathcal{A}$-adjoint of Q if and only if it is a solution of the following operator equation $\mathcal{A}X = Q^*\mathcal{A}$ where X is the unknown. However, it is important to note that the existence and uniqueness of $\mathcal{A}$-adjoint are not guaranteed in general (refer to [1] for more details). According to Douglas's Theorem [2], we can express $\mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ as follows:

$$\mathrm{BL}_{\mathcal{A}}(\mathscr{K}) = \{Q \in \mathrm{BL}(\mathscr{K}) : \mathrm{Ran}(Q^*A) \subseteq \mathrm{Ran}(\mathcal{A})\}.$$

An operator $Q \in \mathrm{BL}(\mathscr{K})$ is considered $\mathcal{A}$-bounded if there exists $\mu > 0$ such that

$$\|Qy\|_{\mathcal{A}} \leq \mu\|y\|_{\mathcal{A}} \quad (y \in \mathscr{K}).$$

Applying Douglas's Theorem [2], once again, we find that the collection of all $\mathcal{A}$-bounded operators is equal to $\mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$. It is widely known that both $\mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$ and $\mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ are subalgebras of $\mathrm{BL}(\mathscr{K})$, but neither one is closed or dense in $\mathrm{BL}(\mathscr{K})$. Furthermore, the following inclusion chains hold:

$$\mathrm{BL}_{\mathcal{A}}(\mathscr{K}) \subseteq \mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K}) \subseteq \mathrm{BL}(\mathscr{K}),$$

with equality if the operator $\mathcal{A}$ is injective and its range is closed in $\mathscr{K}$ (refer to [3] (p. 1463)). Additionally, it is worth noting that, in general, $\mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ is not stable under the involution *, i.e., $Q^* \notin \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$, even if $Q \in \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ (see Example 1.1 in [4]).

According to Douglas's Theorem [2], every $Q \in \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ has a unique $\mathcal{A}$-adjoint, denoted as $Q^\sharp$. The operator $Q^\sharp$ satisfies the property $\mathrm{Ran}(Q^\sharp) \subseteq \overline{\mathrm{Ran}(\mathcal{A})}$, where the overline denotes the closure. In other words, $Q^\sharp$ is the unique solution to the following problem:

$$\mathcal{A}X = Q^*\mathcal{A} \quad \text{and} \quad \mathrm{Ran}(X) \subseteq \overline{\mathrm{Ran}(\mathcal{A})}.$$

The operator $Q^\sharp$ is commonly referred to as the reduced solution of the equation $\mathcal{A}X = Q^*\mathcal{A}$. It is worth noting that $Q^\sharp = \mathcal{A}^\dagger Q^*\mathcal{A}$, where $A^\dagger$ represents the Moore–Penrose inverse of $\mathcal{A}$ (refer to [5–8] for more details). We now present some properties of the operator $Q^\sharp$.

Proposition 1 ([3]). *Let $Q, R \in BL_{\mathcal{A}}(\mathscr{K})$ and $\alpha \in \mathbb{C}$. Then, we have the following statements:*

1. *$Q^\sharp \in BL_{\mathcal{A}}(\mathscr{K})$. Furthermore, $(Q^\sharp)^\sharp = P_{\mathcal{A}}QP_{\mathcal{A}}$, where $P_{\mathcal{A}}$ represents the orthogonal projection onto* $\overline{\mathrm{Ran}(\mathcal{A})}$.
2. $((Q^\sharp)^\sharp)^\sharp = Q^\sharp$.
3. $(QR)^\sharp = R^\sharp Q^\sharp$ *and* $(Q + \alpha R)^\sharp = Q^\sharp + \overline{\alpha}R^\sharp$.

Let $Q \in \mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$, the $\mathcal{A}$-operator seminorm $\|Q\|_{\mathcal{A}}$, the $\mathcal{A}$-numerical radius $\omega_{\mathcal{A}}(Q)$, and the $\mathcal{A}$-Crawford number of Q are, respectively, given by the following:

$$\begin{aligned} \|Q\|_{\mathcal{A}} &:= \sup\{\|Qy\|_{\mathcal{A}} : y \in \mathscr{K} \text{ and } \|y\|_{\mathcal{A}} = 1\}; \\ \omega_{\mathcal{A}}(Q) &:= \sup\{|\langle Qy, y\rangle_{\mathcal{A}}| : y \in \mathscr{K} \text{ and } \|y\|_{\mathcal{A}} = 1\}; \\ \mathrm{c}_{\mathcal{A}}(Q) &:= \inf\{|\langle Qy, y\rangle_{\mathcal{A}}| : y \in \mathscr{K} \text{ and } \|y\|_{\mathcal{A}} = 1\}. \end{aligned}$$

A direct calculation reveals that $\|\cdot\|_{\mathcal{A}}$ and $\omega_{\mathcal{A}}(\cdot)$ are two seminorms defined on $\mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$. For any $Q \in \mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$, it holds that $\|Q\|_{\mathcal{A}} = 0$ if and only if $\mathcal{A}Q = 0$. Additionally, the following properties can be observed:

1. If $Q, R \in \mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$ and $y \in \mathscr{K}$, then
   (i) $\|Q\|_{\mathcal{A}} < +\infty$ and $\|Qy\|_{\mathcal{A}} \leq \|Q\|_{\mathcal{A}}\|y\|_{\mathcal{A}}$;
   (ii) $\dfrac{\|Q\|_{\mathcal{A}}}{2} \leq \omega_{\mathcal{A}}(Q) \leq \|Q\|_{\mathcal{A}}$;
   (iii) $\|QR\|_{\mathcal{A}} \leq \|Q\|_{\mathcal{A}}\|R\|_{\mathcal{A}}$.
2. If $Q \in \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$, then

$$\|Q\|_{\mathcal{A}}^2 = \|Q^\sharp\|_{\mathcal{A}}^2 = \|QQ^\sharp\|_{\mathcal{A}} = \|Q^\sharp Q\|_{\mathcal{A}} = \omega_{\mathcal{A}}(QQ^\sharp) = \omega_{\mathcal{A}}(Q^\sharp Q).$$

For more comprehensive information about the concept of the $\mathcal{A}$-numerical radius and the operator $\mathcal{A}$-seminorm, as well as related results and inequalities, refer to the following sources [9–15], and the references cited therein.

Now, let us review the definitions of certain classes of operators in semi-Hilbert spaces. Some of these classes will be utilized in our investigation.

Definition 2. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$. We say that Q is as follows:*
(i) $\mathcal{A}$-positive if $\langle Qy, y\rangle_{\mathcal{A}} \geq 0$ for every $y \in \mathscr{K}$;
(ii) $\mathcal{A}$-self-adjoint if AQ is self-adjoint;
(iii) $\mathcal{A}$-normal if $Q^\sharp Q = Q^\sharp Q$;
(iv) $\mathcal{A}$-hyponormal if $\|Q^\sharp y\|_{\mathcal{A}} \leq \|Qy\|_{\mathcal{A}}$ for every $y \in \mathscr{K}$;
(v) $\mathcal{A}$-normaloid if $\omega_{\mathcal{A}}(Q) = \|Q\|_{\mathcal{A}}$.

