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Review

Dynamics of Social Influence and Knowledge in Networks: Sociophysics Models and Applications in Social Trading, Behavioral Finance and Business

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Abstract: In this paper we offer a comprehensive review of Sociophysics, focusing on relevant models as well as selected applications in social trading, behavioral finance and business. We discuss three key aspects of social diffusion dynamics, namely Opinion Dynamics (OD), Group Decision-Making (GDM) and Knowledge Dynamics (KD). In the OD case, we highlight special classes of social agents, such as informed agents, contrarians and extremists. As regards GDM, we present state-of-the-art models on various kinds of decision-making processes. In the KD case, we discuss processes of knowledge diffusion and creation via the presence of self-innovating agents. The primary question we wish to address is: to what extent does Sociophysics correspond to social reality? For that purpose, for each social diffusion model category, we present notable Sociophysics applications for real-world socioeconomic phenomena and, additionally, we provide a much-needed critique of the existing Sociophysics literature, so as to raise awareness of certain issues that currently undermine the effective application of Sociophysics, mainly in terms of modelling assumptions and mathematical formulation, on the investigation of key social processes.

Keywords: mathematical modelling; agent-based modelling; sociophysics; econophysics; diffusion processes; opinion dynamics; knowledge dynamics; group decision making; complex networks; social interaction models; behavioral finance

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1. Overview of Sociophysics

Two-and-a-half millennia ago, Greek philosopher Democritus posited the existence of "atoms", tiny indivisible material components, anticipating the emergence of the atomic theory in the 19th century. In ancient and modern forms of the Greek language, the term "atom" (ἄτομον) is interpreted as "indivisible, uncut". The very same etymology is applied to the modern English term "individual" (derived from the Latin term *individuus*, again meaning indivisible), denoting a person. Moreover, in Modern Greek the term άτομο can be used interchangeably, either with a natural or a social connotation.

In a sense, the use of the same term to characterize two fundamentally different concepts (one referring to an inanimate object and the other to a living entity) might seem paradoxical, particularly under the scope of the Enlightenment notion of man as an essentially free creature, unconstrained by the influence of necessity expressed in universal natural laws. However, from the 17th century onwards, thinkers like Thomas Hobbes (1588–1679), David Hume (1711–1776), Henri de Saint-Simon (1760–1825) and others expressed a somewhat strange hope that the social sphere would someday become the domain of a new positive science. The idea of "social physics" appears in explicit form in the work of

French philosopher August Comte (1798–1857), who postulated the existence of objective, unchanging laws regulating the collective behavior of social groups [1]. This idea really acquired traction only in the last quarter of the 20th century with the emergence of concepts like complex systems, self-organization and chaos. Săvoiu and Iorga Simăn [2] aptly define Sociophysics as follows: "*Sociophysics can be described as the sum of activities of searching for fundamental laws and principles that characterize human behavior and result in collective social phenomena*."

Before and after the term "Sociophysics" emerged, various quantitative studies presented impressive and often counter-intuitive findings on social phenomena, such as racial segregation [3], opinion polarization [4,5], social imitation [6], workers' strike processes [7], alliance formation [8] and cultural dissemination [9]. In this context, the behavior of individuals has been implicitly or explicitly compared to that of atoms or molecules floating in a medium, with patterns of repellence and attraction depending on their inherent attributes [10]. Accordingly, Agent-Based Modelling (ABM), Social Network Analysis (SNA) [11] and Network Science in general have been naturally integral to the Sociophysics discipline [12]. Therefore, the underlying principle of Sociophysics is the proposal that relatively simple rules defining the behavior of interacting individuals/agents (micro level) lead to the emergence of a very complex social structure (macro level).

Three of the most promising instantiations of Sociophysics research are Opinion Dynamics (OD), Group Decision Making (GDM) and Knowledge Dynamics (KD). OD, GDM and KD, through the application of ABM and SNA principles, describe how opinion (in the form of ideas, cultural norms, memes, etc.), consensus (i.e., a commonly accepted outlook on how to act) and knowledge (e.g., true information, innovations, empirically justified truths, etc.) are respectively diffused in various social networked environments. In the 21st century information society [13], one of the most prominent and impactful social spaces is the one constituted by the interactions between financial agents (e.g., entrepreneurs, firms, traders, etc.). Econophysics [14,15], the economics-based sub-category of Sociophysics, models the behavior of a not-so-rational homo oeconomicus and the influence that interactions between financial agents have on the determination of asset prices.

Taking the former thoroughly into account, three central questions on the field of Sociophysics and its application in the real social world emerge, namely the following:

Q1: *what are the most important proposed Sociophysics models in the domains of OD, GDM and KD and in what way does each one of them aim to mathematically represent the occurrence of social interaction between agents and the emerging socioeconomic phenomena?*

Q2: *in what forms "bridges" between theoretical Sociophysics models and real-world applications in the domain of Economics and related subdomains such as Social Trading, Behavioral Finance and Business are erected?*

Q3: *given the great importance of Sociophysics, in what ways—known and unknown—does the field fail to accurately capture the inner workings of social reality?*

This review paper aims to provide answers to the above three questions. We believe that our adoption of a wide-ranging scope, taking good care to clarify the distinctions between different categories of social diffusion, and mainly our concurrent attempt to showcase both the advantages (evident in real-world socioeconomic applications) and shortcomings of the Sociophysics field (e.g., misalignment between Sociophysics modelling assumptions and social reality) form a "big picture" that an analysis focused only on particular aspects of Sociophysics may miss. We believe that current analyses on Sociophysics fail to capture in toto the current status of the discipline, its successes and mainly its more elusive shortcomings, especially as regards its ontological distinction from Physics and the related current inadequacies in the correspondence between Sociophysics and social reality. We naturally wish to cover this particular gap by offering a critical view on Sociophysics and its applications.

Accordingly, this review has the following structure: Firstly, the existing literature on diffusion dynamics in social networks is comprehensively discussed. Therefore, important studies, models and selected applications in OD (Section 2), GDM (Section 3) and

KD (Section 4) are presented (Q1) and for each model category certain noteworthy applications in Economics and related subfields are highlighted (Q2). Secondly, a critical evaluation of the existing relevant literature (Q3) is conducted (Section 5), so as to investigate how Sociophysics could further enhance its social relevance and solidify its status as a *mathematics-based applied social science*, whose findings may provide crucial insights on various aspects of social reality.

2. Opinion Dynamics (OD)

2.1. Linear Models

Over the previous decades, a growing body of literature on OD has offered mathematical model formulations simulating how opinion is diffused across social networks. A social network consists of N agents interacting through a set of connections/links [11]. In the simplest case (undirected network), between two communicating agents i and j there exists a single link with an influence weight w_{ij}.

The earlier approaches (from the 1950s to the 1990s) on OD are predominantly linear in nature [16–22], in the sense that communication patterns and interaction structure remains fixed. Among these, of special importance are the *DeGroot model* [18,22] and the *Friedkin-Jensen model* (FJ) [20,21].

In the DeGroot model the evolution of the opinion of agent i, usually taking values in the interval $[0,\ 1]$, is given as a linear combination (in particular a weighted average, since $\sum_{j=1}^{N} w_{ij} = 1$) of the opinions of all other communicating agents:

$$o_i(t+1) = \sum_{j=1}^{N} w_{ij} o_j(t). \tag{1}$$

Hence, the opinion vector O for all agents at time step $t+1$ will be updated accordingly: $O(t+1) = W \cdot O(t)$. Thus, the opinion vector at a later time step τ is of the form: $O(t+\tau) = W^{\tau} \cdot O(t)$. According to [18], if at least one column of matrix W^{τ} contains only positive elements, then consensus is eventually attainable.

The FJ model is a variant of the DeGroot model, in the sense that it takes into account agent i's stubbornness, $\mathfrak{s}_i$, i.e., the tendency to adhere to his initial opinion. The degree of influence from the other agents is set at $1 - \mathfrak{s}_i$. Opinion evolution can be written as:

$$o_i(t+1) = \mathfrak{s}_i o_i(0) + (1 - \mathfrak{s}_i) \sum_{j=1}^{N} w_{ij} o_j(t). \tag{2}$$

The above formula can be expressed in matrix form as $O(t+1) = \mathcal{S} \cdot O(0) + (I - \mathcal{S}) \cdot W \cdot O(t)$, where $\mathcal{S}$ is a $N \times N$ diagonal matrix containing the elements $\mathfrak{s}_i$. In contrast to the DeGroot model, the regularity of matrix W^{τ} does not necessarily entail convergence. Consensus is conditional on the stubbornness level of the agents.

