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Ideals and Homomorphism Theorems of Fuzzy Associative Algebras

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Abstract: Based on the definitions of fuzzy associative algebras and fuzzy ideals, it is proven that the intersections of fuzzy subalgebras are fuzzy subalgebras, and the intersections of fuzzy ideals are fuzzy ideals. Moreover, we prove that the kernels of fuzzy homomorphisms are fuzzy ideals. Using fuzzy ideals, the quotient structures of fuzzy associative algebras are constructed, their corresponding properties are discussed, and their homomorphism theorems are proven.

Keywords: fuzzy associative algebras; fuzzy ideals; fuzzy homomorphisms; quotient algebras; homomorphism theorems

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1. Introduction

Zadeh [1] pioneered the concept of fuzzy sets and laid the foundation for fuzzy mathematics. Following this, Liu [2] introduced the notions of fuzzy invariant subgroups and fuzzy ideals and subsequently discussed several fundamental properties. Ahsan et al. [3–6] conducted extensive research on the structures and properties of fuzzy semirings, integrating fuzzy concepts into semiring structures and catalyzing further research in this area. Liu [7] provided precise definitions of the operations of L-fuzzy ideals in rings. Consequently, numerous researchers have delved into the studies of fuzzy prime ideals in rings. Swamy [8] introduced the concepts of fuzzy ideals and fuzzy prime ideals of rings with truth values in a complete lattice. Furthermore, Malik and Mordeson [9] undertook a thorough examination to characterize all fuzzy prime ideals and confirmed the key properties associated with them. Nanda [10,11] contributed by defining fuzzy fields and subsequently introduced the notions of fuzzy algebras and fuzzy ideals over fuzzy fields. Biswas [12] enhanced the definitions of fuzzy fields and fuzzy linear spaces. Subsequently, Kuraoka and Kuroki [13] introduced fuzzy quotient rings derived from fuzzy ideals and investigated the relationship between fuzzy quotient rings and fuzzy ideals. Gu and Lu [14] raised concerns regarding the validity of Nanda's definition of fuzzy fields, prompting redefinitions of fuzzy fields and fuzzy algebras. Then, they proved that the homomorphic image is a fuzzy algebra. Moreover, researchers have delved into the studies of fuzzy quotient algebras. In subsequent works, scholars primarily focused on exploring fuzzy ideals in semigroups [15–18]. Zhou, Chen, and Chang [19] introduced the concepts of L-fuzzy ideals and L-fuzzy subalgebras. Additionally, Addis, Kausar, and Munir [20] provided the concept of homomorphic kernels on fuzzy rings and proved three homomorphism theorems. Korma, Parimi, and Kifetew [21] conducted a study on the properties of homomorphisms on fuzzy lattices and their quotients. As a result, three isomorphism theorems regarding the quotients of fuzzy lattices were developed by them.

Adak, Nilkamal, and Barman [22] conducted a research on fuzzy semiprime ideals of ordered semigroups. Hamidi and Borumand [23] explored the properties of EQ-algebras. Kumduang and Chinram [24] investigated fuzzy ideals and fuzzy congruences in Menger algebras. Furthermore, various scholars [25–27] have examined alternative approaches to analyzing distinct algebraic structures. Since associative algebra is a very important class of algebraic structures, its theories can be applied to group, ring, and semiring structures. It is an important foundation of modern mathematics. On the other hand, algebraic structures hold a significant position in mathematics with wide-ranging applications in many disciplines such as theoretical physics, computer sciences, information sciences, coding theories, and so on. The study of fuzzy associative algebra is helpful to better understand other fuzzy algebraic structure theories. This serves as ample motivation for us to revisit assorted concepts and findings from the realms of abstract algebras, thereby extending their applications to the broader framework of fuzzy sets.

In this paper, we provide the preliminaries in Section 2. In Section 3, we introduce the concepts of fuzzy subalgebras and fuzzy ideals, and then we discuss their properties. The quotients constructed by fuzzy ideals are presented in Section 4. In Section 5, we provide three isomorphism theorems of fuzzy algebras.

2. Preliminaries

In this section, we provide fundamental theoretical knowledge, serving as the basis for subsequent sections.

Definition 1 ([28]). *Let $(L, \leq)$ be a poset. A poset $(L, \leq, \wedge, \vee)$ is a lattice if any two elements a, b have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$, which we denote as L for short. A lattice L is called a complete lattice if each of its subsets S has $\vee S$ and $\wedge S$, where $\vee S$ and $\wedge S$ represent the least upper bound and the greatest lower bound of all elements in S, respectively. In particular, $\vee\emptyset$ and $\wedge\emptyset$ represent the smallest element 0 and the largest element 1 of L, respectively.*

Definition 2 ([29]). *Let X be a nonempty set and L be a complete lattice. A fuzzy subset of X is a function $\mu : X \to L$, where μ is called the membership function, X is called the carrier of μ, L is called the truth set of μ, and for all x belonging to X, $\mu(x)$ is called the degree of membership of x.*

We use $F_L(X) = \{\mu \mid \mu : X \to L\}$ to represent the set of all membership functions on X.

Definition 3 ([29]). *We define operations $\wedge$, $\vee$ on $F_L(X)$ as follows:*

$$\mu(x) \vee \mu'(x) = \max\{\mu(x),\mu'(x)\},$$
$$\mu(x) \wedge \mu'(x) = \min\{\mu(x),\mu'(x)\},$$
$$\bar{\mu}(x) = 1 - \mu(x),$$

for all $x \in X$, μ, $\mu' \in F_L(X)$.

Definition 4 ([30]). *Let A be a linear space on a field F, in which the multiplication operation is defined as $(\alpha, \beta) \to \alpha\beta$, and it satisfies the axioms*

(1) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$,
(2) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$,
(3) $(k\alpha)\beta = \alpha(k\beta) = k(\alpha\beta)$,
(4) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$,

for all α, β, $\gamma \in A$, $k \in F$; then, A is called an associative algebra over F.

Definition 5. *Let A and B be associative algebras. Then, B is a subalgebra of A if $B \subseteq A$, and every fundamental operation of B is the restriction of the corresponding operation of A.*

3. Fuzzy Subalgebras and Fuzzy Ideals

In this section, we first give the concept of fuzzy associative algebras. Secondly, we define fuzzy subalgebras, fuzzy homomorphisms, and fuzzy ideals in fuzzy associative algebras and prove that the intersections of fuzzy subalgebras are fuzzy subalgebras, the intersections of fuzzy ideals are fuzzy ideals, and both the homomorphic images and preimages of fuzzy ideals are fuzzy ideals.