Observe that if $Q \in \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ is $\mathcal{A}$-self-adjoint or $\mathcal{A}$-normal, then it is $\mathcal{A}$-hyponormal; see Proposition 6 in [12]. Another point to note is that an $\mathcal{A}$-self-adjoint operator is not necessarily $\mathcal{A}$-normal; see Example 5.1 in [4]. The fact that Q is $\mathcal{A}$-self-adjoint does not imply that Q must be equal to $Q^\sharp$; see [15] (p. 161) or [3]. However, it was shown in Lemma 1 in [1] that if $Q \in \mathrm{BL}_{\mathcal{A}}(\mathscr{K})$ is $\mathcal{A}$-self-adjoint, then $Q^\sharp$ is also $\mathcal{A}$-self-adjoint, and we have $(Q^\sharp)^\sharp = Q^\sharp$. The most well-known examples of $\mathcal{A}$-self-adjoint operators are given by $QQ^\sharp$ and $Q^\sharp Q$. Before we move on, it is worth noting that any $\mathcal{A}$-hyponormal operator is an $\mathcal{A}$-normaloid; see Remark 9 in [12]. For more detailed information about these classes of operators and other investigations in the context of semi-Hilbert spaces, refer to [12,16–22] and the references provided therein.

We now proceed to recall that a function $\mathfrak{m} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $(y, z) \mapsto y\mathfrak{m}z$ is said to be mean if it fulfills the following properties:

(C_1) $\min\{y, z\} \leq y\mathfrak{m}z \leq \max\{y, z\}$;
(C_2) $\alpha y\mathfrak{m}\alpha z = \alpha(y\mathfrak{m}z)$ for all $\alpha > 0$;
(C_3) $y\mathfrak{m}z$ is monotone, increasing in both y and z;
(C_4) $y\mathfrak{m}z$ is continuous in y and z.

If $y\mathfrak{m}z = z\mathfrak{m}y$, then $\mathfrak{m}$ is called symmetric mean.

Hereafter, $\mathfrak{m}$ is a symmetric mean. An interpolational path $\mathfrak{m}_\lambda$ for $\mathfrak{m}$ is a continuous map $\lambda \mapsto \mathfrak{m}_\lambda$ from $[0, 1]$ into the set of all means on $\mathbb{R}^+ \times \mathbb{R}^+$, which satisfies the following conditions:

(C_5) $y\mathfrak{m}_0 z = y$, $y\mathfrak{m}_1 z = z$, $y\mathfrak{m}_{\frac{1}{2}} z = y\mathfrak{m}z$;
(C_6) $(y\mathfrak{m}_\lambda z)\mathfrak{m}(y\mathfrak{m}_\mu z) = y\mathfrak{m}_{\frac{\lambda+\mu}{2}} z$;

whenever $y, z \geq 0$ and $\lambda, \mu \in [0, 1]$.

It can be easily demonstrated that the set of all $t \in [0, 1]$ satisfying the following equation:

$$(y\mathfrak{m}_\alpha z)\mathfrak{m}_t(y\mathfrak{m}_\beta z) = y\mathfrak{m}_{(1-t)\alpha+t\beta} z \tag{2}$$

for every $\alpha, \beta \in [0, 1]$ and every $y, z \geq 0$, forms a convex subset of $[0, 1]$, which includes both 0 and 1. Consequently, the equality (2) holds for all $\alpha, \beta, t \in [0, 1]$ (see Lemma 1 in [23]).

The most well-known examples of symmetric means are the arithmetic mean ∇, the geometric mean $\sharp$, and the harmonic mean $!$. They are defined as follows:

$$y\nabla z := \frac{y+z}{2};$$
$$y\sharp z := \sqrt{yz};$$
$$y!z := \left(\frac{y^{-1}+z^{-1}}{2}\right)^{-1};$$

where $y, z > 0$. Their associated interpolational paths are, respectively, given by the following:

$$y\nabla_\lambda z := (1-\lambda)y + \lambda z;$$
$$y\sharp_\lambda z := y^{1-\lambda} z^{\lambda};$$
$$y!_\lambda z := \left[(1-\lambda)y^{-1} + \lambda z^{-1}\right]^{-1};$$

where $y, z > 0$ and $\lambda \in [0,1]$. It is well known that these interpolational paths satisfy the following inequalities:

$$y!_\lambda z \le y\sharp_\lambda z \le y\nabla_\lambda z \ \ (y, z > 0 \text{ and } 0 \le \lambda \le 1).$$

For a more in-depth understanding of means theory, we recommend consulting the following references [23–27].

In their recent work, Conde et al. [28] introduced a novel quantity that lies between the numerical radius, $\omega(Q)$, and the operator norm, $\|Q\|$, of an operator, $Q \in \mathrm{BL}(\mathscr{K})$, using the concept of interpolational paths. Specifically, if $Q \in \mathrm{BL}(\mathscr{K})$, and $\mathfrak{m}_\lambda$ represents an interpolational path of the symmetric mean m, the quantity $\|Q\|_{\mathfrak{m}_\lambda}$ is defined as follows:

$$\|Q\|_{\mathfrak{m}_\lambda} = \sup\left\{\sqrt{|\langle Qy, y\rangle|^2 \mathfrak{m}_\lambda \|Qy\|^2},\ y \in \mathscr{K}, \|y\| = 1\right\}.$$

The primary objective of this study is to introduce and investigate a new concept for the class of $\mathcal{A}$-bounded operators that encompasses and extends the aforementioned definition.

2. Main Results

In this section, we outline our contributions. To begin with, drawing inspiration from the recent research by Conde et al. [28], we introduce the following definition:

Definition 3. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous increasing and bijective function, and let $\mathfrak{m}_\lambda$ be an interpolational path of the symmetric mean, $\mathfrak{m}$. For an operator $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$, we define*

$$\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} := \sup\left\{f^{-1}\Big(f\big(|\langle Qy, y\rangle_{\mathcal{A}}|\big)\mathfrak{m}_\lambda f\big(\|Qy\|_{\mathcal{A}}\big)\Big) : y \in \mathscr{K}, \|y\|_{\mathcal{A}} = 1\right\},$$

where f^{-1} represents the inverse function of f.

Obviously, $\|Q\|_{\mathcal{A},\mathfrak{m}_0,f} = \omega_{\mathcal{A}}(Q)$ and $\|Q\|_{\mathcal{A},\mathfrak{m}_1,f} = \|Q\|_{\mathcal{A}}$. This shows that the quantity $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ generalizes the $\mathcal{A}$-norm operator and the $\mathcal{A}$-numerical radius. Moreover, when $A = I$ and $f(x) = x^2$ $(x \ge 0)$, this quantity coincides with the one defined in [28]. When $p \in [0, +\infty)$ and $f_p(x) = x^p$ $(x \ge 0)$, we simply write $\|\cdot\|_{\mathcal{A},\mathfrak{m}_0,p}$ instead of $\|\cdot\|_{\mathcal{A},\mathfrak{m}_0,f_p}$.