2.2. Non-Linear Models

A rising number of non-linear approaches have also made their appearance in the literature. Non-linearity may emerge when influence weights and/or specific communication interactions are conditioned on agent opinions in previous time steps.

Non-linear models can be divided into three main categories: continuous, discrete and mixed. In continuous models, the opinion of the agents can take any value in a particular interval, e.g., $[0,\ 1]$ or $[-1,\ 1]$. In discrete models, the opinion of the agents can take only specific values, mostly of binary form (e.g., 0/1, yes/no, support/opposition, buy/sell). Finally, in mixed models, the continuous opinions of the agents are expressed as discrete choices. For all the above cases, there are also multi-dimensional (vector) representations of agent opinions [23,24].

2.2.1. Continuous Models

Among the non-linear continuous models found in the OD literature, the most prominent ones belong to the *Bounded Confidence* class of models [25–27]. Bounded Confidence models operate under the assumption that social influence is possible only when the opinions $o_i(t)$ and $o_j(t)$ of two neighboring agents i and j are sufficiently close, below a certain tolerance threshold ε:

$$\left|o_j(t) - o_i(t)\right| < \varepsilon. \tag{3}$$

The most well-known Bounded Confidence models are the *Hegselmann-Krause* (HK) *model* [28–30] and the *Deffuant-Weisbuch* (DW) *model* [31–33]. Their primary distinction lies in the communication assumption employed in each case: in the HK model, each agent communicates simultaneously with all sufficiently like-minded neighbors, while in the DW model communication is realized between two random like-minded neighbors at a time.

Hence, in the HK model, the opinion of agent i evolves in the following manner:

$$o_i(t+1) = \frac{1}{\left|S_i^{\varepsilon}\right|} \sum_{j \in S_i^{\varepsilon}} o_j(t), \tag{4}$$

where $S_i^{\varepsilon} = \left\{ j : \left|o_j(t) - o_i(t)\right| < \varepsilon \right\}$ is the set containing those neighbors j of agent i with a sufficiently close opinion to his own.

On the other hand, in the DW model, when two agents communicate and condition (3) is satisfied, a mutual opinion update takes place in the following fashion:

$$\begin{aligned} o_i(t+1) &= o_i(t) + \mu\big(o_j(t) - o_i(t)\big), \\ o_j(t+1) &= o_j(t) + \mu\big(o_i(t) - o_j(t)\big), \end{aligned} \tag{5}$$

with μ a parameter determining the rate of convergence between agents i and j. When the tolerance threshold ε is high enough, widespread opinion convergence is possible, whereas when ε is low, opinion clustering among groups of agents is observed.

2.2.2. Mixed Models

Sophisticated mixed models, such as the *continuous opinion and discrete actions* (CODA) *model* [34–36] have also been developed. In the prominent CODA model, an agent i internalizes his preference at time step t in the form of a continuous probability $p_i(t)$ and expresses to other agents his quantized binary opinion $o_i(t) = sign(p_i(t) - 0.5)$, taking values -1 or 1. Whenever an agent i is influenced by an agent j, his internalized preference, expressed in log-odds form as $u_i(t) = \log\left(\frac{p_i(t)}{1-p_i(t)}\right)$, shifts to $u_i(t+1) = u_i(t) \pm a_i = u_i(t) + o_j(t)a_i$ with a_i the susceptibility parameter of agent i to the opinion $o_j(t)$ of his neighbor j. Consequently, $p_i(t+1) \neq p_i(t)$ and thus it is probable that, under the influence of his neighbor, agent i will change his professed opinion $o_i(t+1) \neq o_i(t)$. In this manner, opinion diffusion can take place.

2.2.3. Discrete Models

As regards discrete OD modelling, much attention has been given to the *voter model* [37,38], the *Sznajd model* [39–42] and the *majority rule model* [43–46].

In the voter, model agents are placed in a square lattice. Agent opinion $o_i(t)$ is of strictly binary type, i.e., 0 or 1. At each time step, an agent i will randomly adopt the opinion $o_j(t)$ of a neighbor j, i.e., $o_i(t+1) = o_j(t)$.

In the Sznajd model, agents are placed on a line. At each step, a pair of neighboring agents i and $i+1$ is randomly selected. If their opinions are identical, then the adjacent agents $i-1$ and $i+2$ will be jointly influenced as follows: $o_{i-1}(t+1) = o_{i+2}(t+1) = o_{i+1}(t) = o_i(t)$. On the other hand, if their opinions differ, then each agent will influence only his other neighbor, i.e., $o_{i-1}(t+1) = o_i(t)$ and $o_{i+2}(t+1) = o_{i+1}(t)$.

The majority rule model concerns voting at different hierarchical levels. In the simplest case, at each hierarchical level the population is randomly clustered in groups of three agents. In each group, the three members vote between two candidates (type A or type B). The voting result at hierarchical level 1 influences the composition of the population at level 2. After clustering in groups of three at level 2, the same process is to be repeated until vote convergence is observed. The probability for a type A candidate to be elected at level $n+1$ is as follows:

$$p_A(n+1) = p_A^3(n) + 3p_A^2(n)(1 - p_A(n)), \tag{6}$$

where $p_A(n)$ is the probability that a type A candidate was elected at level n.

It can be shown that when $p_A(0) < \frac{1}{2}$, the probability sequence $p_A(n)$ converges to zero. Hence, the existence of A is eliminated. On the other hand, when $p_A(0) > \frac{1}{2}$, the sequence converges to 1 and the prevalence of A is certain. The key finding is that, although the electoral process operates in a fully democratic manner, the condition $p_A(0) > p_B(0)$ (or the reverse) is sufficient for a "totalitarian" result to eventually emerge. One of the two political positions (A or B) will be all but excluded, regardless of the amount of initial support for B or A.

Counter-intuitive findings, as the above, demonstrate the usefulness of Sociophysics for the disclosure of well-hidden "holes" in seemingly robust social or political processes that qualitative studies are often unable to detect. Certain "bottom-up" multi-stage electoral systems like the ones found in socialist or even liberal democratic states (e.g., the US presidential electoral system) operate in a manner which bears certain similarities to the one proposed in the majority rule model.

2.3. Special Classes of Social Agents

Certain continuous OD models focus on the impact of special classes of social agents on the opinion diffusion process in the network. Such agents operate in a particular manner that distinguish them from other agents they interact with. These special categories of agents (informed, contrarian and extremist) are discussed below.

2.3.1. Informed Agents

It has been noted that apart from the existence of *key opinion leaders* or *influentials* who spread their opinions to the rest of the population [47–50], the presence of a «critical mass of easily influenced individuals» is also essential for effective social influence [51]. From this critical mass a cascading effect begins, leading to widespread social influence.

Along these lines, a large segment of the available literature on OD focuses on the impact of *informed agents*, i.e., individuals with hidden agendas who act as secret advertisers of ideas and norms to the rest of the population. Prominent studies on informed agents [52,53] are grounded on findings derived from studies on animal population dynamics [54,55], suggesting that the collective behavior of social groups can be guided by a small fraction of purposeful agents.

For example, in [52], taking into account the bounded confidence assumption (3), opinion diffusion from regular agent j to regular agent i is realized as follows: $o_i(t+1) = w_{ii}(t)o_i(t) + w_{ji}(t)o_j(t)$, where $w_{ji}(t)$ denotes the interpersonal social influence of agent j on agent i and $w_{ii}(t)$ denotes the tendency of agent i to adhere to his opinion at time t. However, for an informed agent κ, a third parameter, $w_\kappa^g(t)$, indicating hidden devotion to the pursuance of a pre-specified goal o^g is added. Therefore, opinion diffusion from regular agent λ to informed agent κ is realized as follows:

$$o_\kappa(t+1) = w_{\kappa\kappa}(t)o_\kappa(t) + w_{\lambda\kappa}(t)o_\lambda(t) + w_\kappa^g(t)o^g. \tag{7}$$

In the general case, an informed agent may also become influenced by other agents. Still, the presence of an unchanging pre-set goal makes possible to steer public opinion using only a small set of informed agents in the social network, especially when these are well-connected individuals. Note that informed agents are not necessarily prominent members of society.

An even more fleshed out approach on the processes of social change induced by informed agents, or rather *change agents*, has been presented in [56]. Change agents mimic the opinions of their neighbors and, at the same time, attempt to divert them towards a preferred direction. They gradually shift their neighbors' opinions towards a pre-set goal, so as to commence a cascading diffusion of social influence. Therefore, the strategy of "salami slicing" is incorporated into the opinion diffusion equation of a change agent.