Definition 6. *Let A be an associative algebra over the number field F and L be a complete lattice. $\mu_A \in F_L(A)$ is a fuzzy algebra on A, satisfying*

(1) $\mu_A(a_1) \wedge \mu_A(a_2) \leq \mu_A(a_1 + a_2)$,

(2) $\mu_A(a_1) \wedge \mu_A(a_2) \leq \mu_A(a_1 \cdot a_2)$,

(3) $\mu_A(a_1) \leq \mu_A(k \cdot a_1)$,

(4) $\mu_A(e) = 1$;

for all a_1, $a_2 \in A$, $k \in F$, e is a constant in A, and we denote it as (A, μ_A).

Remark 1. *If A is a ring, then it satisfies (1), (2), and (4) of Definition 6; if A is a group, then it satisfies (1), (4) or (2), (4) of Definition 6; if A is a semiring, then it satisfies (1), (2), and (4) of Definition 6, and it is a commutative semigroup under addition; and if A is an associative algebra, then it is a commutative group under addition, and the associative algebra has one more scalar multiplication operation than a semiring.*

Definition 7. *Let A be an associative algebra, B be a subalgebra of A, L be a complete lattice, and $\mu_{A|_B} \in F_L(A)$; then, $(B, \mu_{A|_B})$ is a fuzzy subalgebra of (A, μ_A).*

Definition 8. *Let (A, μ_A), (B, μ_B) be fuzzy algebras and a function $\alpha : A \to B$ be a homomorphism from A to B. A mapping $\alpha : (A, \mu_A) \to (B, \mu_B)$ is called a fuzzy homomorphism from (A, μ_A) to (B, μ_B) if*

$$\mu_A(a) \leq \mu_B(\alpha(a)),$$

for all $a \in A$.

Example 1. *The addition, multiplication, and scalar multiplication of polynomial sets over a field F form associative algebras.*

Let $f_1(x) = a_1x^2 + b_1x + c_1$ and $f_2(x) = a_2x^2 + b_2x + c_2$; then, $(f_1(x), \mu_1)$ and $(f_2(x), \mu_2)$ are fuzzy algebras. Suppose that $\alpha : (f_1(x), \mu_1) \to (f_2(x), \mu_2)$ and $\mu_2(f_2(x)) = 1.5\Big(\mu_1(f_1(x))\Big)$, for any $ax^2 + bx + c \in f_1(x)$ and $\mu_2(\alpha(ax^2 + bx + c)) = 1.5\mu_1(ax^2 + bx + c)$; thus, α is a fuzzy homomorphism.

Remark 2. *(1) A fuzzy homomorphism $\alpha : (A, \mu_A) \to (B, \mu_B)$ is called a fuzzy monomorphism from (A, μ_A) to (B, μ_B) if $\alpha : A \to B$ is an injection;*

(2) A fuzzy homomorphism $\alpha : (A, \mu_A) \to (B, \mu_B)$ is called a fuzzy epimorphism from (A, μ_A) to (B, μ_B) if $\alpha : A \to B$ is a surjection;

(3) A fuzzy homomorphism $\alpha : (A, \mu_A) \to (B, \mu_B)$ is called a fuzzy isomorphism from (A, μ_A) to (B, μ_B) if $\alpha : A \to B$ is a bijection.

Remark 3. *(1) For all $a \in A$, $\mu_B(\alpha(a)) = \bigvee \mu_A\Big(\alpha^{-1}(\alpha(a))\Big)$ if a mapping $\alpha : (A, \mu_A) \to (B, \mu_B)$ is a fuzzy homomorphism;*

(2) For all $a \in A$, $\mu_B(\alpha(a)) = \mu_A(a)$ if a mapping $\alpha : (A, \mu_A) \to (B, \mu_B)$ is a fuzzy isomorphism.

Definition 9. *Let A be an associative algebra, R be a subalgebra of A, L be a complete lattice, and a fuzzy set of R be a function $\mu_R : R \to L$. Then,*

(1) (R, μ_R) is a fuzzy left ideal of (A, μ_A) if $\mu_R(a \cdot b) \geq \mu_R(b)$ for all $a \in A$, $b \in R$;

(2) (R, μ_R) is a fuzzy right ideal of (A, μ_A) if $\mu_R(a \cdot b) \geq \mu_R(a)$ for all $a \in R$, $b \in A$;

(3) (R, μ_R) is a fuzzy ideal of (A, μ_A) if $\mu_R(a \cdot b) \geq \mu_R(a) \vee \mu_R(b)$ for all $a \in R$, $b \in R$.

Theorem 1. *Let (A, μ_A) be a fuzzy algebra and $\{(B_i, \mu_{B_i}) | i \in I\}$ be a set of fuzzy subalgebras of (A, μ_A). Then, $\bigwedge\limits_{i \in I} (B_i, \mu_{B_i})$ is a fuzzy subalgebra of (A, μ_A).*

Proof. It is obvious that $\bigwedge\limits_{i\in I} B_i \subset A$ and $\bigwedge\limits_{i\in I} B_i$ is a subalgebra of A. Then, we have

$$\begin{aligned}\bigwedge_{i\in I}\mu_{B_i}(a_i+b_i) &\geq \bigwedge_{i\in I}\left(\mu_{B_i}(a_i)\wedge\mu_{B_i}(b_i)\right)\\ &=\bigwedge_{i\in I}\mu_{B_i}(a_i)\wedge\bigwedge_{i\in I}\mu_{B_i}(b_i),\end{aligned}$$

$$\begin{aligned}\bigwedge_{i\in I}\mu_{B_i}(a_i\cdot b_i) &\geq \bigwedge_{i\in I}\left(\mu_{B_i}(a_i)\wedge\mu_{B_i}(b_i)\right)\\ &=\bigwedge_{i\in I}\mu_{B_i}(a_i)\wedge\bigwedge_{i\in I}\mu_{B_i}(b_i),\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{i\in I}\mu_{B_i}(ka_i) &= \mu_{B_1}(ka_1)\wedge\ldots\wedge\mu_{B_n}(ka_n)\\ &\geq \mu_{B_1}(a_1)\wedge\ldots\wedge\mu_{B_n}(a_n)\\ &=\bigwedge_{i\in I}\mu_{B_i}(a_i);\end{aligned}$$

for all $a_i, b_i \in B_i$, $k \in F$, and $\bigwedge\limits_{i\in I}\mu_{B_i}(e) = 1$.