By a direct application of the inequality (1), we can derive the following inequalities:

$$\omega_A(Q) \le \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} \le \|Q\|_{\mathcal{A}} \ \ (\lambda \in [0,1]). \tag{3}$$

The primary objective of this paper is to investigate this newly introduced quantity.

Throughout the rest of the paper, we assume that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous increasing and bijective function, and $\mathfrak{m}_\lambda$ represents an interpolational path of the symmetric mean, $\mathfrak{m}$.

We commence our study by examining the following fundamental properties of $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ which can be readily derived from its definition.

Proposition 2. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{H})$ and $\lambda \in [0,1]$. Then, the following properties hold:*

1. $\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} = 0$ *if and only if* $\mathcal{A}Q = 0$.
2. *If f is multiplicative (i.e., $f(yz) = f(y)f(z)$ for $y, z \geq 0$), then for every $\mu \in \mathbb{C}$, we have* $\|\mu Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} = |\mu|\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$.
3. *If Q is $\mathcal{A}$-normaloid, then*

$$\omega_{\mathcal{A}}(Q) = \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} = \left\|Q^\sharp\right\|_{\mathcal{A},\mathfrak{m}_\lambda,f} = \|Q\|_{\mathcal{A}}.$$

In particular, for any $R \in BL_{\mathcal{A}}(\mathscr{H})$, we have the following:

$$\|R\|_{\mathcal{A}}^2 = \|RR^\sharp\|_{\mathcal{A},\mathfrak{m}_\lambda,f} = \|R^\sharp R\|_{\mathcal{A},\mathfrak{m}_\lambda,f}.$$

4. *If Q is $\mathcal{A}$-hyponormal, then* $\|Q^\sharp\|_{\mathcal{A},\mathfrak{m}_\lambda,f} \leq \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$.
5. *Let ∇ be the arithmetic mean. If f is concave, then for every $\lambda \in [0,1]$, we have*

$$\|Q\|_{\mathcal{A},\nabla_\lambda,f} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,1}.$$

In particular, if $p \in [0,1]$, then $\|Q\|_{\mathcal{A},\nabla_\lambda,p} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,1}$ for all $\lambda \in [0,1]$.

By using Equality (2), we derive the following inequalities.

Proposition 3. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{H})$. Then, for each $\alpha, \beta, t \in [0,1]$, we have the following:*

$$\|Q\|_{\mathcal{A},\mathfrak{m}_{(1-t)\alpha+t\beta},f} \leq f^{-1}\Big(f(\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha})\mathfrak{m}_t f(\|Q\|_{\mathcal{A},\mathfrak{m}_\beta})\Big) \leq \max\{\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha}; \|Q\|_{\mathcal{A},\mathfrak{m}_\beta}\}.$$

Proof. Let $y \in \mathscr{H}$, such that $\|y\|_{\mathcal{A}} = 1$ and $\alpha, \beta, t \in [0,1]$. Set

$$a = f^{-1}\Big[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\alpha f(\|Qy\|_{\mathcal{A}})\Big],$$

and

$$b = f^{-1}\Big[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\beta f(\|Qy\|_{\mathcal{A}})\Big].$$

It follows from Equality (2) that

$$\begin{aligned}
&f^{-1}\Big(f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_{(1-t)\alpha+t\beta} f(\|Qy\|_{\mathcal{A}})\Big) \\
&= f^{-1}\bigg(\Big[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\alpha f(\|Qy\|_{\mathcal{A}})\Big]\mathfrak{m}_t\Big[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\beta f(\|Qy\|_{\mathcal{A}})\Big]\bigg) \\
&= f^{-1}\bigg\{f(a)\mathfrak{m}_t f(b)\bigg\} \\
&\leq f^{-1}\Big(f(\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f})\mathfrak{m}_t f(\|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f})\Big) \ \text{[by } (C_3)] \\
&\leq \max\{\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f}; \|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f}\} \ \text{[by } (C_1)].
\end{aligned}$$

So, by taking the supremum over all $y \in \mathscr{H}$ with $\|y\|_{\mathcal{A}} = 1$, we obtain the desired inequality. □

Evidently, the following inequality is fulfilled:

$$\|Q\|_{\mathcal{A},\nabla_\lambda,f} \le f^{-1}\big(f(\omega_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\big), \tag{4}$$

where $\lambda \in [0,1]$ and $Q \in \mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$. The following result provides a necessary and sufficient condition for the inequality (4) to be an equality.

Proposition 4. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$ and $\lambda \in [0,1]$. The following statements are equivalent:*

1. $\|Q\|_{\mathcal{A},\nabla_\lambda,f} = f^{-1}\big(f(\omega_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\big)$;
2. *There exists a sequence $\{v_j\}$ of elements of $\mathscr{K}$, with $\|v_j\|_{\mathcal{A}} = 1$, such that*

$$\lim_{j\to\infty} |\langle Qv_j, v_j\rangle_{\mathcal{A}}| = \omega_{\mathcal{A}}(Q) \ \text{ and } \ \lim_{j\to\infty} \|Qv_j\|_A = \|Q\|_{\mathcal{A}}. \tag{5}$$

3. *Q is $\mathcal{A}$-normaloid.*

Proof. $(1) \Longrightarrow (2)$. By the definition of $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$, there exists a sequence $\{v_j\}$ of elements of $\mathscr{K}$, with $\|v_j\|_{\mathcal{A}} = 1$, such that

$$\lim_{j\to\infty} f^{-1}\Big[f\big(|\langle Qv_j, v_j\rangle_{\mathcal{A}}|\big)\nabla_\lambda f\big(\|Qv_j\|_{\mathcal{A}}\big)\Big] = f^{-1}\big(f(\omega_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\big).$$

This implies that

$$\lim_{j\to\infty}(1-\lambda)f\big(|\langle Qv_j, v_j\rangle_{\mathcal{A}}|\big) + \lambda f\big(\|Qv_j\|_{\mathcal{A}}\big) = (1-\lambda)f(\omega_{\mathcal{A}}(Q)) + \lambda f(\|Q\|_{\mathcal{A}}).$$

Hence, we have the following:

$$\lim_{j\to\infty} |\langle Qv_j, v_j\rangle_{\mathcal{A}}| = \omega_{\mathcal{A}}(Q) \ \text{ and } \ \lim_{j\to\infty} \|Qv_j\|_A = \|Q\|_{\mathcal{A}}.$$

This gives the desired result.
$(2) \Longrightarrow (3)$. Assume that there exists a sequence $\{v_j\}$ of elements of $\mathscr{K}$, with $\|v_j\|_{\mathcal{A}} = 1$, satisfying (5). So, by applying Proposition 4 in [12], we conclude that Q is $\mathcal{A}$-normaloid.
$(3) \Longrightarrow (1)$. By the definition of $\omega_A(\cdot)$, we can find a sequence $\{v_j\}$ of elements of $\mathscr{K}$, with $\|v_j\|_{\mathcal{A}} = 1$, such that

$$\lim_{j\to\infty} |\langle Qv_j, v_j\rangle_{\mathcal{A}}| = \omega_{\mathcal{A}}(Q).$$

By using the fact that Q is $\mathcal{A}$-normaloid and the Cauchy–Schwarz inequality (1), we can infer that

$$\lim_{j\to\infty} \|Qv_j\|_A = \|Q\|_{\mathcal{A}}.$$

On the other hand, for each j, we have the following:

$$f^{-1}\Big[f\big(|\langle Qv_j, v_j\rangle_{\mathcal{A}}|\big)\nabla_\lambda f\big(\|Qv_j\|_{\mathcal{A}}\big)\Big] \le \|Q\|_{\mathcal{A},\nabla_\lambda,f} \le f^{-1}\big(f(\omega_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\big).$$

So, by letting $j \to \infty$, we conclude that $\|Q\|_{\mathcal{A},\nabla_\lambda,f} = f^{-1}\big(f(\omega_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\big)$.
This completes the proof. □

The following result shows the convexity of the map $\lambda \mapsto f\big(\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}\big)$.