Whereas opinion diffusion for regular agents is of a form similar to other DW-derived implementations, for a change agent κ, opinion update operates in the following manner:

$$o_\kappa(t+1) = \bar{o}_\kappa^{out}(t) + s_\kappa(t)\big(o^g - \bar{o}_\kappa^{out}(t)\big), \tag{8}$$

where $\bar{o}_\kappa^{out}(t) = \frac{1}{|S_\kappa^+|} \sum_{\lambda \in S_\kappa^+} o_\lambda(t)$ is the average opinion of all out-neighbors of κ to whom agent κ is connected with a positive outgoing (directed) link weight $w_{\kappa \to \lambda}$. This set is denoted by $S_\kappa^+ = \{\lambda : \, w_{\kappa\lambda} = w_{\kappa \to \lambda} > 0\}$. Additionally, $s_\kappa(t)$ is a time-dependent slicing parameter lying in the interval $[0, \, 1]$, indicating "thin" (gradual) shift of the change agent's professed opinion. In this way, the change agent κ is able to induce opinion change to his out-neighbors, whose average opinion $\bar{o}_\kappa^{out}(t)$ is incorporated in the change agent's public stance.

2.3.2. Contrarian Agents

Contrarians are agents who tend to disagree with the majority. According to [57], one way to model the influence of contrarians is to simply refine the majority rule model found in [43–45]. Contrarian agents are presumed to be found in the general population with a density a. In this way, they might induce a shift in electoral outcomes of lower-level groups. Therefore, (6) is modified accordingly:

$$p_A(t+1) = (1-a)\Big[p_A^3(t) + 3p_A^2(t)(1-p_A(t))\Big] + a\Big[p_B^3(t) + 3p_B^2(t)(1-p_B(t))\Big], \tag{9}$$

where $p_A(n)$ and $p_B(n)$ are the probabilities that a candidate of type A or B is elected at level n.

A positive outcome of the presence of contrarian agents is that the totalitarian outcome of the majority rule model is averted for small values of a. For example, $a = 0.1$ leads to opinion convergence of the form 0.85 to 0.15 in favor of outcome A, not 1 to 0.

2.3.3. Extremist Agents

Extremists are agents whose opinion lies on the edges of the opinion interval [58–61]. For example, if agent opinions take values in the interval $[-1, \, 1]$, then agent opinion for an extremist agent takes values very close or equal to 1 or -1.

In [59], it is proposed that if extremist agents are present in a social network, their extremist views can eventually become widely accepted, provided that there exists opinion uncertainty among regular agents. Uncertainty is modeled assuming the existence of opinion segments $s_i = [o_i - u_i, \, o_i + u_i\,]$, where o_i indicates opinion and u_i uncertainty. The segment overlap h_{ji} between two agents i and j is: $h_{ji} = \min\big(o_j + u_j, \, o_i + u_i\big) - \max\big(o_j - u_j, \, o_i - u_i\big)$, whereas the non-overlapping part is $2u_j - h_{ji}$.

Subtraction of the two quantities divided by u_j gives the relative agreement term $\frac{h_{ji}}{u_j} - 1$. Hence, on the condition that $h_{ji} > u_j$, opinion diffusion from an agent j to a neighbor agent i is possible and the DW Equation (5) for agent i take the form:

$$\begin{aligned} o_i(t+1) &= o_i(t) + \mu \cdot \left(\frac{h_{ji}(t)}{u_j(t)} - 1 \right) \big(o_j(t) - o_i(t) \big), \\ u_i(t+1) &= u_i(t) + \mu \cdot \left(\frac{h_{ji}(t)}{u_j(t)} - 1 \right) \big(u_j(t) - u_i(t) \big), \end{aligned} \tag{10}$$

where μ is a convergence parameter determining the speed of the diffusion process.

Due to the fact that an extremist agent j has narrower opinion uncertainty u_j than a regular agent, the condition $h_{ji} > u_j$ is usually fulfilled when regular agents interact with extremists. Hence, diffusion and, in some cases, prevalence of extremist opinions becomes possible.

2.4. Applications in Behavioral Finance/Social Trading

One of the most fruitful applications of OD is in the field of behavioral finance/social trading [62]. The application of OD in finance, aiming to model the behavior of financial social agents, has a rather strong empirical foundation due to the existence of sociological studies [63,64], suggesting that price formation is partially dependent on the social influence that financial agents (firms, traders, entrepreneurs, investors, finfluencers, etc.) exert on each other.

Financial agents act semi-rationally at best and, due to their unavoidable social embeddedness, they are heavily influenced by the preferences of their peers. Personal bias also seems to have a rather strong impact on individual behavior. Thus, it seems that financial agents do not always follow a rationalist modus operandi; behavioral short-termism is common [65]. However, despite their bounded rationality [66], agent behavior can still be mathematically modeled and simulated via the appropriate application of ABM principles. This is one of the most important benefits of the Sociophysics approach and its economics-oriented sub-field, *Econophysics*.

2.4.1. Discrete Models

Certain discrete OD models applied on behavioral finance are based on previous studies from the field. One notable example is found in [67], where a generalized version of the financial *Brock-Hommes model* [68], which assumes that all investors interact with each other, is applied on several different network topologies.

In [67], the opinion (investment behavior) of agent i at time $t+1$ shifts according to the opinions of his neighbors on previous time t. The opinion $o_i(t)$ of agent i is binary with 0 indicating a "fundamentalist" behavior and 1 indicating a "chartist" behavior. A chartist is an investor who evaluates assets based on a technical analysis of historical price trends. The basic assumption is that all necessary information is already incorporated into the current price. A fundamentalist is an investor who evaluates assets based on α fundamental analysis. The goal of a fundamentalist is to determine whether an asset is overvalued, undervalued or fairly valued by the market, and then make investment decisions based on this analysis.

The probability $p_i^c(t+1)$ that agent i will be of the chartist type at time $t+1$ is as follows:

$$p_i^c(t+1) = o_i(t) \left(\prod_{j \in S_i} o_j(t) \right) + \left[o_i(t) \left(1 - \prod_{j \in S_i} o_j(t) \right) + (1 - o_i(t)) \left(1 - \prod_{j \in S_i} (1 - o_j(t)) \right) \right] P_\beta^{c>f}(t+1).$$

The above formula can be summarized as follows:

$$p_i^c(t+1) = \begin{cases} 1 & when\ o_i(t) = 1\ and\ o_j(t) = 1\ \forall j \in S_i \\ 0 & when\ o_i(t) = 0\ and\ o_j(t) = 0\ \forall j \in S_i \\ P_\beta^{c>f}(t+1)\ otherwise & \end{cases} \tag{11}$$

where S_i is the set of neighbors of agent i, and $P_\beta^{c>f}(t+1)$ is a logistic probability indicating the extent that the chartist position is preferable over the fundamentalist one:

$$P_\beta^{c>f}(t+1) = \frac{1}{1 + e^{\beta[U^f(t+1) - U^c(t+1)]}} \tag{12}$$

Here, β is a parameter indicating intensity of choice due to idiosyncratic preferences and personality biases and $U^f(t+1)$, $U^c(t+1)$ are utility indices resulting from a fundamentalist (f) or chartist (c) position, respectively, taking into account the asset performance in previous time steps.

According to (11), $p_i^c(t+1) = 1$ when all neighbors $j \in S_i$ of agent i and i himself are of the chartist type, and $p_i^c(t+1) = 0$ when all neighbors $j \in S_i$ and i himself are of the fundamentalist type. When the opinions of neighbors are not all of the same type, then the probability that agent i is of the chartist type is: $p_i^c(t+1) = P_\beta^{c>f}(t+1)$.

In an another relevant study [69], a three-state probabilistic OD model is proposed in which opinion represents available options (buy, sell or do nothing) forming the set $s = \{1,\ 2,\ 3\}$. Agents are split into two categories, noise "traders" and noise "contrarians". Noise traders tend to align with the local majority of their neighbors. Noise contrarians take into account all agents (not only their neighbors) and tend to adopt the opinion of the minority. The status of each agent (trader or contrarian) is unchanging in time. The probability that an agent (trader or contrarian) will not follow the above-described behavior is p. For example, assuming that the local majority of a noise trader at time t prefers the "buy" option (1), the probabilities at time $t+1$ are as follows:

$$\begin{aligned} P\left(o_i^{trader}(t+1) = 1 \middle| \left(N_i^1 > N_i^2\right) \wedge \left(N_i^1 > N_i^3\right)\right) &= 1 - p, \\ P\left(o_i^{trader}(t+1) = 2 \middle| \left(N_i^1 > N_i^2\right) \wedge \left(N_i^1 > N_i^3\right)\right) &= \frac{p}{2}, \\ P\left(o_i^{trader}(t+1) = 3 \middle| \left(N_i^1 > N_i^2\right) \wedge \left(N_i^1 > N_i^3\right)\right) &= \frac{p}{2}, \end{aligned} \tag{13}$$

where

$$N_i^1 = \left|\{j : j \in S_i\ with\ o_j(t) = 1\}\right|, N_i^2 = \left|\{j : j \in S_i\ with\ o_j(t) = 2\}\right|, N_i^3 = \left|\{j : j \in S_i\ with\ o_j(t) = 3\}\right|, \tag{14}$$

the number of neighbors of agent i having the corresponding states 1, 2 and 3. Due to its probabilistic character, the model is of a similar nature to other similar approaches on the majority rule model [70–73], first described in Section 2.2.3.