In conclusion, $\bigwedge\limits_{i\in I}(B_i, \mu_{B_i})$ is a fuzzy subalgebra of (A, μ_A). □

Theorem 2. *Let (A, μ_A) be a fuzzy algebra and $\{(R_i, \mu_{R_i})|i \in I\}$ be a set of fuzzy ideals of (A, μ_A). Then, $\bigwedge\limits_{i\in I}(R_i, \mu_{R_i})$ is a fuzzy ideal of (A, μ_A).*

Proof. It is easy to obtain that $\bigwedge\limits_{i\in I} R_i$ is a subalgebra of A. Then, we have

$$\begin{aligned}\bigwedge_{i\in I}\mu_{R_i}(a_i\cdot b_i) &\geq \bigwedge_{i\in I}\left(\mu_{R_i}(a_i)\vee\mu_{R_i}(b_i)\right)\\ &=\bigwedge_{i\in I}\mu_{R_i}(a_i)\vee\bigwedge_{i\in I}\mu_{R_i}(b_i),\end{aligned}$$

for all a_i, $b_i \in R_i$.

In conclusion, $\bigwedge\limits_{i\in I}(R_i, \mu_{R_i})$ is a fuzzy ideal of (A, μ_A). □

Remark 4. *Let (A, μ_A) be a fuzzy algebra and $\{(B_i, \mu_{B_i})|i \in I\}$ be a set of fuzzy subalgebras of (A, μ_A) [respectively, let $\{(R_i, \mu_{R_i})|i \in I\}$ be a set of fuzzy ideals of (A, μ_A)]. Then, $\bigvee\limits_{i\in I}(B_i, \mu_{B_i})$ may not be a fuzzy subalgebra of (A, μ_A) [respectively, $\bigvee\limits_{i\in I}(R_i, \mu_{R_i})$ may not be a fuzzy ideal of (A, μ_A)].*

Example 2. *Consider polynomial algebras in Example 1, where addition, multiplication, and scalar multiplication are defined in a conventional manner. Consider that two of these fuzzy subalgebras, $(F1, \mu_1)$ and $(F2, \mu_2)$ are sets of fuzzy polynomial algebras:*

(1) $(F1, \mu_1)$: The fuzzy degree of the fuzzy subsets of all constant polynomials is 1; the fuzzy degree of the other polynomials is 0.

(2) $(F2, \mu_2)$: The fuzzy degree of fuzzy subsets of all linear polynomials is 1; the fuzzy degree of the other polynomials is 0.

Obviously, both $(F1, \mu_1)$ and $(F2, \mu_2)$ are fuzzy subalgebras. However, $(F1, \mu_1) \vee (F2, \mu_2)$ is not a fuzzy subalgebra; for example, the fuzzy degree of quadratic polynomial x^2 is 0 in $(F1, \mu_1) \vee (F2, \mu_2)$; however, x^2 is neither a constant polynomial nor a linear polynomial.

Remark 5. *One can provide an example of fuzzy ideals by following the construction method described in Example 2.*

Theorem 3. *Let (A, μ_A), (B, μ_B) be fuzzy algebras, $f : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy epimorphism, and (R, μ_R) be a fuzzy ideal of (A, μ_A). Then, $(f(R), \mu_{f(R)})$ is a fuzzy ideal of (B, μ_B).*

Proof. It is easy to obtain that $f(R)$ is a subalgebra of B.

Suppose that a, $b \in R$; thus, $f(a)$, $f(b) \in f(R)$. Then,

$$\begin{aligned}
\mu_{f(R)}\big(f(a) \cdot f(b)\big) &= \bigvee_{f(z)=f(a)\cdot f(b)} \mu_R(z) \\
&= \bigvee_{z=a\cdot b} \mu_R(a \cdot b) \\
&\geq \bigvee_{z=a\cdot b} \big(\mu_R(a) \vee \mu_R(b)\big) \\
&= \big(\bigvee \mu_R(a)\big) \vee \big(\bigvee \mu_R(b)\big) \\
&= \mu_{f(R)}\big(f(a)\big) \vee \mu_{f(R)}\big(f(b)\big).
\end{aligned}$$

In conclusion, $(f(R), \mu_{f(R)})$ is a fuzzy ideal of (B, μ_B). □

Theorem 4. *Let (A, μ_A), (B, μ_B) be fuzzy algebras, $f : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy homomorphism, and (R, μ_R) be a fuzzy ideal of (B, μ_B). Then, $(f^{-1}(R), \mu_{f^{-1}(R)})$ is a fuzzy ideal of (A, μ_A).*

Proof. It is easy to obtain that $f^{-1}(R)$ is a subalgebra of A.

Suppose that a, $b \in f^{-1}(R)$; thus, $f(a)$, $f(b) \in R$. Then, from Remark 3(1), we have

$$\begin{aligned}
\bigvee \mu_{f^{-1}(R)}(a \cdot b) &= \mu_R\big(f(a \cdot b)\big) \\
&= \mu_R\big(f(a) \cdot f(b)\big) \\
&\geq \mu_R\big(f(a)\big) \vee \mu_R\big(f(b)\big) \\
&= \bigvee \mu_{f^{-1}(R)}(a) \vee \bigvee \mu_{f^{-1}(R)}(b).
\end{aligned}$$

In conclusion, $(f^{-1}(R), \mu_{f^{-1}(R)})$ is a fuzzy ideal of (A, μ_A). □

4. Quotients of Fuzzy Algebras

In this section, we define the quotients constructed by fuzzy ideals and establish the existences of fuzzy homomorphisms and fuzzy isomorphisms between these quotient structures.

Definition 10. *Let (A, μ_A) be a fuzzy algebra and (R, μ_R) be a fuzzy ideal of (A, μ_A). We define an addition, a multiplication, and a scalar multiplication operations on A/R as follows:*

(1) $(a \cdot R) + (b \cdot R) = (a + b) \cdot R$,

(2) $(a \cdot R) \cdot (b \cdot R) = (a \cdot b) \cdot R$,

(3) $k(a \cdot R) = (ka) \cdot R$,

for all a, $b \in A$, $k \in F$.

Theorem 5. *Let (A, μ_A) be a fuzzy algebra, (R, μ_R) be a fuzzy ideal of (A, μ_A). There exists an $a \in A$ such that $\mu_A(a) = 1$, $\mu_{A/R}$ is defined by*

$$\mu_{A/R}(a'/R) = \begin{cases} 1, & a' \in R, \\ \sup\limits_{b\in R} \mu_A(a' \cdot b), & a' \notin R, \end{cases};$$

then, $(A/R, \mu_{A/R})$ is a fuzzy algebra, which is called a fuzzy quotient algebra of (A, μ_A).