Proposition 5. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. Then, for each $\alpha, \beta, t \in [0,1]$, we have the following:*

$$f\big(\|Q\|_{\mathcal{A},\mathfrak{m}_{(1-t)\alpha+t\beta},f}\big) \le (1-t)f\big(\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f}\big) + tf\big(\|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f}\big).$$

Proof. Observe first that for all $\lambda \in [0,1]$, we have the following:

$$f(\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}) = \sup\{f(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_\lambda f(\|Qy\|_{\mathcal{A}}) : y \in \mathscr{K}, \|y\|_{\mathcal{A}} = 1\},$$

since f is continuous and increasing.

Now, let $\alpha, \beta, t \in [0,1]$, and let $y \in \mathscr{K}$, such that $\|y\|_{\mathcal{A}} = 1$. By using Equality (2) and Theorem 2 in [23], we obtain the following:

$$\begin{aligned}
&f(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_{(1-t)\alpha+t\beta}f(\|Qy\|_{\mathcal{A}})\\
&= \Big(f(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_\alpha f(\|Qy\|_{\mathcal{A}})\Big)\mathfrak{m}_t\Big(f(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_\beta f(\|Qy\|_{\mathcal{A}})\Big)\\
&\le (1-t)f(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_\alpha f(\|Qy\|_{\mathcal{A}}) + tf(|\langle Qy,y\rangle_{\mathcal{A}}|)\mathfrak{m}_\beta f(\|Qy\|_{\mathcal{A}})\\
&\le (1-t)f(\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f}) + tf(\|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f}).
\end{aligned}$$

Hence, by taking the supremum over all $y \in \mathscr{K}$ with $\|y\|_{\mathcal{A}} = 1$, we reach the required inequality. □

The following result is a direct consequence of Proposition 5.

Corollary 1. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. If the function f is concave, then the function $\lambda \mapsto \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ is convex.*

In particular, if $p \in [0,1]$, then the function $\lambda \mapsto \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,p}$ is convex.

Proof. Since the function, f, is concave, we have that the function f^{-1} is convex. So, by using the fact that the function f^{-1} is increasing and Proposition 5, we obtain the following:

$$\begin{aligned}
\|Q\|_{\mathcal{A},\mathfrak{m}_{(1-t)\alpha+t\beta},f} &\le f^{-1}\Big((1-t)f(\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f}) + tf(\|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f})\Big)\\
&\le (1-t)\|Q\|_{\mathcal{A},\mathfrak{m}_\alpha,f} + t\|Q\|_{\mathcal{A},\mathfrak{m}_\beta,f},
\end{aligned}$$

for all $\alpha, \beta, t \in [0,1]$. This gives the desired result. □

Another consequence of Proposition 5 is the following result:

Corollary 2. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. If $f(x) = \log(x)$ $(x > 0)$, then the function $\lambda \mapsto \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,\log}$ is log-convex, i.e., for each $\alpha, \beta, t \in [0,1]$, we have the following:*

$$\|Q\|_{\mathcal{A},\mathfrak{m}_{(1-t)\alpha+t\beta},\log} \le \|Q\|^{1-t}_{\mathcal{A},\mathfrak{m}_\alpha,\log}\,\|Q\|^{t}_{\mathcal{A},\mathfrak{m}_\beta,\log}.$$

The following result presents an improvement of (3).

Proposition 6. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. Then, for each $\lambda \in [0,1]$ and each positive integer $l \ge 1$, we have the following:*

$$\begin{aligned}
f(\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}) &\le f(\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}) + \sum_{n=0}^{l-1} r_n(\lambda)\sum_{k=1}^{2^n}\Delta_{\mathcal{A},Q,f}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\lambda)\\
&\le \lambda f(\|Q\|_{\mathcal{A}}) + (1-\lambda)f(\omega_{\mathcal{A}}(Q))\\
&\le f(\|Q\|_{\mathcal{A}}),
\end{aligned}$$

where

$$\Delta_{\mathcal{A},Q,f}(n,k) = \left[f\left(\|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{k-1}{2^n}},f}\right) + f\left(\|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{k}{2^n}},f}\right)\right] - 2f\left(\|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{2k-1}{2^{n+1}}},f}\right),$$

$\chi_I(\cdot)$ represents the characteristic function of an interval I, $r_0(\lambda) = \min\{\lambda; 1-\lambda\}$ and $r_n(\lambda) = \min\{2r_{n-1}(\lambda); 1-2r_{n-1}(\lambda)\}$ for each integer $n \ge 1$.

Proof. By combining Proposition 5 with Theorem 2.4 in [29], we obtain the desired result. □

The following result is similar to Proposition 6.

Proposition 7. *Let $Q \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. If the function f is concave, then for each $\lambda \in [0,1]$ and each positive integer $N \geq 1$, we have the following:*

$$\begin{aligned}\|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} &\leq \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f} + \sum_{n=0}^{N-1} r_n(\lambda) \sum_{k=1}^{2^n} \delta_{\mathcal{A},Q,f}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\lambda) \\ &\leq \lambda\|Q\|_{\mathcal{A}} + (1-\lambda)\omega_{\mathcal{A}}(Q) \\ &\leq \|Q\|_{\mathcal{A}},\end{aligned}$$

where

$$\delta_{\mathcal{A},Q}(n,k) = \left[\|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{k-1}{2^n}},f} + \|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{k}{2^n}},f}\right] - 2\|Q\|_{\mathcal{A},\mathfrak{m}_{\frac{2k-1}{2^{n+1}}},f}.$$

Proof. The result follows immediately by combining Corollary 1 with Theorem 2.4 in [29]. □

In Proposition 2, we show that the new quantity $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ satisfies the properties of the seminorm, except the triangular inequality. This leads us to wonder the following: *Under what conditions does the quantity $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ satisfy the triangle inequality?* In the following, we present an answer to this question in special cases. The first result is similar to the triangular inequality.