2.4.2. Continuous Models

As regards asset price dynamics, opinion (investment behavior) may take continuous values. For example, in [74] the expectation of agent i about the future price of an asset is influenced by the opinion of all other N agents, including himself:

$$E_i(t) = \sum_{j=1}^{N} w_{ij} o_j(t). \tag{15}$$

The above linear formula is similar to the DeGroot opinion update in (1). Each agent invests a relative amount of wealth $w_i(t)$ equal to the product of his price expectation

$E_i(t)$ with his capital ratio $c_i(t)$, denoting the capital share of agent i over the total capital invested in this specific asset. The asset price at time t is defined as:

$$v(t) = \sum_{i=1}^{N} w_i(t) = \sum_{i=1}^{N} E_i(t)c_i(t). \tag{16}$$

The opinion of agent i about the future asset price is a function of the price $v(t)$ in the previous m_i time steps, i.e., $o_i(t+1) = f(v(t),\ v(t-1), \ldots, v(t-m_i))$.

Non-linear formulations have also been proposed. In a notable recent study [75], the OD Equation (5) is used for the description of asset price dynamics. The excess demand $D^{\mu}(t)$ for an asset μ is defined as the sum of the distances between the opinion $o_i^{\mu}(t) \in [0,\ 1]$ of agent i about the price of asset μ and its normalized price $\mathcal{p}^{\mu}(t) \in [0,\ 1]$. The normalized price is defined as $\mathcal{p}^{\mu}(t) = \frac{p^{\mu}(t)}{\sum_{\xi} p^{\xi}(t)}$, where $p^{\mu}(t)$ is the price of asset μ. Hence, the excess demand is $D^{\mu}(t) = |\varepsilon| \sum_i \left(o_i^{\mu}(t) - \mathcal{p}^{\mu}(t) \right)$, where $\varepsilon \sim N(0, \sigma)$ is a random noise element having a "quenching" influence on excess demand. Accordingly, price dynamics can be described in the following manner:

$$\mathcal{p}^{\mu}(t+1) = \begin{cases} 1 & if\ \mathcal{p}^{\mu}(t) + D^{\mu}(t) > 1 \\ \mathcal{p}^{\mu}(t) + D^{\mu}(t) & if\ \mathcal{p}^{\mu}(t) + D^{\mu}(t) \in [0,\ 1] \\ 0 & if\ \mathcal{p}^{\mu}(t) + D^{\mu}(t) < 0 \end{cases}. \tag{17}$$

In [76,77], another non-linear model is developed, in which network structure is considered to be evolving in time. Each agent i is characterized by the following two attributes: his opinion $o_i(t)$ representing his attitude on risk-taking and his reputation $r_i(t)$ as an investor among his peers. The reputation $r_i(t)$ depends on his wealth $w_i(t)$ and his charisma c_i, as follows: $r_i(t) = (1-h)w_i(t) + hc_i$, with $h \in [0,\ 1]$ a coefficient indicating irrationality. Wealth evolution is dependent on his previous wealth level $w_i(t)$ and his risk-taking attitude $o_i(t)$ as follows: $w_i(t+1) = f(w_i(t), o_i(t))$. Opinion update, bearing similarities to both the HK (4) and the FJ model (2), is written as

$$o_i(t+1) = \begin{cases} \mathcal{s}_i o_i(0) + (1-\mathcal{s}_i) \frac{1}{|S_i(t)|} \sum\limits_{j \in S_i(t)} o_j(t),\ |S_i(t)| > 0 \\ o_i(0),\ |S_i(t)| = 0 \end{cases} \tag{18}$$

where $\mathcal{s}_i$ is the stubbornness of agent i and $S_i(t) = \{j : a_{ji}(t) = 1\}$ is the set of neighbors j of agent i at time t, with $a_{ji}(t)$ the corresponding time-dependent element of the adjacency matrix:

$$a_{ji}(t) = \begin{cases} 1,\ \sigma_{ji}(t) > 0.5 \\ 0,\ elsewhere \end{cases} \tag{19}$$

where $\sigma_{ji}(t)$ is the activation function, determining whether the corresponding link $a_{ji}(t)$ becomes activated or deactivated. The dynamics of $\sigma_{ji}(t)$ depend on the reputation difference $r_j(t) - r_i(t)$ between agents j and i, as follows:

$$\varphi\left(\frac{d^2\sigma_{ji}}{dt^2}, \frac{d\sigma_{ji}}{dt}, \sigma_{ji}(t) \right) = (-1)^{a_{ji}(t)} \max\left\{ 0,\ (-1)^{a_{ji}(t)} \left(r_j(t) - r_i(t) \right) \right\}. \tag{20}$$

The reputation difference $r_j(t) - r_i(t)$ is analogous to the opinion difference found in (3), used in the original HK model and other non-linear models. However, contrary to the original HK model, network structure evolves due to the presence of a *co-evolution* mechanism [78–80]. Specifically, the reputation of an agent increases due to wealth accumulation. This leads to shifts in network structure through the activation or deactivation of specific network links. This has an immediate effect on the formation of the opinions of the agents, which, in turn, influences wealth accumulation for each agent and so on.

Note that the formulation presented above aims to incorporate both: (a) the rational aspect of social influence, i.e., wealth $w_i(t)$, considered to be the "hard power" element in the model, and (b) the irrational aspect of social influence, i.e., charisma c_i, considered to be the "soft power" element, indicating the psychological tendency to be influenced by those who possess social status.

3. Group Decision Making (GDM)

In the field of Sociophysics, Group Decision Making (GDM) in a social network usually takes place between multiple collaborating agents of often diverse backgrounds, ideas, knowledge and authority, each having a different opinion on a specific subject [81–84]. Thus, GDM implies the existence of collective efforts aiming for the achievement of consensus in a non-competitive setting on the part of all network agents, all consciously committed in the decision-making process.

On the other hand, OD primarily focuses on processes of undeliberate shift in perceptions, with maybe only a minority of agents intently aiming for the promotion of specific desired opinions and ideas (e.g., informed/change agents) to unaware regular agents. In OD, even if consensus appears, in the form of opinion uniformity, as a macroscopic emergent phenomenon, this does not necessarily imply intentionality and commitment to the attainment of consensus on the part of the agents, as in the GDM case. Even if each agent is assumed to purposefully select whom they will interact with, according to specific selection criteria, pursuance of purpose remains, at most, an individual, not collective, endeavor. Thus, consensus in OD, when it emerges, is merely a byproduct of interactions. What might interest the researcher is the eventual opinion distribution (steady state) among the agents, i.e., uniformity, polarization or fragmentation of opinion. On the contrary, consensus in GDM is what is actively sought after. Here, what matters is the achievement of an eventual collective and negotiated opinion value and the conditions for its implementation. Note that in OD such equations referring to some steady state (such as (23) and (31), below), are naturally absent, since opinion diffusion is a "purpose-free" process, which, as noted above, does not entail collective pursuance of purpose from the part of the agents.

However, despite their heavy contextual differences, heavy influence from OD to GDM in terms of methodology and mathematical formulation is nearly inevitable due to the fact that both involve the promotion of personal views. Thus, OD models are related to GDM models but only in terms of similar social diffusion equations. More specifically, an OD model can be converted to a GDM model if the element of consensus pursuance is added into the existing model architecture leading to further mathematical elaboration. Note that for this reason the distinction between OD and GDM may not be directly perceivable when looking only at certain parts of the mathematical formulation. Thus, the difference in the social context of interaction is not immediately apparent from the social diffusion equations used, which are, after all, frequently used in similar or identical form in GDM as well as OD. Nonetheless, the collaborative aspect of GDM is still apparent as it emerges from the way a proposed GDM model operates in its totality, taking into account all equations describing its modus operandi. This collaborative aspect is, of course, directly enforced by the developer of the model.

In order to showcase the distinctive features of GDM, prominent examples from the available literature are presented and discussed below.