Remark 6. *First, we prove that the operations on A/R are well-defined.*

Let $a', a'', b', b'' \in A$, $r \in R$, $a' \cdot r$ and $b' \cdot r$ belong to the same class, $a'' \cdot r$ and $b'' \cdot r$ belong to the same class, thus,

$$\mu_A(a' \cdot r) = \mu_A(b' \cdot r), \mu_A(a'' \cdot r) = \mu_A(b'' \cdot r),$$

$$\sup_{r \in R} \mu_A(a' \cdot r) = \sup_{r \in R} \mu_A(b' \cdot r), \text{ and } \sup_{r \in R} \mu_A(a'' \cdot r) = \sup_{r \in R} \mu_A(b'' \cdot r).$$

(1) If $a', b' \in R$, $a'', b'' \in R$, then

$$\mu_A\big((a' \cdot a'') \cdot r\big) = 1 = \mu_A\big((b' \cdot b'') \cdot r\big).$$

(2) If $a', b' \notin R$, $a'', b'' \notin R$, then

$$\sup_{r \in R} \mu_A\big((a' \cdot a'') \cdot r\big) = \sup_{r \in R} \mu_A\big(a' \cdot (a'' \cdot r)\big) = \sup_{\substack{a'' \cdot r = \bar{r}, \\ \bar{r} \in R}} \mu_A(a' \cdot \bar{r}),$$

$$\sup_{r \in R} \mu_A\big((b' \cdot b'') \cdot r\big) = \sup_{r \in R} \mu_A\big(b' \cdot (b'' \cdot r)\big) = \sup_{\substack{b'' \cdot r = \bar{\bar{r}}, \\ \bar{\bar{r}} \in R}} \mu_A(b' \cdot \bar{\bar{r}}),$$

and $a' \cdot \bar{r} \in a' \cdot R$, $b' \cdot \bar{\bar{r}} \in b' \cdot R$, then $\sup_{\substack{a'' \cdot r = \bar{r}, \\ \bar{r} \in R}} \mu_A(a' \cdot \bar{r}) = \sup_{\substack{b'' \cdot r = \bar{\bar{r}}, \\ \bar{\bar{r}} \in R}} \mu_A(b' \cdot \bar{\bar{r}})$, thus, $\sup_{r \in R} \mu_A\big((a' \cdot a'') \cdot r\big)$ $= \sup_{r \in R} \mu_A\big((b' \cdot b'') \cdot r\big)$, we can obtain that the multiplication operation is well-defined. In the same way, we can obtain that addition and scalar multiplication operations are well-defined.

Next, we prove Theorem 5.

Proof. Let us prove that the result under multiplication is true.

Suppose that $a_1, a_2 \in A$.

(1) If $a_1, a_2 \in R$, then

$$\mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big) = \mu_{A/R}\big((a_1 \cdot a_2) \cdot R\big) = 1;$$

thus, $\mu_{A/R}(a_1 \cdot R) \wedge \mu_{A/R}(a_2 \cdot R) \leq \mu_{A/R}((a_1 \cdot R) \cdot (a_2 \cdot R))$.

(2) If $a_1 \in R$, $a_2 \notin R$, then

$$\begin{aligned}
\mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big) &= \sup_{b \in R} \mu_A\big((a_1 \cdot a_2) \cdot b\big) \\
&\geq \sup_{a_1 \in R, a_2 \notin R} \mu_A(a_1 \cdot a_2) \wedge \sup_{b \in R} \mu_A(b) \\
&\geq \sup_{a_1 \in R, a_2 \notin R} \mu_A(a_1 \cdot a_2) \wedge 1 \\
&= \mu_{A/R}(a_2 \cdot R).
\end{aligned}$$

In addition, $\mu_{A/R}(a_1 \cdot R) = 1$; then, $\mu_{A/R}(a_1 \cdot R) \wedge \mu_{A/R}(a_2 \cdot R) = \mu_{A/R}(a_2 \cdot R)$. Thus, $\mu_{A/R}(a_1 \cdot R) \wedge \mu_{A/R}(a_2 \cdot R) \leq \mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big)$.

In conclusion, the result under multiplication is true.

Similarly, we can prove that the results under addition and scalar multiplication are true.

Thus, $(A/R, \mu_{A/R})$ is a fuzzy algebra. □

Remark 7. *The definition of $\mu_{A/R}$ in Theorem 5 conforms to the Zadeh extension principle.*

Theorem 6. *Let (A, μ_A) be a fuzzy algebra; there exists an $a \in A$ such that $\mu_A(a) = 1$. Let (R, μ_R) be a fuzzy ideal of (A, μ_A) and $(A/R, \mu_{A/R})$ be a fuzzy quotient algebra of (A, μ_A). $\mu_{A/R}$ is defined by*

$$\mu_{A/R}(a'/R) = \begin{cases} 1, & a' \in R, \\ \sup_{b \in R} \mu_A(a' \cdot b), & a' \notin R, \end{cases}.$$

We define a mapping as follows:

$$v : (A, \mu_A) \to (A/R, \mu_{A/R}), v(a') = a'/R,$$

for all $a' \in A$; then, v is a fuzzy homomorphism.

Proof. First, it is easy to obtain that v is a homomorphism.

Next, we prove that v is a fuzzy homomorphism.

(1) If a_1, $a_2 \in R$, then

$$\mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big) = \mu_{A/R}\big((a_1 \cdot a_2) \cdot R\big) = 1.$$

Thus, $\mu_A(a_1) \wedge \mu_A(a_2) \leq \mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big)$.

(2) If $a_1 \in R$, $a_2 \notin R$, then

$$\begin{aligned}\mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big) &= \sup_{b \in R} \mu_A\big((a_1 \cdot a_2) \cdot b\big) \\ &\geq \sup_{a_1 \in R} \mu_A(a_1) \wedge \sup_{a_2 \notin R} \mu_A(a_2) \wedge \sup_{b \in R} \mu_A(b) \\ &= \sup_{a_1 \in R} \mu_A(a_1) \wedge \sup_{a_2 \notin R} \mu_A(a_2) \wedge 1 \\ &\geq \mu_A(a_1) \wedge \mu_A(a_2).\end{aligned}$$

Thus, $\mu_A(a_1) \wedge \mu_A(a_2) \leq \mu_{A/R}\big((a_1 \cdot R) \cdot (a_2 \cdot R)\big)$.

In conclusion, the result under multiplication is true. Similarly, we can prove that the results under addition and scalar multiplication are true.

Hence, v is a fuzzy homomorphism. □

Theorem 7. *Let (A, μ_A), (B, μ_B) be fuzzy algebras, $f' : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy homomorphism, (R, μ_R) and $(R', \mu_{R'})$ be fuzzy ideals of (A, μ_A) and (B, μ_B), respectively, and $(A/R, \mu_{A/R})$ and $(B/R', \mu_{B/R'})$ be fuzzy quotient algebras of (A, μ_A) and (B, μ_B), respectively. A mapping $f : (A/R, \mu_{A/R}) \to (B/R', \mu_{B/R'})$ is defined as follows:*

$$f(a \cdot R) = b \cdot R', f(R) \subseteq R', f(a \cdot R) = f(a) \cdot R,$$

for all $a \cdot R \in A/R$, $b \cdot R' \in B/R'$, $\mu_{B/R'}$ is defined by

$$\mu_{B/R'}(b/R') = \begin{cases} 1, & b \in R', \\ \sup\limits_{f(a/R) = b/R'} \mu_A(a \cdot R), & b \notin R', \end{cases};$$

then, f is a fuzzy homomorphism.

$$\begin{array}{ccc} (A, \mu_A) & \xrightarrow{f'} & (B, \mu_B) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ (A/R, \mu_{A/R}) & \xrightarrow{f} & (B/R', \mu_{B/R'}) \end{array}$$

Proof. First, it is easy to obtain that f is a homomorphism.