Theorem 1. *Let $Q, R \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$ and $\lambda \in [0,1]$. If f is concave and multiplicative, then we have the following:*

$$f\left(\|Q+R\|_{\mathcal{A},\nabla_\lambda,f}\right) \leq f\left(\|Q\|_{\mathcal{A},\nabla_\lambda,f}\right) + f\left(\|R\|_{\mathcal{A},\nabla_\lambda,f}\right). \tag{6}$$

In particular, if $f(2) = 2$, we have the following:

$$\|Q+R\|_{\mathcal{A},\nabla_\lambda,f} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,f} + \|R\|_{\mathcal{A},\nabla_\lambda,f}. \tag{7}$$

Proof. Let us first show that $f(0) = 0$. Since f is multiplicative, we have that $f(0) = f(0)^2$ and $f(1) = f(1)^2$. By using the fact that f is injective, we find that there are two possible cases, either $f(0) = 1$ and $f(1) = 0$ or $f(0) = 0$ and $f(1) = 1$. We can easily see that the first case is impossible because f is increasing. Therefore, we only have the second case, namely $f(0) = 0$ and $f(1) = 1$. Consequently, $f(0) = 0$.

Now, we will show that f is sub-additive; that is, $f(x+y) \leq f(x) + f(y)$ for every $x, y \geq 0$. Observe first that from the concavity of f and the fact that $f(0) = 0$, we can conclude the following:

$$f(\lambda t) \geq \lambda f(t) \quad (\lambda \in [0,1], t \geq 0). \tag{8}$$

Now, let $x, y \geq 0$. The result is clearly trivial when $x = 0$ or $y = 0$, since $f(0) = 0$. Suppose now that $x, y > 0$. Set $\lambda_1 = \dfrac{x}{x+y}$ and $\lambda_2 = \dfrac{y}{x+y}$. Obviously, $\lambda_1, \lambda_2 \in (0,1)$, $\lambda_1 + \lambda_2 = 1$, $f(x) = f(\lambda_1(x+y))$, and $f(y) = f(\lambda_2(x+y))$. Therefore, combining this fact with the inequality (8), we have that $f(x+y) \leq f(x) + f(y)$, and so f is sub-additive.

We now return to the proof of our inequality. Let $y \in \mathscr{K}$, such that $\|y\|_{\mathcal{A}} = 1$. Then, we have the following:

$$\begin{aligned}&f\left(|\langle (Q+R)y, y\rangle_{\mathcal{A}}|\right)\nabla_\lambda f\left(\|(Q+R)y\|_{\mathcal{A}}\right) \\ &\leq f\left(|\langle Qy, y\rangle_{\mathcal{A}}| + |\langle Ry, y\rangle_{\mathcal{A}}|\right)\nabla_\lambda f\left(\|Qy\|_{\mathcal{A}} + \|Ry\|_{\mathcal{A}}\right) \\ &\leq \left[f\left(|\langle Qy, y\rangle_{\mathcal{A}}|\right) + f\left(|\langle Ry, y\rangle_{\mathcal{A}}|\right)\right]\nabla_\lambda\left[f\left(\|Qy\|_{\mathcal{A}}\right) + f\left(\|Ry\|_{\mathcal{A}}\right)\right] \\ &= f\left(|\langle Qy, y\rangle_{\mathcal{A}}|\right)\nabla_\lambda f\left(\|Qy\|_{\mathcal{A}}\right) + f\left(|\langle Ry, y\rangle_{\mathcal{A}}|\right)\nabla_\lambda f\left(\|Ry\|_{\mathcal{A}}\right).\end{aligned}$$

Hence, by taking the supremum over all $y \in \mathscr{K}$ with $\|y\|_{\mathcal{A}} = 1$, we have the inequality (6).

Now, let us assume that $f(2) = 2$. By using the inequality (6) and the fact that the function f^{-1} is convex and multiplicative, we can infer the following:

$$\begin{aligned}
\|Q+R\|_{\mathcal{A},\nabla_\lambda,f} &\leq f^{-1}\Big(f(\|Q\|_{\mathcal{A},\nabla_\lambda,f}) + f(\|R\|_{\mathcal{A},\nabla_\lambda,f})\Big) \\
&= f^{-1}(2) f^{-1}\left(\frac{f(\|Q\|_{\mathcal{A},\nabla_\lambda,f}) + f(\|R\|_{\mathcal{A},\nabla_\lambda,f})}{2}\right) \\
&\leq f^{-1}(2)\,\frac{f^{-1}\Big(f(\|Q\|_{\mathcal{A},\nabla_\lambda,f})\Big) + f^{-1}\Big(f(\|R\|_{\mathcal{A},\nabla_\lambda,f})\Big)}{2} \\
&= 2\,\frac{\|Q\|_{\mathcal{A},\nabla_\lambda,f} + \|R\|_{\mathcal{A},\nabla_\lambda,f}}{2} \\
&= \|Q\|_{\mathcal{A},\nabla_\lambda,f} + \|R\|_{\mathcal{A},\nabla_\lambda,f}.
\end{aligned}$$

This gives the inequality (7). $\square$

The following result is a direct application of Theorem 1.

Corollary 3. *Let $Q, R \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. If $p \in [0,1]$, then for every $\lambda \in [0,1]$, we have the following:*

$$\big(\|Q+R\|_{\mathcal{A},\nabla_\lambda,p}\big)^p \leq \big(\|Q\|_{\mathcal{A},\nabla_\lambda,p}\big)^p + \big(\|R\|_{\mathcal{A},\nabla_\lambda,p}\big)^p.$$

In particular, for each $\lambda \in [0,1]$, $\|\cdot\|_{\mathcal{A},\nabla_\lambda,1}$ is a seminorm on $BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$.

The next result asserts that the quantity $\|\cdot\|_{\mathcal{A},\nabla,2}$ defines a seminorm on $\mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$.

Theorem 2. *Let $Q, R \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. Then, we have the following:*

$$\|Q+R\|_{\mathcal{A},\nabla,2} \leq \|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2}.$$

Proof. Let $y \in \mathscr{K}$, such that $\|y\|_{\mathcal{A}} = 1$. Obviously

$$|\langle (Q+R)y, y)\rangle_{\mathcal{A}}|^2 \nabla \|(Q+R)y\|_{\mathcal{A}}^2 \leq \frac{(|\langle Qy,y\rangle_{\mathcal{A}}| + |\langle Ry,y\rangle_{\mathcal{A}}|)^2 + (\|Qy\|_{\mathcal{A}} + \|Ry\|_{\mathcal{A}})^2}{2}.$$

On the other hand, by using the Cauchy–Schwarz inequality, we have the following:

$$\begin{aligned}
&(|\langle Qy,y\rangle_{\mathcal{A}}| + |\langle Ry,y\rangle_{\mathcal{A}}|)^2 + (\|Qy\|_{\mathcal{A}} + \|Ry\|_{\mathcal{A}})^2 \\
&= \|Qy\|_{\mathcal{A}}^2 + |\langle Qy,y\rangle_{\mathcal{A}}|^2 + \|Ry\|_{\mathcal{A}}^2 + |\langle Ry,y\rangle_{\mathcal{A}}|^2 \\
&\qquad + 2(|\langle Qy,y\rangle_{\mathcal{A}}||\langle Ry,y\rangle_{\mathcal{A}}| + \|Qy\|_{\mathcal{A}}\|Ry\|_{\mathcal{A}}) \\
&\leq \|Qy\|_{\mathcal{A}}^2 + |\langle Qy,y\rangle_{\mathcal{A}}|^2 + \|Ry\|_{\mathcal{A}}^2 + |\langle Ry,y\rangle_{\mathcal{A}}|^2 \\
&\qquad + 2\left(\sqrt{|\langle Qy,y\rangle_{\mathcal{A}}|^2 + \|Qy\|_{\mathcal{A}}^2}\sqrt{|\langle Ry,y\rangle_{\mathcal{A}}|^2 + \|Ry\|_{\mathcal{A}}^2}\right).
\end{aligned}$$