3.1. Decision Making via Agent-to-Agent Influence

A significant number of GDM models [85–87] incorporate an agent-to-agent interaction mechanism for decision making. A notable model is the one found in [84], in which the bounded confidence assumption in (3) is adopted, although certain distinguishing elements are present.

The opinion $o_i(t)$ of agent i is taking values in the interval $[0,\ 1]$. Each pair of agents i and j is influenced by each other, depending on their opinion distance $d_{ij}(t) = \left|o_i(t) - o_j(t)\right|$, which takes values in $[0,\ 1]$. Three separate cases are distinguished:

- When $d_{ij}(t) < \varphi$, the agents are deemed to be in agreement and therefore there is no further need to influence each other.
- When $d_{ij}(t) \in [\varphi,\ \varepsilon)$, with consensus threshold $\varphi \in [0,\ \varepsilon]$, the agents continue to influence each other and they have to compromise. Only in this case opinion update for both agents takes place.
- When $d_{ij}(t) \geq \varepsilon$, with confidence threshold $\varepsilon \in [0,\ 1]$, no influence takes place between the agents as they belong to a separate opinion cluster.

For an agent i at each time step t the set $S_i = \left\{j : d_{ij}(t) \in [\varphi,\ \varepsilon)\right\}$ is determined. This set contains all agents j that agent i has to negotiate with. Among these agents, a particular agent κ is selected for interaction if $d_{i\kappa}(t) = \max\limits_{j \in S_i}\left\{d_{ij}(t)\right\}$. Resembling the DW formulation in (5), between agents i and κ, opinion diffusion operates in the following manner:

$$\begin{aligned} o_i(t+1) &= \mu_i(t) o_i(t) + (1 - \mu_i(t)) o_\kappa(t), \\ o_\kappa(t+1) &= \mu_\kappa(t) o_\kappa(t) + (1 - \mu_\kappa(t)) o_i(t), \end{aligned} \tag{21}$$

where $\mu_i(t),\ \mu_\kappa(t) \in [0,\ 1]$ are the time-dependent convergence parameters of agents i and κ:

$$\begin{aligned} \mu_i(t) &= 1 - \frac{DEG_\kappa(t)}{(DEG_i(t) + DEG_\kappa(t)) \cdot \eth_i}, \\ \mu_\kappa(t) &= 1 - \frac{DEG_i(t)}{(DEG_i(t) + DEG_\kappa(t)) \cdot \eth_\kappa} \end{aligned} \tag{22}$$

The degree centrality of agent i, indicating his credibility, is denoted by $DEG_i(t)$. The reasoning for this, is the argument that the more one is followed, the more his opinion tends to be respected. Additionally, $\eth_i$ and $\eth_k$, are constants indicating stubbornness, taking values equal to 2, 3 or 4.

At time step t_{eq} the network reaches an equilibrium as all opinion distances belong to the interval $\Delta = [0,\ \varphi) \cup [\varepsilon,\ 1]$. This means that $S_i = \varnothing$, for all agents i. The final aggregated opinion O_{eq} is calculated as the weighted average of all final agent opinions giving the solution to the GDM problem:

$$O_{eq} = \frac{\sum_{i=1}^{N} o_i\left(t_{eq}\right) \cdot DEG_i\left(t_{eq}\right)}{\sum_{i=1}^{N} DEG_i\left(t_{eq}\right)}. \tag{23}$$

3.2. Decision Making via Clustering

When the population of the interacting agents is large, usually greater than 11 agents [88] or more [89,90], decision making can be realized in clusters of agents. For example, in [90], the agents are placed in Z distinct clusters $G_z, z = 1, \ldots, Z$ with populations of $|G_z|$ agents. Each agent i has its own confidence threshold ε_i. All agents are faced with a set of m decision alternatives, namely: $X = \{x_1,\ x_2, \ldots, x_m\}$. Each alternative x_p has n attributes, i.e., $x_p = \left[a_p^1,\ a_p^2, \ldots, a_p^n\right]$. Attributes are split into two categories: (a) benefit attributes and (b) cost attributes. It is desirable that benefit attributes take high values while cost attributes take low values.

The opinion of agent i on the attribute a_p^q of the alternative x_p is denoted as $o_i^{p(q)}(t)$, taking values in the interval $[0,\ 1]$. Therefore, the agent i of cluster G_γ is characterized by the opinion matrix $O_{i[\gamma]}(t) = \left[o_{i[\gamma]}^{p(q)}(t)\right]_{m \times n}$ and his prestige/reputation $r_{i[\gamma]} = \frac{\overline{\omega}_i}{\sum\limits_{j \in G_\gamma} \overline{\omega}_j}$, where $\overline{\omega}_i$ is the mean value of two centralities, degree and closeness. The (relative) prestige/reputation of cluster G_γ is as follows:

$$r_\gamma = \frac{|G_\gamma|^2}{\sum_{z=1}^{Z}\left(|G_z|^2\right)}. \tag{24}$$

Due to the presence of a clustering mechanism, consensus is gradually formed in hierarchical levels of interaction (micro-, meso- and macro-). The mathematical formulations for each level are summarized in Table 1. Consensus has to be first implemented in the cluster (meso) level. The opinion matrix of the cluster G_γ is $O_{[\gamma]}(t) = \sum_{i=1}^{|G_\gamma|}\left[r_{i[\gamma]}\cdot O_{i[\gamma]}(t)\right] = \left[o_{[\gamma]}^{p(q)}(t)\right]_{m\times n}$. The difference between the opinion matrix $O_{i[\gamma]}(t)$ of agent i and the opinion matrix $O_{[\gamma]}(t)$ of the cluster G_γ to which agent i belongs, is calculated via the Manhattan distance:

$$d_{i[\gamma]} = d\left(O_{i[\gamma]}(t), O_{[\gamma]}(t)\right) = \frac{1}{mn}\sum_{p=1}^{m}\sum_{q=1}^{n}\left|o_{i[\gamma]}^{p(q)}(t) - o_{[\gamma]}^{p(q)}(t)\right|. \tag{25}$$

Table 1. Mathematical Formulation of Opinion, Distance and Consensus at different hierarchical levels of interaction.

Level	Opinion	Distance	Consensus
Micro	$O_{i[\gamma]}(t) = \left[o_{i[\gamma]}^{p(q)}(t)\right]_{m\times n}$	Agent i to agent j: $d_{ij[\gamma]} = d\left(O_{j[\gamma]}(t),\ O_{i[\gamma]}(t)\right)$ Agent i to cluster G_γ: $d_{i[\gamma]} = d\left(O_{i[\gamma]}(t), O_{[\gamma]}(t)\right)$	Agent i to cluster G_γ: $C\left(O_{i[\gamma]}(t)\right) = 1 - d\left(O_{i[\gamma]}(t), O_{[\gamma]}(t)\right)$
Meso	$O_{[\gamma]}(t) = \sum_{i=1}^{\lvert G_\gamma\rvert}\left[r_{i[\gamma]}\cdot O_{i[\gamma]}(t)\right] = \left[o_{[\gamma]}^{p(q)}(t)\right]_{m\times n}$	-	$C\left(O_{[\gamma]}(t)\right) = \sum_{i=1}^{\lvert G_\gamma\rvert}\left[r_{i[\gamma]}\cdot C\left(O_{i[\gamma]}(t)\right)\right]$
Macro	$O(t) = \sum_{z=1}^{Z} r_z O_{[z]}(t) = \left[o^{p(q)}(t)\right]_{m\times n}$	-	$C(O(t)) = \sum_{z=1}^{Z} r_z\cdot C\left(O_{[z]}(t)\right)$

The consensus level of agent i and the consensus level of the cluster G_γ are given correspondingly: $C\left(O_{i[\gamma]}(t)\right) = 1 - d\left(O_{i[\gamma]}(t), O_{[\gamma]}(t)\right) = 1 - d_{i[\gamma]}$ and $C\left(O_{[\gamma]}(t)\right) = \sum_{i=1}^{|G_\gamma|}\left[r_{i[\gamma]}\cdot C\left(O_{i[\gamma]}(t)\right)\right]$. The group/network consensus level is defined as follows:

$$C(O(t)) = \sum_{z=1}^{Z} r_z\cdot C\left(O_{[z]}(t)\right). \tag{26}$$

When $C(O(t)) > \varphi$, where φ is an arbitrary selected consensus threshold, the consensus reaching process is completed. Otherwise, the agents belonging to the cluster with the lowest consensus level, say G_γ, need to properly readjust their opinions (Table 2). In this cluster G_γ, the opinion update of agent i is the following:

$$O_{i[\gamma]}(t+1) = I_i O_{i[\gamma]}(t) + (1 - I_i)\cdot\frac{1}{|S_i \cap E_i|}\sum_{j\in S_i\cap E_i} O_{j[\gamma]}(t), \tag{27}$$

where I_i is the opinion inertia of agent i, incorporating his self-confidence, S_i is the set of neighbors of agent i, while E_i is the set of in-cluster agents having similar opinion to agent i. More specifically, E_i is the set of agents of the cluster G_γ between whom and agent i the Manhattan distance is smaller or equal than the confidence bound ε_i of agent i:

$$d_{ij[\gamma]} = d\left(O_{j[\gamma]}(t),\ O_{i[\gamma]}(t)\right) = \frac{1}{mn}\sum_{p=1}^{p}\sum_{q=1}^{q}\left|o_{j[\gamma]}^{p(q)}(t) - o_{i[\gamma]}^{p(q)}(t)\right| \le \varepsilon_i. \tag{28}$$

Table 2. In-Cluster Opinion Diffusion for an agent i of cluster G_γ.