Next, we prove that f is a fuzzy homomorphism.

Let us prove that the result under multiplication is true.

(1) If $b \in R'$, then $\mu_{B/R'}(b \cdot R') = 1$; thus, $\mu_{A/R}(a \cdot R) \leq \mu_{B/R'}(b \cdot R')$.

(2) If $b \notin R'$, then

$$\begin{aligned}\mu_{B/R'}(b \cdot R') &= \sup_{f(a \cdot R) = b \cdot R'} \mu_{A/R}(a \cdot R) \\ &= \bigvee_{f(a \cdot R) = b \cdot R'} \mu_{A/R}(a \cdot R) \\ &\geq \mu_{A/R}(a \cdot R).\end{aligned}$$

In conclusion, the result under multiplication is true.

Similarly, we can prove that the results under addition and scalar multiplication are true.

Hence, f is a fuzzy homomorphism. $\square$

Theorem 8. *Let (A, μ_A), (B, μ_B) be fuzzy algebras, $f : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy homomorphism, and (R, μ_R) be a fuzzy ideal of (B, μ_B). Thus, $(f^{-1}(R), \mu_{f^{-1}(R)})$ is a fuzzy ideal of (A, μ_A), and $(A/f^{-1}(R), \mu_1)$ is a fuzzy quotient algebra. We define a mapping as follows:*

$$\alpha : (A/f^{-1}(R), \mu_1) \to (B/R, \mu_2), \alpha\big(a/f^{-1}(R)\big) = b/R,$$

for all $a/f^{-1}(R) \in A/f^{-1}(R)$, and μ_2 is defined as follows:

$$\mu_2(b/R) = \begin{cases} 1, & b \in R, \\ \sup\limits_{\alpha(a'/f^{-1}(R))=b'/R, b' \in R} \mu_1\big((a \cdot a') \cdot f^{-1}(R)\big), & b \notin R, \cdot \end{cases}$$

If $b \in R$, then $\mu_1\big(\alpha^{-1}(b/R)\big) = 1$, and there exists an $a''/f^{-1}(R) \in A/f^{-1}(R)$ such that $\mu_1\big(a''/f^{-1}(R)\big) = 1$; then, α is a fuzzy isomorphism.

Proof. First, we prove that α is a homomorphism.

Suppose that $a_1/f^{-1}(R)$, $a_2/f^{-1}(R)$,$(a_1 \cdot a_2)/f^{-1}(R)$,$(a_1 + a_2)/f^{-1}(R) \in A/f^{-1}(R)$, $b_1/R, b_2/R, (b_1 + b_2)/R$,$(b_1 \cdot b_2)/R$,$b/R, b^*/R \in B/R$,$\alpha\big(a_1/f^{-1}(R)\big) = b_1 \cdot R, \alpha\big(a_2/f^{-1}(R)\big) = b_2 \cdot R$,$\alpha\big((a_1 \cdot a_2)/f^{-1}(R)\big) = b \cdot R, \alpha\big((a_1 + a_2)/f^{-1}(R)\big) = b^* \cdot R, b_1 \cdot b_2 = b, b_1 + b_2 = b^*$; then, we have

$$\begin{aligned} \alpha\Big(\big(a_1/f^{-1}(R)\big) \cdot \big(a_2/f^{-1}(R)\big)\Big) &= \alpha\big((a_1 \cdot a_2)/f^{-1}(R)\big) \\ &= b \cdot R \\ &= (b_1 \cdot b_2) \cdot R \\ &= (b_1 \cdot R) \cdot (b_2 \cdot R) \\ &= \alpha\big(a_1/f^{-1}(R)\big) \cdot \alpha\big(a_2/f^{-1}(R)\big), \end{aligned}$$

$$\begin{aligned} \alpha\Big(\big(a_1/f^{-1}(R)\big) + \big(a_2/f^{-1}(R)\big)\Big) &= \alpha\big((a_1 + a_2)/f^{-1}(R)\big) \\ &= b^* \cdot R \\ &= (b_1 + b_2) \cdot R \\ &= (b_1 \cdot R) + (b_2 \cdot R) \\ &= \alpha\big(a_1/f^{-1}(R)\big) + \alpha\big(a_2/f^{-1}(R)\big), \end{aligned}$$

and

$$\begin{aligned} \alpha\Big(k\big(a_1/f^{-1}(R)\big)\Big) &= \alpha\Big(k\big(a_1/f^{-1}(R)\big)\Big) \\ &= (kb_1) \cdot R \\ &= k\alpha\big(a_1/f^{-1}(R)\big). \end{aligned}$$

Thus, α is a homomorphism.

Next, we prove that α is a bijection.

(i) For any $a_1/f^{-1}(R), a_2/f^{-1}(R) \in A/f^{-1}(R)$, if $a_1/f^{-1}(R) \neq a_2/f^{-1}(R)$, then $\alpha\big(a_1/f^{-1}(R)\big) \neq \alpha\big(a_2/f^{-1}(R)\big)$; thus, α is an injection.

(ii) For any $b/R \in B/R$, there exists an $a_1/f^{-1}(R) \in A/f^{-1}(R)$ such that $\alpha\big(a_1/f^{-1}(R)\big) = b_1/R$; thus, α is a surjection.

From the above proof, we can obtain that α is an isomorphism.

Finally, we prove that α is a fuzzy isomorphism.

Suppose that $b_1, b_2 \in B$; then,

(1) If $b_1, b_2 \in R$, then

$$\mu_2\big((b_1 \cdot R) \cdot (b_2 \cdot R)\big) = \mu_2\big((b_1 \cdot b_2) \cdot R\big) = 1 = \mu_1\big((a_1 \cdot a_2) \cdot f^{-1}(R)\big).$$

Thus, $\mu_2\big((b_1 \cdot R) \cdot (b_2 \cdot R)\big) = \mu_1\big((a_1 \cdot a_2) \cdot f^{-1}(R)\big)$.