This implies the following:

$$\begin{aligned}
&|\langle (Q+R)y, y)\rangle_{\mathcal{A}}|^2 \nabla \|(Q+R)y\|_{\mathcal{A}}^2 \\
&\leq |\langle Qy,y\rangle_{\mathcal{A}}|^2 \nabla \|Qy\|_{\mathcal{A}}^2 + |\langle Ry,y\rangle_{\mathcal{A}}|^2 \nabla \|Ry\|_{\mathcal{A}}^2 \\
&\qquad + 2\sqrt{|\langle Qy,y\rangle_{\mathcal{A}}|^2 \nabla \|Qy\|_{\mathcal{A}}^2}\sqrt{|\langle Ry,y\rangle_{\mathcal{A}}|^2 \nabla \|Ry\|_{\mathcal{A}}^2} \\
&\leq \|Q\|_{\mathcal{A},\nabla,2}^2 + \|R\|_{\mathcal{A},\nabla,2}^2 + 2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2}.
\end{aligned}$$

Consequently,

$$\|Q+R\|_{\mathcal{A},\nabla,2}^2 \leq (\|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2})^2.$$

This completes the proof. $\square$

It is often of interest to investigate equality cases. The following theorem provides a characterization for the equality $\|Q+R\|_{\mathcal{A},\nabla,2} = \|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2}$ to hold in $\mathrm{BL}_{\sqrt{\mathcal{A}}}(\mathscr{K})$.

Theorem 3. *Let $Q, R \in BL_{\sqrt{\mathcal{A}}}(\mathscr{K})$. Then, the following conditions are equivalent.*

(1) $\|Q+R\|_{\mathcal{A},\nabla,2} = \|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2}$.

(2) *There exists a sequence $\{\nu_j\}$ of $\mathcal{A}$-unit vectors in $\mathscr{K}$, i.e., vectors satisfying $\|\nu_j\|_{\mathcal{A}} = 1$ for all j, such that*

$$\lim_{j\to\infty} \mathfrak{Re}(\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}) = 2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2}.$$

Proof. (2) $\Rightarrow$ (1) Notice first that if $AQ = 0$ or $AR = 0$, then trivially, the equality $\|Q+R\|_{\mathcal{A},\nabla,2} = \|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2}$ holds. Assume that $AQ \neq 0$ or $AR \neq 0$. Suppose that there exists a sequence $\{\nu_j\}$ of $\mathcal{A}$-unit vectors in $\mathscr{K}$, such that

$$\lim_{j\to\infty} \mathfrak{Re}(\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}) = 2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2}.$$

So, for every $n \in \mathbb{N}$, we obtain the following:

$$\begin{aligned}
&\mathfrak{Re}^2\left(\frac{1}{2}\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \frac{1}{2}\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}\right) \\
&= \left|\frac{1}{2}\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \frac{1}{2}\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}\right|^2 \\
&\qquad - \Im^2\left(\frac{1}{2}\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \frac{1}{2}\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}\right) \\
&\leq \left|\frac{1}{2}\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \frac{1}{2}\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}\right|^2 \\
&\leq \left(\frac{1}{2}|\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}}| + \frac{1}{2}|\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}||\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}|\right)^2 \\
&\leq \left(\frac{1}{2}\|R\nu_j\|_{\mathcal{A}}\|Q\nu_j\|_{\mathcal{A}} + \frac{1}{2}|\langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}||\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}|\right)^2.
\end{aligned}$$

Notice that

$$\left(\frac{|\langle Q\nu_j, \nu_j\rangle_{\mathcal{A}}|^2 + \|Q\nu_j\|_{\mathcal{A}}^2}{2}\right) \leq \|Q\|_{\mathcal{A},\nabla,2}^2.$$

So, by applying the Cauchy–Schwarz inequality, we observe the following:

$$\begin{aligned}
&\frac{1}{4}\mathfrak{Re}^2(\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}) \\
&\leq \left(\frac{|\langle Q\nu_j, \nu_j\rangle_{\mathcal{A}}|^2 + \|Q\nu_j\|_{\mathcal{A}}^2}{2}\right)\left(\frac{|\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}|^2 + \|R\nu_j\|_{\mathcal{A}}^2}{2}\right) \\
&\leq \|Q\|_{\mathcal{A},\nabla,2}^2\left(\frac{|\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}|^2 + \|R\nu_j\|_{\mathcal{A}}^2}{2}\right) \\
&\leq \|Q\|_{\mathcal{A},\nabla,2}^2\|R\|_{\mathcal{A},\nabla,2}^2.
\end{aligned}$$

So, by letting j go to ∞ in the above inequalities, and then taking into consideration the following fact:

$$\lim_{j\to\infty} \mathfrak{Re}(\langle R\nu_j, Q\nu_j\rangle_{\mathcal{A}} + \langle \nu_j, Q\nu_j\rangle_{\mathcal{A}}\langle R\nu_j, \nu_j\rangle_{\mathcal{A}}) = 2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2},$$

we conclude the following:

$$\|Q\|^2_{\mathcal{A},\nabla,2}\lim_{j\to\infty}\left(\frac{\left|\langle Rv_j,v_j\rangle_{\mathcal{A}}\right|^2+\|Rv_j\|^2_{\mathcal{A}}}{2}\right)=\|Q\|^2_{\mathcal{A},\nabla,2}\|R\|^2_{\mathcal{A},\nabla,2}.$$

Since $AQ\neq 0$, then $\|Q\|_{\mathcal{A},\nabla,2}\neq 0$. Hence, we infer the following:

$$\lim_{j\to\infty}\left(\frac{\left|\langle Rv_j,v_j\rangle_{\mathcal{A}}\right|^2+\|Rv_j\|^2_{\mathcal{A}}}{2}\right)=\|R\|^2_{\mathcal{A},\nabla,2}. \tag{9}$$

Similarly, we may prove the following:

$$\lim_{j\to\infty}\left(\frac{\left|\langle Qv_j,v_j\rangle_{\mathcal{A}}\right|^2+\|Qv_j\|^2_{\mathcal{A}}}{2}\right)=\|Q\|^2_{\mathcal{A},\nabla,2}. \tag{10}$$