Condition	Case	Subcase	Influence from	Opinion Update for Agent i
$[C(O(t)) < \varphi] \wedge \left[C\left(O_{[\gamma]}(t)\right) = \min\limits_{z \in \{1,\ldots,Z\}}\left\{C\left(O_{[z]}(t)\right)\right\}\right]$	$S_i \cap E_i \neq \varnothing$	-	all agents $j \in S_i \cap E_i$	$O_{i[\gamma]}(t+1) = I_i O_{i[\gamma]}(t) + \frac{1-I_i}{\lvert S_i \cap E_i \rvert} \sum\limits_{j \in S_i \cap E_i} O_{j[\gamma]}(t)$
	$S_i \cap E_i = \varnothing$	$S_i \neq \varnothing$	The agent κ, specified as: $O_{\kappa[\gamma]}(t) = arg\left[\max\limits_{j \in S_i}\left\{C\left(O_{j[\gamma]}(t)\right)\right\}\right]$	$O_{i[\gamma]}(t+1) = I_i O_{i[\gamma]}(t) + (1-I_i) O_{\kappa[\gamma]}(t)$
		$(S_i = \varnothing) \wedge (E_i \neq \varnothing)$	The agent κ, specified as: $O_{\kappa[\gamma]}(t) = arg\left[\max\limits_{j \in E_i}\left\{C\left(O_{j[\gamma]}(t)\right)\right\}\right]$	
		$(S_i = \varnothing) \wedge (E_i = \varnothing)$	A hypothetical agent κ, defined as: $O_{\kappa[\gamma]}(t) = O_{i[\gamma]}(t) + \frac{\varepsilon_i}{d_{is[\gamma]}}\left(O_{s[\gamma]}(t) - O_{i[\gamma]}(t)\right)$ with $O_{s[\gamma]}(t) = arg\left[\max\limits_{j \in G_\gamma}\left\{C\left(O_{j[\gamma]}(t)\right)\right\}\right]$	

Due to the fact that confidence bounds ε_i vary, an agent j may be included in the set E_i of agent i but the opposite is not necessarily true. Note that distance measures apply only for the micro case and not for the meso (cluster level) and macro (group level) cases (Table 1). Nonetheless, further elaboration is possible, at least for the meso case, taking also into account inter-cluster distances for the formation of consensus, not only the reputation weights r_z (see below).

In case $S_i \cap E_i = \varnothing$ agent i updates his opinion via taking into account the opinion of a single agent κ:

$$O_{i[\gamma]}(t+1) = I_i O_{i[\gamma]}(t) + (1 - I_i) O_{\kappa[\gamma]}(t) \tag{29}$$

Depending on the subcase (see Table 2), agent κ can be: (a) the neighbor with the greatest consensus level in the cluster of agent i (when $S_i \neq \varnothing$), (b) the agent with the greatest consensus level in the cluster of agent i among agents with opinion distance satisfying the confidence bound of agent i (when $(S_i = \varnothing) \wedge (E_i \neq \varnothing)$) and (c) a hypothetical agent whose opinion distance from agent i is equal to ε_i (when $(S_i = \varnothing) \wedge (E_i = \varnothing)$).

Opinions matrices $O_{[z]}(t)$ of all clusters $G_1, G_2, \ldots, G_Z$ are aggregated in the global opinion matrix $O(t) = \sum_{z=1}^{Z} r_z O_{[z]}(t) = \left[o^{p(q)}(t)\right]_{m \times n}$. When consensus has been globally achieved at time t_{eq}, i.e., $C\left(t_{eq}\right) > \varphi$, the evaluation $\mathscr{E}\left(x_p\right)$ of alternative x_p is computed as:

$$\mathscr{E}\left(x_p\right) = \sum_{q=1}^{n} w^{p(q)} o^{p(q)}\left(t_{eq}\right) \tag{30}$$

where $w^{p(q)}$ is weight of attribute q of alternative p and $o^{p(q)}\left(t_{eq}\right)$ is the aggregated group/network opinion on this particular attribute–alternative pair. The final group/network decision is

$$D = \max_{p \in \{1,\ldots,m\}}\left\{\mathscr{E}\left(x_p\right)\right\} \tag{31}$$

3.3. Decision Making via Consultation

Shared mental models [91,92] were initially proposed to describe shared cognition in dyads of agents. These models were afterwards expanded at the group/team level. Thus, so-called "team mental models" have been used to study collective shared understanding, beliefs and knowledge inside groups of agents. In this way, valuable insights can be extracted, concerning the way team members perceive and interpret knowledge, make decisions and collaborate with each other [92,93].

In [94], a consultation process in a group of agents is studied. The primary aim of the model is the optimization of a true utility function $U(\mathcal{p})$, to which no agent has access. The true utility function $U(\mathcal{p})$ takes values in [0, 1], where $\mathcal{p} = [x_1,\ x_2,\ \ldots,\ x_m]$ is a plan of action defined on a m-dimensional problem space. Each dimension x_p represents an "aspect" of the problem. Each plan $\mathcal{p}$ corresponds to an opinion set $o(\mathcal{p}) = \{\mathcal{p}, U(\mathcal{p})\}$. Each agent i has two types of memory, namely:

- The Peer-derived memory $M_i^{peer} = \left\{o_{ij(\delta)}\right\}_{(N-1)\times c}$ which is derived from the opinion sets of the other $N-1$ agents. Each agent i has a limited memory capacity c. The elements of the peer-derived memory M_i^{peer} are the opinion sets $o_{ij(\delta)} = \left\{\mathcal{p}_{ij(\delta)}, U_j\left(\mathcal{p}_{ij(\delta)}\right)\right\}$, where $\mathcal{p}_{ij(\delta)}$ is the δ-th plan of agent i derived from agent j and $U_j\left(\mathcal{p}_{ij(\delta)}\right)$ is the utility assigned to the plan $\mathcal{p}_{ij(\delta)}$, according to the individual utility function $U_j(\mathcal{p})$ of agent j (see below).
- The Self-derived memory $M_i^{self} = \left\{o_{ii(\delta)}\right\}_{1\times s}$ which consists of s opinion sets. Each opinion set $o_{ii(\delta)} = \left\{\mathcal{p}_{ii(\delta)}, U\left(\mathcal{p}_{ii(\delta)}\right) + \varepsilon\right\}$ consists of a plan, randomly derived from the m-dimensional problem space and $U\left(\mathcal{p}_{ii(\delta)}\right) + \varepsilon$ is the utility assigned by the true utility function $U(\mathcal{p})$ to the plan $\mathcal{p}_{ii(\delta)}$, adding a noise component $\varepsilon \in [-\eta,\ \eta]$, with $\eta \in [0,\ 1]$.

At each time step (a consultation round), a discussion among the agents takes place and each agent adopts a certain number of other opinions from his peers. The memory capacity c constrains the number of opinions derived from colleagues that an agent can hold at the same time. Thus, older opinions are gradually replaced by newer ones following the First-in-First-out (FiFo) principle. Note that the self-derived opinion sets do not get affected by this "forgetting" process. For each agent i the individual utility function $U_i(\mathcal{p})$ is defined as

$$U_i(\mathcal{p}) = \sum_j \left(\sum_\delta \left(CL\left(\mathcal{p}_{ij(\delta)}, \mathcal{p}\right) \cdot U_j\left(\mathcal{p}_{ij(\delta)}\right)\right)\right), \tag{32}$$

where $U_j\left(\mathcal{p}_{ij(\delta)}\right)$ is the utility value of agent i for the δ-th plan $\mathcal{p}_{ij(\delta)}$ derived from agent j, and $D\left(\mathcal{p}_{ij(\delta)}, \mathcal{p}\right) = \frac{cl\left(\mathcal{p}_{ij(\delta)}, \mathcal{p}\right)}{\sum_j\left(\sum_\delta\left[cl\left(\mathcal{p}_{ij(\delta)}, \mathcal{p}\right)\right]\right)}$ is the normalized value of the inverse square distance (closeness) $cl\left(\mathcal{p}_{ij(\delta)}, \mathcal{p}\right) = \left|\mathcal{p}_{ij(\delta)} - \mathcal{p}\right|^{-2}$ between the plan $\mathcal{p}_{ij(\delta)}$ of agent i derived from agent j and a plan under consideration $\mathcal{p}$.