(2) If $b_1 \in R$, $b_2 \notin R$, since $(A/f^{-1}(R), \mu_1)$ is a fuzzy algebra, we have $\mu_1\big((a_1 \cdot a_2 \cdot a') \cdot f^{-1}(R)\big) \geq \mu_1\big(a_1 \cdot f^{-1}(R)\big) \wedge \mu_1\big(a_2 \cdot f^{-1}(R)\big) \wedge \mu_1\big(a' \cdot f^{-1}(R)\big)$; thus,

$$\begin{aligned}
\mu_2\big((b_1 \cdot R) \cdot (b_2 \cdot R)\big) &= \mu_2\big((b_1 \cdot b_2) \cdot R\big) \\
&= \sup \mu_1\big((a_1 \cdot a_2 \cdot a') \cdot f^{-1}(R)\big) \\
&\geq \sup_{\alpha\left(a_1 \cdot f^{-1}(R)\right) = b_1 \cdot R,} \mu_1\big(a_1 \cdot f^{-1}(R)\big) \\
&\wedge \sup_{\alpha\left(a_2 \cdot f^{-1}(R)\right) = b_2 \cdot R} \mu_1\big(a_2 \cdot f^{-1}(R)\big) \\
&\wedge \sup_{\alpha\left(a' \cdot f^{-1}(R)\right) = b' \cdot R, b' \in R} \mu_1\big(a' \cdot f^{-1}(R)\big) \\
&= 1 \wedge \sup_{\alpha\left(a_2 \cdot f^{-1}(R)\right) = b_2 \cdot R} \mu_1\big(a_2 \cdot f^{-1}(R)\big) \wedge 1 \\
&= \sup_{\alpha\left(a_2 \cdot f^{-1}(R)\right) = b_2 \cdot R} \mu_1\big(a_2 \cdot f^{-1}(R)\big) \\
&\geq \mu_1\big(a_2 \cdot f^{-1}(R)\big).
\end{aligned}$$

From the definition of fuzzy algebras, we have $\mu_1\big((a_1 \cdot a_2) \cdot f^{-1}(R)\big) \geq \mu_1\big(a_1 \cdot f^{-1}(R)\big) \wedge \mu_1\big(a_2 \cdot f^{-1}(R)\big)$; thus, $\mu_1\big((a_1 \cdot a_2) \cdot f^{-1}(R)\big) \leq \mu_2\big((b_1 \cdot R) \cdot (b_2 \cdot R)\big)$.

Conversely, whether $b_1 \cdot b_2 \in R$ or $b_1 \cdot b_2 \notin R$, there always exists an $(a_1 \cdot a_2) \cdot f^{-1}(R) \in A/f^{-1}(R)$ such that $\mu_1\big((a_1 \cdot a_2) \cdot f^{-1}(R)\big) \leq \mu_2\big((b_1 \cdot R) \cdot (b_2 \cdot R)\big)$.

In conclusion, the result under multiplication is true.

Similarly, we can prove that the results under addition and scalar multiplication are true.

In conclusion, α is a fuzzy isomorphism. □

5. Homomorphism Theorems

In this section, we give the concept of homomorphic kernels and prove that they are fuzzy ideals. In addition, three homomorphism theorems are proved.

Definition 11. *Let (A,μ_A),(B,μ_B) be fuzzy algebras, $\alpha : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy homomorphism, and L be a complete lattice. Then, the kernel of α is defined as follows:*

$$Ker\alpha = \{a \in A \mid \alpha(a) = 0\}, \mu : Ker\alpha \to L, \mu(a) = 1,$$

which we denote as $(Ker\alpha,\mu)$ for short.

Example 3. *Let (A,μ_A),(B,μ_B) be fuzzy matrices, $\alpha : (A, \mu_A) \to (B, \mu_B)$, and for all (a, μ_a), $(c, \mu_c) \in (A, \mu_A)$, $\alpha\big((a, \mu_a)\big) = (a, \mu_a) \cdot (c, \mu_c) = (b, \mu_b)$; then, $Ker\alpha = \{(c, \mu_c) \in (A, \mu_A) | (a, \mu_a) \cdot (c, \mu_c) = 0\}$, and the 0 here represents the null matrix.*

Theorem 9. *$(Ker\alpha,\mu)$ is a fuzzy ideal of (A, μ_A).*

Proof. Suppose that $a, b \in Ker\alpha$; then,

$$\mu(a \cdot b) \geq \mu(a) \wedge \mu(b) = 1 \wedge 1 = 1 = \mu(a) \vee \mu(b).$$

Thus, $(Ker\alpha,\mu)$ is a fuzzy ideal of (A, μ_A). □

Theorem 10. *Let (A, μ_A), (B, μ_B) be fuzzy algebras and $\alpha : (A, \mu_A) \to (B, \mu_B)$ be a fuzzy epimorphism. There exists an $a \in A$ such that $\mu_A(a) = 1$, $(A/Ker\alpha, \mu_{A/Ker\alpha})$ is a fuzzy quotient algebra of (A, μ_A), and $\mu_{A/Ker\alpha}$ is defined by*

$$\mu_{A/Ker\alpha}(a'/Ker\alpha) = \begin{cases} 1, & a' \in a/Ker\alpha, \\ \sup\limits_{b \in a/Ker\alpha} \mu_A(a' \cdot b), & a' \notin a/Ker\alpha, \end{cases}.$$

If $a' \in a/Ker\alpha$, then, $\mu_B(\alpha(a')) = 1$. $v : (A, \mu_A) \to (A/Ker\alpha, \mu_{A/Ker\alpha})$ is a fuzzy homomorphism, and $v(a') = a'/Ker\alpha$ for all $a' \in A$. We define a mapping as follows:

$$\beta : (A/Ker\alpha, \mu_{A/Ker\alpha}) \to (B, \mu_B), \beta\big(a'/Ker\alpha\big) = \alpha\big(a'\big),$$

for all $a' \in A$; then, β is a fuzzy isomorphism.

$$\begin{tikzcd}
(A,\mu_A) \arrow[rr, "\alpha"] \arrow[dr, "v"'] & & (B,\mu_B) \\
& (A/Ker\alpha,\mu_{A/Ker\alpha}) \arrow[ur, dotted, "\beta"'] &
\end{tikzcd}$$

Proof. Suppose that $a_1, a_2 \in A$. We can obtain that β is a homomorphism using Theorem 6. We only need to prove that β is a bijection and $\mu_B\Big(\beta\big((a_1 \cdot a_2)/Ker\alpha\big)\Big) = \mu_{A/Ker\alpha}\big((a_1 \cdot a_2)/Ker\alpha\big)$.

First, we prove that β is a bijection.

(i) For any a_1, $a_2 \in A$, if $a_1/Ker\alpha \neq a_2/Ker\alpha$, then, $\alpha(a_1) \neq \alpha(a_2)$; thus, β is an injection.