So, by applying Theorem 2 and using Equation (10) with Equation (9), we infer the following:

$$\begin{aligned}
&(\|Q\|_{\mathcal{A},\nabla,2}+\|R\|_{\mathcal{A},\nabla,2})^2\\
&=\|Q\|^2_{\mathcal{A},\nabla,2}+2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2}+\|R\|^2_{\mathcal{A},\nabla,2}\\
&=\lim_{j\to\infty}\left(\frac{\left|\langle Qv_j,v_j\rangle_{\mathcal{A}}\right|^2+\|Qv_j\|^2_{\mathcal{A}}}{2}\right)+\lim_{j\to\infty}\left(\frac{\left|\langle Rv_j,v_j\rangle_{\mathcal{A}}\right|^2+\|Rv_j\|^2_{\mathcal{A}}}{2}\right)\\
&+\lim_{j\to\infty}\mathfrak{Re}(\langle Rv_j,Qv_j\rangle_{\mathcal{A}}+\langle v_j,Qv_j\rangle_{\mathcal{A}}\langle Rv_j,v_j\rangle_{\mathcal{A}})\\
&=\lim_{j\to\infty}\frac{1}{2}\left(\left|\left\langle (Q+R)v_j,v_j\right)\right\rangle_{\mathcal{A}}\right|^2+\|(Q+R)v_j\|^2_{\mathcal{A}}\right)\\
&\leq\|Q+R\|^2_{\mathcal{A},\nabla,2}\\
&\leq(\|Q\|_{\mathcal{A},\nabla,2}+\|R\|_{\mathcal{A},\nabla,2})^2.
\end{aligned}$$

Thus, we deduce $\|Q+R\|_{\mathcal{A},\nabla,2}=\|Q\|_{\mathcal{A},\nabla,2}+\|R\|_{\mathcal{A},\nabla,2}$ as desired.

(1) $\Rightarrow$ (2): By the hypothesis, there exists a sequence, $\{v_j\}$, of $\mathcal{A}$-unit vectors in $\mathscr{K}$, such that

$$\lim_{j\to\infty}\frac{1}{2}\left(|\langle (Q+R)v_j,v_j)\rangle_{\mathcal{A}}|^2+\|(Q+R)v_j\|^2_{\mathcal{A}}\right)=(\|Q\|_{\mathcal{A},\nabla,2}+\|R\|_{\mathcal{A},\nabla,2})^2.$$

On the other hand, for all $n\in\mathbb{N}$, we have the following:

$$\begin{aligned}
&\frac{1}{2}\left|\left\langle (Q+R)v_j,v_j\right)\right\rangle_{\mathcal{A}}\Big|^2+\frac{1}{2}\|(Q+R)v_j\|^2_{\mathcal{A}}\\
&\quad=\frac{1}{2}\|Qv_j\|^2_{\mathcal{A}}+\mathfrak{Re}(\langle Rv_j,Qv_j\rangle_{\mathcal{A}}+\langle v_j,Qv_j\rangle_{\mathcal{A}}\langle Rv_j,v_j\rangle_{\mathcal{A}})\\
&\qquad\qquad+\frac{1}{2}\|Rv_j\|^2_{\mathcal{A}}+\frac{1}{2}|\langle Qv_j,v_j\rangle_{\mathcal{A}}|^2+\frac{1}{2}|\langle Rv_j,v_j\rangle_{\mathcal{A}}|^2\\
&\quad\leq\|Q\|^2_{\mathcal{A},\nabla,2}+\|R\|^2_{\mathcal{A},\nabla,2}+\mathfrak{Re}(\langle Rv_j,Qv_j\rangle_{\mathcal{A}}+\langle v_j,Qv_j\rangle_{\mathcal{A}}\langle Rv_j,v_j\rangle_{\mathcal{A}})\\
&\quad\leq\|Q\|^2_{\mathcal{A},\nabla,2}+\|R\|^2_{\mathcal{A},\nabla,2}+|\langle Rv_j,Qv_j\rangle_{\mathcal{A}}+\langle v_j,Qv_j\rangle_{\mathcal{A}}\langle Rv_j,v_j\rangle_{\mathcal{A}}|.
\end{aligned}$$

By using the Cauchy–Schwarz inequality, we infer the following:

$$\begin{aligned}
&\frac{1}{2}\left|\langle (Q+R)v_j, v_j)\rangle_{\mathcal{A}}\right|^2 + \frac{1}{2}\|(Q+R)v_j\|_{\mathcal{A}}^2 \\
&\quad \le \|Q\|_{\mathcal{A},\nabla,2}^2 + \|R\|_{\mathcal{A},\nabla,2}^2 + (\|Rv_j\|_{\mathcal{A}}\|Qv_j\|_{\mathcal{A}} + |\langle v_j, Qv_j\rangle_{\mathcal{A}}||\langle Rv_j, v_j\rangle_{\mathcal{A}}|) \\
&\quad \le \|Q\|_{\mathcal{A},\nabla,2}^2 + \|R\|_{\mathcal{A},\nabla,2}^2 + 2\sqrt{\frac{|\langle Qv_j, v_j\rangle_{\mathcal{A}}|^2 + \|Qv_j\|_{\mathcal{A}}^2}{2}}\sqrt{\frac{|\langle Rv_j, v_j\rangle_{\mathcal{A}}|^2 + \|Rv_j\|_{\mathcal{A}}^2}{2}} \\
&\quad \le (\|Q\|_{\mathcal{A},\nabla,2} + \|R\|_{\mathcal{A},\nabla,2})^2.
\end{aligned}$$

Taking limits as j approaches infinity, we can conclude the following:

$$\lim_{j\to\infty} \mathfrak{Re}(\langle Rv_j, Qv_j\rangle_{\mathcal{A}} + \langle v_j, Qv_j\rangle_{\mathcal{A}}\langle Rv_j, v_j\rangle_{\mathcal{A}}) = 2\|Q\|_{\mathcal{A},\nabla,2}\|R\|_{\mathcal{A},\nabla,2},$$

as required. Thus, the proof is complete. □

The results below provide lower bounds for $\|\cdot\|_{\mathcal{A},\mathfrak{m}_\lambda,f}$ in terms of the A-Crawford number $c_A(\cdot)$.

Theorem 4. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$ and $\lambda \in [0,1]$. Then, we have the following inequality:*

$$\max\left\{f^{-1}\left[f(w_{\mathcal{A}}(Q))\mathfrak{m}_\lambda f\left(\sqrt{c_{\mathcal{A}}(Q^\sharp Q)}\right)\right]; f^{-1}\left[f(c_{\mathcal{A}}(Q))\mathfrak{m}_\lambda f(\|Q\|_{\mathcal{A}})\right]\right\} \le \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}. \tag{11}$$

Proof. Let $y \in \mathscr{K}$, such that $\|y\|_A = 1$. Then, we have the following:

$$\begin{aligned}
f^{-1}\left[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\lambda f(\|Qy\|_A)\right] &= f^{-1}\left[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\lambda f\left(\sqrt{\langle Q^\sharp Qy, y\rangle_{\mathcal{A}}}\right)\right] \\
&\ge f^{-1}\left[f(|\langle Qy, y\rangle_{\mathcal{A}}|)\mathfrak{m}_\lambda f\left(\sqrt{c_{\mathcal{A}}(Q^\sharp Q)}\right)\right].
\end{aligned}$$