Additionally, a tentative group plan $\mathcal{p}_\gamma = \left[x_{1(\gamma)},\ x_{2(\gamma)},\ \ldots,\ x_{m(\gamma)}\right]$ is discussed, while each agent i has his own personal plan $\mathcal{p}_i^{max} = \left[x_{1(i)}^{max},\ x_{2(i)}^{max},\ \ldots,\ x_{m(i)}^{max}\right]$, which is the one with the maximum utility according to his individual utility function. At each time step, an agent κ is randomly selected to speak to the group and suggest a revision of the group plan. He presents an opinion $o_{\kappa\{\xi\}}^{\text{sug}} = \left\{\mathcal{p}_{\kappa\{\xi\}}^{\text{sug}}, U_\kappa\left(\mathcal{p}_{\kappa\{\xi\}}^{\text{sug}}\right)\right\}$, where $\mathcal{p}_{\kappa\{\xi\}}^{\text{sug}} = \left[x_{1(\gamma)},\ \ldots,\ x_{\xi(\kappa)}^{max}, \ldots,\ x_{m(\gamma)}\right]$ is a newly suggested plan. Here the ξ-th aspect $x_{\xi[\gamma]}$ of the current group plan $\mathcal{p}_{[\gamma]}$ has been replaced by the newly suggested aspect $x_{\xi(\kappa)}^{max}$ of the personal plan of agent κ, which he personally considers it important due to its maximization of the utility gain: $\Delta U_{\kappa,\xi} = \max\limits_{\lambda\in\{1,2,\ldots,m\}}\left\{U_\kappa\left(\mathcal{p}_{\kappa\{\lambda\}}^{\text{sug}}\right) - U_\kappa\left(\mathcal{p}_{[\gamma]}\right)\right\}$.

An agent i is convinced only if $U_i\left(\mathcal{p}_{\kappa\{\xi\}}^{\text{sug}}\right) > U_i\left(\mathcal{p}_i^{max}\right)$. Then he updates his personal plan accordingly, thus $\mathcal{p}_i^{new\_max} = \mathcal{p}_{\kappa\{\xi\}}^{\text{sug}}$. Afterwards, the revised individual plan $\mathcal{p}_i^{new\_max}$ might become further refined via a local search for an even better plan $\mathcal{p}^*$ in a sphere of radius r centered on $\boldsymbol{p}_i^{new\_max}$, i.e., $\left|\boldsymbol{p}^* - \boldsymbol{p}_i^{new\_max}\right| < r$. In this way, it is possible that further optimization of utility may be achieved.

If the condition $U_i\left(\boldsymbol{p}_{\kappa\{\xi\}}^{\text{sug}}\right) > U_i\left(\boldsymbol{p}_i^{max}\right)$ is not fulfilled, he still may accept the proposal with probability:

$$P\left(\boldsymbol{p}_i^{max} \to \boldsymbol{p}_{\kappa\{\xi\}}^{\text{sug}}\right) = \exp\left(-\frac{\left|\boldsymbol{p}_{\kappa\{\xi\}}^{\text{sug}} - \boldsymbol{p}_i^{max}\right|^2}{T_i}\right), \tag{33}$$

where T_i is the "temperature" (openness) of agent i, indicating his tolerance for accepting new proposals of lower utility value than the utility value of his own personal plan.

Revision of the group plan $\boldsymbol{p}_{[\gamma]}$ is implemented if the condition $\frac{\sigma}{N-1} > \theta$ is satisfied, where θ is the acceptance threshold and σ is the number of supporters, excluding the speaker. The algorithm is completed when the group finds a plan with a utility value equal to the maximum utility value of the true utility function.

3.4. Application in Economics

GDM has been sparsely applied on Economics [95,96]. For instance, in [95], a GDM model is proposed for simulating decision making among multiple stakeholders (agents) on the implementation of financial aid programs in rural China. The participating stakeholders-agents, such as development banks, local government institutions and credit suppliers, express their opinions in heterogeneous mathematical formulations.

As in the case of [84,90,94], a distinct decision-making mechanism is proposed via which agents collectively decide on a set of m alternatives $X = \{x_1, x_2, \ldots, x_m\}$. Each agent can express his opinion in one of the following formats:

- Preference Utility: The opinion o_i of agent i is expressed as a utility vector $o_i = u_i = \left[u_{1(i)}, u_{2(i)}, \ldots, u_{m(i)}\right]$ containing utility values for each alternative x_p with $p = 1, 2, \ldots, m$. The largest element indicates agent i's top preference. Relative utilities $\upsilon_{pq(i)} = \frac{u_{p(i)}}{u_{q(i)}}$ with $p, q = 1, 2, \ldots, m$ are encoded in a $m \times m$ matrix, indicating the preference of x_p over x_q as evaluated by agent i. All relative utilities are encoded in the opinion matrix $O_i = U_i = \left[\upsilon_{pq(i)}\right]_{m\times m}$. The normalized opinion matrix is $\mathcal{O}_i = \mathcal{U}_i = \left[\frac{\upsilon_{pq(i)}}{\sum_{\xi=1}^{m}\upsilon_{\xi q(i)}}\right]_{m\times m}$.
- Preference Ranking: The opinion o_i of agent i is expressed as a ranking vector $o_i = r_i = \left[r_{1(i)}, r_{2(i)}, \ldots, r_{m(i)}\right]$, whose elements is an ordinal variable, indicating the ranking of each alternative. If $r_{p(i)} = 1$ then the alternative x_p is agent i's top preference. The normalized opinion matrix is $\mathcal{O}_i = \mathcal{R}_i = \left[\frac{\boldsymbol{r}_{pq(i)}}{\sum_{\xi=1}^{m}\boldsymbol{r}_{\xi q(i)}}\right]_{m\times m}$, where $\boldsymbol{r}_{pq(i)} = \frac{m-r_{p(i)}}{m-r_{q(i)}}$ indicates the preference of x_p over x_q as evaluated by agent i.
- Multiplicative Preference Relation: The opinion o_i of agent i is expressed via a Pairwise Comparison Matrix $O_i = A_i = \left[a_{pq(i)}\right]_{m\times m}$ with elements taking values in the interval $\left[\frac{1}{9}, 9\right]$, satisfying the condition $a_{pq(i)} \cdot a_{qp(i)} = 1$. When $a_{pq(i)} < 1$ then alternative x_p is favored over x_q while when $a_{pq(i)} > 1$ the reverse is true. The greater the distance from 1 (i.e., closer to 1/9 or closer to 9), the more preferable the corresponding alternative is. The normalized opinion matrix is $\mathcal{O}_i = \mathcal{A}_i = \left[\frac{a_{pq(i)}}{\sum_{\xi=1}^{m}a_{\xi q(i)}}\right]_{m\times m}$.
- Additive Preference Relation: The opinion o_i of agent i is expressed via a Pairwise Comparison Matrix $O_i = B_i = \left[b_{pq(i)}\right]_{m\times m}$, with elements taking values in the interval $[0, 1]$ and satisfying the condition $b_{pq(i)} + b_{qp(i)} = 1$. When $b_{pq(i)} > 0.5$, then alternative x_p is favored over x_q while when $b_{pq(i)} < 0.5$, the reverse is true. The greater the

distance from 0.5 (i.e., closer to 1 or closer to 0), the more preferable the corresponding alternative is. The normalized opinion matrix is $\mathcal{O}_i = \mathcal{B}_i = \left[\frac{b_{pq(i)}}{\sum_{\xi=1}^{m} b_{\xi q(i)}}\right]_{m\times m}$.