(ii) For any $c \in B$, since α is surjective, there exists an $a' \in A$ such that $\alpha(a') = c$. Since $a'/Ker\alpha \in A/Ker\alpha$, then $\beta(a'/Ker\alpha) = \alpha(a') = c$; thus, β is a surjection.

Next, we prove that $\mu_B\Big(\beta\big((a_1 \cdot a_2)/Ker\alpha\big)\Big) = \mu_{A/Ker\alpha}\big((a_1 \cdot a_2)/Ker\alpha\big)$.

(1) If $a_1, a_2 \in a/Ker\alpha$, then $\mu_{A/Ker\alpha}\big((a_1 \cdot Ker\alpha) \cdot (a_2 \cdot Ker\alpha)\big) = \mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big) = 1$. In this case, $\mu_B\big(\alpha(a_1 \cdot a_2)\big) = 1$.

Thus, $\mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big) = \mu_B\Big(\beta\big((a_1 \cdot a_2) \cdot Ker\alpha\big)\Big)$.

(2) If $a_1 \in a/Ker\alpha$, $a_2 \notin a/Ker\alpha$, then

$$\begin{aligned}
\mu_{A/Ker\alpha}\big((a_1 \cdot Ker\alpha) \cdot (a_2 \cdot Ker\alpha)\big) &= \mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big) \\
&= \sup_{b \in a/Ker\alpha} \mu_A\big((a_1 \cdot a_2) \cdot b\big) \\
&\geq \sup_{a_1 \in a/Ker\alpha} \mu_A(a_1) \wedge \sup_{a_2 \notin a/Ker\alpha} \mu_A(a_2) \wedge \sup_{b \in a/Ker\alpha} \mu_A(b) \\
&= \sup_{a_1 \in a/Ker\alpha} \mu_A(a_1) \wedge \sup_{a_2 \notin a/Ker\alpha} \mu_A(a_2) \wedge 1 \\
&\geq \mu_A(a_1) \wedge \mu_A(a_2).
\end{aligned}$$

For any $a_1, a_2 \in A$,

$$\begin{aligned}
\mu_B\big(\alpha(a_1 \cdot a_2)\big) &= \vee \mu_A\Big(\alpha^{-1}\big(\alpha(a_1 \cdot a_2)\big)\Big) \\
&\geq \mu_A(a_1) \wedge \mu_A(a_2);
\end{aligned}$$

then, $\mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big) \leq \mu_B\Big(\beta\big((a_1 \cdot a_2) \cdot Ker\alpha\big)\Big)$.

Let $\beta' : (B, \mu_B) \to (A/Ker\alpha, \mu_{A/Ker\alpha})$, and $\beta'(\alpha(a')) = a'/Ker\alpha$ for all $a' \in A$.

(3) If $a_1, a_2 \in a/Ker\alpha$, then the process of the proof is similar to (1).

(4) If $a_1 \in a/Ker\alpha$, $a_2 \notin a/Ker\alpha$, then

$$\begin{aligned}
\mu_{A/Ker\alpha}\big((a_1 \cdot Ker\alpha) \cdot (a_2 \cdot Ker\alpha)\big) &= \mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big) \\
&= \sup_{b \in a/Ker\alpha} \mu_A((a_1 \cdot a_2) \cdot b) \\
&\geq \sup_{a_1 \in a/Ker\alpha} \mu_A(a_1) \wedge \sup_{a_2 \notin a/Ker\alpha} \mu_A(a_2) \wedge \sup_{b \in a/Ker\alpha} \mu_A(b) \\
&= 1 \wedge \sup_{a_2 \notin a/Ker\alpha} \mu_A(a_2) \wedge 1 \\
&\geq \mu_A(a_2).
\end{aligned}$$

From the definition of μ_B, $\mu_B(\alpha(a_1 \cdot a_2)) \geq \mu_A(a_1) \wedge \mu_A(a_2)$, we have

$$\mu_B\Big(\beta\big((a_1 \cdot a_2) \cdot Ker\alpha\big)\Big) = \mu_B(\alpha(a_1 \cdot a_2)) \leq \mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big).$$

Hence, $\mu_B\Big(\beta\big((a_1 \cdot a_2) \cdot Ker\alpha\big)\Big) = \mu_{A/Ker\alpha}\big((a_1 \cdot a_2) \cdot Ker\alpha\big)$.

In conclusion, the result under multiplication is true. Similarly, we can prove that the results under addition and scalar multiplication are true; thus, β is a fuzzy isomorphism. □

Theorem 11. *Let (A, μ_A) be a fuzzy algebra, (R_1, μ_{R_1}) and (R_2, μ_{R_2}) be fuzzy ideals of (A, μ_A), (R_2, μ_{R_2}) be a fuzzy subalgebra of (R_1, μ_{R_1}), $(A/R_1, \mu_1)$ and $(A/R_2, \mu_2)$ be fuzzy quotient algebras of (A, μ_A), $\big((A/R_2)/(R_1/R_2), \mu_3\big)$ be a fuzzy quotient algebra of $(A/R_2, \mu_2)$. There exists an $(a''/R_2)/(R_1/R_2) \in (A/R_2)/(R_1/R_2)$ such that $\mu_3\big((a''/R_2)/(R_1/R_2)\big) = 1$, μ_1 is defined as follows:*

$$\mu_1(a/R_1) = \begin{cases} 1, & a \in (a''/R_2)/(R_1/R_2), \\ \sup\limits_{a' \in (a''/R_2)/(R_1/R_2)} \mu_3\Big(\big((a \cdot a')/R_2\big)/(R_1/R_2)\Big), & a \notin (a''/R_2)/(R_1/R_2), \end{cases}.$$

We define a mapping as follows:

$$\alpha : \big((A/R_2)/(R_1/R_2), \mu_3\big) \to (A/R_1, \mu_1), \alpha\big((a/R_2)/(R_1/R_2)\big) = a/R_1,$$

for all $(a/R_2)/(R_1/R_2) \in (A/R_2)/(R_1/R_2)$; then, α is a fuzzy isomorphism.

Proof. For any $a_1, a_2 \in A$, $a_1/R_1, a_2/R_1 \in A/R_1$,$(a_1/R_2)/(R_1/R_2)$,$(a_2/R_2)/(R_1/R_2) \in (A/R_2)/(R_1/R_2)$, we have $\alpha\big((a_1/R_2)/(R_1/R_2)\big) = \alpha\big((a_2/R_2)/(R_1/R_2)\big) \Leftrightarrow a_1/R_1 = a_2/R_1$; thus, α is a well-defined bijection.

Similarly, we can obtain that α is a homomorphism using Theorem 6; thus, α is an isomorphism.

Next, we prove that α is a fuzzy isomorphism.