Taking supremum over all $y \in \mathscr{K}$ with $\|y\|_A = 1$, we have the following:

$$f^{-1}\left[f(w_{\mathcal{A}}(Q))\mathfrak{m}_\lambda f\left(\sqrt{c_{\mathcal{A}}(Q^\sharp Q)}\right)\right] \le \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}. \tag{12}$$

On the other hand, let $\{y_n\}$ be a sequence of elements of $\mathscr{K}$, with $\|y_n\|_{\mathcal{A}} = 1$, such that $\lim_{n\to\infty}\|Qy_n\|_A = \|Q\|_A$. Fix $n \in \mathbb{N}$. By the definition of $c_{\mathcal{A}}(\cdot)$, we have the following inequalities:

$$\begin{aligned}
f^{-1}\left[f(c_{\mathcal{A}}(Q))\mathfrak{m}_\lambda f(\|Qy_n\|_A)\right] &\le f^{-1}\left[f(|\langle Qy_n, y_n\rangle_{\mathcal{A}}|)\mathfrak{m}_\lambda f(\|Qy_n\|_A)\right] \\
&\le \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}.
\end{aligned}$$

Letting $n \to +\infty$ yields the following:

$$f^{-1}\left[f(c_{\mathcal{A}}(Q))\mathfrak{m}_\lambda f(\|Q\|_{\mathcal{A}})\right] \le \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,f}. \tag{13}$$

Consequently, we can achieve the desired inequality by combining Equations (12) and (13). □

The following result is an easy consequence of Theorem 4:

Corollary 4. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$ and $\lambda \in [0,1]$. Then, for every $p \in (0,+\infty)$, we have the following inequality:*

$$\max\left\{\left[w_{\mathcal{A}}(Q)^p \mathfrak{m}_\lambda c_{\mathcal{A}}^{\frac{p}{2}}(Q^\sharp Q)\right]^{\frac{1}{p}};\left[c_{\mathcal{A}}^p(Q)\mathfrak{m}_\lambda \|Q\|_{\mathcal{A}}^p\right]^{\frac{1}{p}}\right\} \leq \|Q\|_{\mathcal{A},\mathfrak{m}_\lambda,p}. \tag{14}$$

Another consequence of Theorem 4 is the following result:

Corollary 5. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$ and $\lambda \in [0,1]$. Then, we have the following:*

$$f^{-1}\Big(\min\{\lambda;1-\lambda\}f(\|Q\|_{\mathcal{A}})\Big) \leq \|Q\|_{\mathcal{A},\nabla_\lambda,f}.$$

In particular, for each $p \in (0,+\infty)$, we have the following:

$$\min\{\lambda^{\frac{1}{p}};(1-\lambda)^{\frac{1}{p}}\}\|Q\|_{\mathcal{A}} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,p}.$$

Proof. According to Theorem 4, we can derive the following:

$$\begin{aligned}
&f^{-1}\Big(\min\{\lambda;1-\lambda\}f(\|Q\|_{\mathcal{A}})\Big)\\
&= \max\Big(f^{-1}\Big(\min\{\lambda;1-\lambda\}f(\omega_A(Q))\Big);f^{-1}\Big(\min\{\lambda;1-\lambda\}f(\|Q\|_{\mathcal{A}})\Big)\Big)\\
&\leq \max\Big\{f^{-1}\Big((1-\lambda)f(\omega_A(Q))\Big);f^{-1}\Big(\lambda f(\|Q\|_{\mathcal{A}})\Big)\Big\}\\
&\leq \max\Big\{f^{-1}\Big[f(w_{\mathcal{A}}(Q))\nabla_\lambda f\Big(\sqrt{c_{\mathcal{A}}(Q^\sharp Q)}\Big)\Big];f^{-1}\Big[f(c_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}})\Big]\Big\}\\
&\leq \|Q\|_{\mathcal{A},\nabla_\lambda,f}.
\end{aligned}$$

This gives the desired result. □

In the case where $\mathfrak{m} = \nabla$, we obtain the following estimation:

Proposition 8. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$ and $\lambda \in [0,1]$. Then, we have the following inequality:*

$$\max\Big\{f^{-1}\Big[f(\omega_A(Q))^{1-\lambda}f\Big(c_{\mathcal{A}}(Q^\sharp Q)\Big)^\lambda\Big];f^{-1}\Big[f(c_{\mathcal{A}}(Q))^{1-\lambda}f(\|Q\|_{\mathcal{A}})^\lambda\Big]\Big\} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,f}.$$

Proof. Using Young's inequality, one obtains the following inequalities:

$$f(\omega_A(Q))^{1-\lambda}f\Big(c_{\mathcal{A}}(Q^\sharp Q)\Big)^\lambda \leq f(w_{\mathcal{A}}(Q))\nabla_\lambda f\Big(\sqrt{c_{\mathcal{A}}(Q^\sharp Q)}\Big) \tag{15}$$

and

$$f(c_{\mathcal{A}}(Q))^{1-\lambda}f(\|Q\|_{\mathcal{A}})^\lambda \leq f(c_{\mathcal{A}}(Q))\nabla_\lambda f(\|Q\|_{\mathcal{A}}). \tag{16}$$

Therefore, by combining inequalities (16) and (15) with (11), we obtain the required result. □

An immediate consequence of Proposition 8 is as follows:

Corollary 6. *Let $Q \in BL_{\mathcal{A}}(\mathscr{K})$ and $\lambda \in [0,1]$. Then, for every $p \in (0,+\infty)$, we have the following inequality:*

$$\max\Big\{\omega_A^{1-\lambda}(Q)c_{\mathcal{A}}^\lambda(Q^\sharp Q);c_{\mathcal{A}}^{1-\lambda}(Q)\|Q\|_{\mathcal{A}}^\lambda\Big\} \leq \|Q\|_{\mathcal{A},\nabla_\lambda,p}.$$

3. Conclusions

In this paper, we introduced and investigated a novel operator seminorm, $\|Q\|_{A,\mathfrak{m}_\lambda,f}$, which provides a link between the $\mathcal{A}$-numerical radius and the operator $\mathcal{A}$-seminorm for an $\mathcal{A}$-bounded operator, Q, on a complex Hilbert space. Our analysis has revealed several interesting properties and relationships associated with this new concept.

Moreover, we believe that this work can serve as a starting point for further research in this area. The connections established with existing literature open up avenues for exploring deeper connections and applications of these seminorms in various mathematical contexts. Future work could focus on extending these results to more general settings or investigating specific applications in the operator theory and related areas.

Overall, the results presented in this paper contribute to the understanding of operator seminorms and their connections to other important concepts in functional analysis, paving the way for future research and exploration in this fascinating field.

Author Contributions: This work was a collaborative effort among all the authors. Each author made substantial and equal contributions to the writing of this article. All authors have actively participated in the manuscript's development and have read and approved the final published version. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R514), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors would like to extend their sincere appreciation to the anonymous referees for their invaluable comments and suggestions, which greatly contributed to the enhancement of our article. Additionally, the first author would like to acknowledge the support received from Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R514), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors have no conflicts of interest to declare.

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