(1) If $a_1, a_2 \in (a''/R_2)/(R_1/R_2)$, then

$$\mu_1\big((a_1 \cdot R_1) \cdot (a_2 \cdot R_1)\big) = \mu_1\big((a_1 \cdot a_2) \cdot R_1\big) = 1 = \mu_3\Big(\big((a_1 \cdot a_2)/R_2\big)/(R_1/R_2)\Big).$$

Thus, $\mu_1\big((a_1 \cdot R_1) \cdot (a_2 \cdot R_1)\big) = \mu_3\Big(\big((a_1 \cdot a_2)/R_2\big)/(R_1/R_2)\Big)$.

(2) If $a_1 \in (a''/R_2)/(R_1/R_2)$, $a_2 \notin (a''/R_2)/(R_1/R_2)$, since $\big((A/R_2)/(R_1/R_2), \mu_3\big)$ is a fuzzy algebra, we have $\mu_3\Big(\big((a_1 \cdot a_2 \cdot a')/R_2\big)/(R_1/R_2)\Big) \geq \mu_3\big((a_1/R_2)/(R_1/R_2)\big) \wedge$

$\mu_3\big((a_2/R_2)/(R_1/R_2)\big) \wedge \mu_3\big((a'/R_2)/(R_1/R_2)\big)$; thus,

$$
\begin{aligned}
\mu_1((a_1 \cdot R_1) \cdot (a_2 \cdot R_1)) &= \mu_1((a_1 \cdot a_2) \cdot R_1) \\
&= \sup_{a' \in (a''/R_2)/(R_1/R_2)} \mu_3\Big(((a_1 \cdot a_2 \cdot a')/R_2)/(R_1/R_2)\Big) \\
&\geq \sup_{a_1 \in (a''/R_2)/(R_1/R_2)} \mu_3\big((a_1/R_2)/(R_1/R_2)\big) \\
&\wedge \sup_{a_2 \notin (a''/R_2)/(R_1/R_2)} \mu_3\big((a_2/R_2)/(R_1/R_2)\big) \\
&\wedge \sup_{a' \in (a''/R_2)/(R_1/R_2)} \mu_3\big((a'/R_2)/(R_1/R_2)\big) \\
&= \sup_{a_1 \in (a''/R_2)/(R_1/R_2)} \mu_3\big((a_1/R_2)/(R_1/R_2)\big) \\
&\wedge \sup_{a_2 \notin (a''/R_2)/(R_1/R_2)} \mu_3\big((a_2/R_2)/(R_1/R_2)\big) \wedge 1 \\
&= \sup_{a_1 \in (a''/R_2)/(R_1/R_2)} \mu_3\big((a_1/R_2)/(R_1/R_2)\big) \\
&\wedge \sup_{a_2 \notin (a''/R_2)/(R_1/R_2)} \mu_3\big((a_2/R_2)/(R_1/R_2)\big) \\
&\geq \mu_3\big((a_1/R_2)/(R_1/R_2)\big) \wedge \mu_3\big((a_2/R_2)/(R_1/R_2)\big).
\end{aligned}
$$

Thus, $\mu_3\big((a_1 \cdot R_2)/(R_1/R_2)\big) \wedge \mu_3\big((a_2 \cdot R_2)/(R_1/R_2)\big) \leq \mu_1\big((a_1 \cdot a_2)/R_1\big)$.

Conversely, whether $a_1 \cdot a_2 \in (a''/R_2)/(R_1/R_2)$ or $a_1 \cdot a_2 \notin (a''/R_2)/(R_1/R_2)$, there always exists an $a \in (a''/R_2)/(R_1/R_2)$ such that $\mu_1(a/R_1) = \mu_1\big((a_1 \cdot a_2)/R_1\big) = \mu_3\Big(((a_1 \cdot a_2)/R_2)/(R_1/R_2)\Big)$.

In conclusion, the result under multiplication is true. Similarly, we can prove that the results under addition and scalar multiplication are true.

Hence, α is a fuzzy isomorphism. □

Theorem 12. *Let (A, μ_A) be a fuzzy algebra, (H, μ_H) be a fuzzy algebra of (A, μ_A), and (R, μ_R) be a fuzzy ideal of (A, μ_A); then, $(HR/R, \mu_4)$ and $(H/H \cap R, \mu_5)$ are fuzzy quotient algebras. We define a mapping as follows:*

$$\alpha' : (HR/R, \mu_4) \to (H/H \cap R, \mu_5), \alpha(hr/r) = h/h \cap r,$$

for all $hr/r \in HR/R$, there exists an $h''r/r \in HR/R$ such that $\mu_4(h''r/r) = 1$, and $\mu_5(h/h \cap r)$ is defined by

$$\mu_5(h/h \cap r) = \begin{cases} 1, & h \in h''r/r, \\ \sup_{h' \in h''r/r} \mu_4\big((hr \cdot h'r)/r\big), & h \notin h''r/r, \end{cases};$$

then, similar to the proof of Theorem 11, we can obtain that α' is a fuzzy isomorphism.

6. Conclusions

In this paper, we discussed the properties of fuzzy ideals and quotients of fuzzy associative algebras. In Section 3, we provided the concepts of fuzzy associative algebras, fuzzy homomorphisms, and fuzzy ideals over a common number field. In Theorems 1 and 2, we proved that the intersections of the subalgebras were fuzzy subalgebras and the intersections of fuzzy ideals were fuzzy ideals. In Theorems 3 and 4, we showed that if $f : (A, \mu_A) \to (B, \mu_B)$ is a fuzzy epimorphism, then the homomorphic images and preimages of fuzzy ideals are fuzzy ideals. In Section 4, we defined an addition, a multiplication, and a scalar multiplication operation on quotient structures constructed by fuzzy ideals. We proved that the quotient structures created by fuzzy ideals were fuzzy algebras and there were fuzzy homomorphisms between fuzzy algebras and its fuzzy quotient algebras. In Theorem 7, we proved that if (R, μ_R) and $(R', \mu_{R'})$ are fuzzy ideals of (A, μ_A) and (B, μ_B), respectively, then $f : (A/R, \mu_{A/R}) \to (B/R', \mu_{B/R'})$ is a fuzzy homomorphism. In Section 5, we defined the concepts of *kernels* in fuzzy homomorphisms, and in Theorem 9, we proved that the *kernels* were fuzzy ideals. In particular, we proved that if $\alpha : (A, \mu_A) \to (B, \mu_B)$ is a fuzzy epimorphism, then $A/Ker\alpha$ is isomorphic to (B, μ_B). Moreover, we proved two other homomorphism theorems.

This work helps us to better understand other specific fuzzy algebra structure theories and provides important theoretical support for the study of other algebraic theories. On this basis, the classification and representation of fuzzy associative algebras can be studied in the future.

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