Depending on the form of the normalized opinion matrix $\mathcal{O}_i$, each agent is placed in one of the following four separate sets: $S_u = \{i : \mathcal{O}_i = \mathcal{U}_i\}$, $S_r = \{i : \mathcal{O}_i = \mathcal{R}_i\}$, $S_A = \{i : \mathcal{O}_i = \mathcal{A}_i\}$, $S_B = \{i : \mathcal{O}_i = \mathcal{B}_i\}$, with $S_u \cup S_r \cup S_A \cup S_B = S$ the union of all sets containing all agents. A weight c_i, indicating the cooperativeness of agent i for reaching consensus, is assigned to each normalized opinion matrix $\mathcal{O}_i$. Group opinion is defined as the collective opinion vector $o_{group} = arg \max\limits_{o\in\mathbb{R}^m}\{G(o)\} = \left[o_{1(group)}, o_{2(group)}, \ldots, o_{m(group)}\right]$, where:

$$G(o) = \sum_{i\in S_u}\left(c_i\sum_{q=1}^{m}\left\langle \upsilon_{[q](i)}, o\right\rangle\right) + \sum_{i\in S_r}\left(c_i\sum_{q=1}^{m}\left\langle r_{[q](i)}, o\right\rangle\right) + \sum_{i\in S_A}\left(c_i\sum_{q=1}^{m}\left\langle a_{[q](i)}, o\right\rangle\right) + \sum_{i\in S_B}\left(c_i\sum_{q=1}^{m}\left\langle b_{[q](i)}, o\right\rangle\right) \tag{34}$$

The cosine similarity $\left\langle o_{[q](i)}, o\right\rangle$, with the opinion matrix $\mathcal{O}_i$ element $o_{[q](i)}$ taking one of the following forms, namely $\upsilon_{[q](i)}, r_{[q](i)}, a_{[q](i)}, b_{[q](i)}$, is defined as:

$$\left\langle o_{[q](i)}, o\right\rangle = \frac{\sum_{p=1}^{m}\left(o_{pq(i)}\cdot o_p\right)}{\sqrt{\sum_{p=1}^{m} o_{pq(i)}^2}\sqrt{\sum_{p=1}^{m} o_p^2}}. \tag{35}$$

The quantity $\sum_{q=1}^{m}\left\langle o_{[q](i)}, o\right\rangle$ is the sum of the above cosine similarity measures between the columns $o_{[q](i)}$ of the opinion matrix of agent i and an opinion vector $o = [o_1, o_2, \ldots, o_m]$. The consensus level $\frac{G(o_{group})}{|S|\cdot m}$ expresses the normalized similarity between the group opinion o_{group} and the opinions o_i of the agents. When $\frac{G(o_{group})}{|S|\cdot m} < \varepsilon$, where ε is an arbitrary threshold, the consensus level is not acceptable and, therefore, an opinion update for all agents is required. Then, an agent i will randomly select a new opinion $o_{pq(i)}^{new}$ in the interval $\left[\min\left\{o_{pq\_group(i)}, o_{pq(i)}\right\}, \max\left\{o_{pq\_group(i)}, o_{pq(i)}\right\}\right]$, where $o_{pq\_group(i)} = f_i\left(o_{p(group)}, o_{q(group)}\right)$ is the collective opinion about x_p in terms of x_q in a form understandable to agent i, according to his opinion format.

During the above-described consensus reaching process, agents whose opinions o_i diverge further from the group opinion o_{group} are identified and clustered together. Such non-cooperative agents are called to modify their opinion o_i. In case they persist in their divergence from the group opinion, they forfeit a part of their social influence in the GDM process as their individual cooperative weight c_i is lowered after each iteration but never completely nullified. At the start of the GDM process, all agents have equal cooperative weights c_i.

4. Knowledge Dynamics (KD)

Alongside OD and GDM, in the field of Sociophysics, Knowledge Dynamics (KD) is also intensively studied so as to investigate how knowledge is diffused in complex networks. Knowledge is distinguished from opinion as it requires a justification founded on logical conclusions about observations [97], not subjective and ad-hoc approximations of reality. As in the case of OD and GDM, Network Science has been utilized for modelling knowledge diffusion processes in complex networks [98].

4.1. Knowledge Exchange

One of the most notable agent-based approaches for knowledge diffusion in complex networks is based on the knowledge exchange principle [99,100]. In [99], agents are characterized by a knowledge vector containing several knowledge items $s = 1, 2, \ldots, N$. The corresponding knowledge level of agent i for item s at time step t is denoted by $k_i^s(t)$.

The exchange of knowledge between two neighboring agents is based on the "quid pro quo" principle ("win–win" interaction). This means that every possible interaction between two agents must result in mutual benefit, i.e., knowledge gain. The only prerequisite for knowledge exchange between agents i and j is the existence of a relative knowledge deficit in at least two different items. This condition is widely adopted in many knowledge diffusion models [101–103].

For simplicity, it is assumed that agents possess knowledge in only two items. Suppose that there is a relative knowledge deficit for agent i in item $s = 1$ and for agent j in item $s = 2$, i.e., $k_i^1(t) < k_j^1(t)$ and $k_j^2(t) < k_i^2(t)$, correspondingly. Knowledge diffusion is formulated as follows:

$$\begin{aligned} &For\ item\ s = 1: \left\{ \begin{array}{c} k_i^1(t+1) = k_i^1(t) + a\left(k_j^1(t) - k_i^1(t)\right) \\ k_j^1(t+1) = k_j^1(t) \end{array} \right\} because\ k_i^1(t) < k_j^1(t). \\ &For\ item\ s = 2: \left\{ \begin{array}{c} k_i^2(t+1) = k_i^2(t) \\ k_j^2(t+1) = k_j^2(t) + a\left(k_i^2(t) - k_j^2(t)\right) \end{array} \right\} because\ k_j^2(t) < k_i^2(t), \end{aligned} \tag{36}$$

where $0 < a < 1$ is a constant (similar to the convergence parameter μ in the DW model), indicating the capacity for knowledge absorption.

It is important to highlight that knowledge diffusion is conditioned on the existence of a mutual need to increase the present knowledge levels $k_i^1(t)$ and $k_j^2(t)$. We can contrast this "quid pro quo" principle ("win–win" interaction), with the widely used bounded confidence condition (3) in OD. The key difference is that in KD there is an objective superiority of agent i in comparison to agent j in terms of knowledge in a specific knowledge item, whereas in OD there is no objective ranking of the agents involved in a social interaction. The mathematical implication of this remark is that in KD knowledge diffusion in a single knowledge item is unidirectional, namely from the more knowledgeable agent to the less knowledgeable one. On the contrary, in OD social influence may be bidirectional, in the sense a single social interaction may result into a shift of opinion for both interacting agents.

Additionally, it must be noted that in ON diffusion constraints are set by the psychosocial inability to communicate with someone with radically different posture on various issues (heterophobia), whereas in the context of KD diffusion constraints are set by objective knowledge gaps between agents. Someone who is vastly inferior in knowledge than someone highly knowledgeable (experts) simply cannot absorb the available knowledge. Thus, when communication fails, this is due to objective, not intersubjective, reasons.

4.2. Self-Innovating Agents

Central to the study of knowledge networks is the potential presence of *self-innovating agents* [97,101,103,104]. Self-innovating agents can be compared to the informed agents in OD. Contrary to informed agents, who may act as stealth manipulators of social change, self-innovating agents are generally regarded as beneficial sources guiding the knowledge enlightenment of their peers.

The concept of self-innovating agents was firstly proposed in [101]. Here, knowledge is a scalar variable. At each time step, a self-innovating agent i is selected to innovate, i.e., produce new knowledge. The process of innovation is formulated as $k_i(t) = k_i(t-1)(1+\beta_i)$, where $\beta_i > 0$ is the innovation ability of agent i. As knowledge is of scalar form, not vector, as in the case of knowledge exchange, there is no prerequisite condition for mutual knowledge gain. Therefore, knowledge diffusion from the self-innovating agent i to a regular neighboring agent j can be realized only if $k_i(t) > k_j(t)$, with the knowledge diffusion formula taking the following form:

$$k_j(t+1) = k_j(t) + a_j\big(k_i(t) - k_j(t)\big), \tag{37}$$

where $0 < a_j < 1$ is the knowledge absorptive captivity of agent j.

Knowledge level is normalized taking values in the interval $[-1, 1]$. The innovation parameter of a self-innovating agent i is regarded as his self-weight, i.e., $\beta_i = w_{ii}$, which in [97] takes even negative values indicating self-destructive thinking. In addition to the presence of self-destructive agents, knowledge "destruction" can take place due to the presence of unreliable knowledge channels (misinformation) or targeted attacks on the knowledge network (disinformation).

Note that in most KD models [99,101], agents select randomly other agents for knowledge acquisition. This "randomness" assumption for social interaction is also found in most OD models [31–33]. However, and as we have already noted in the introduction, humans (or even other intelligent entities, artificial or otherwise) are not material atoms. Their actions have purpose. Only in certain recent model proposals [105–107], agents are treated as boundedly rational purposeful individuals [66] limited by their uncertainty about the knowledge level of their peers and/or the reliability and efficiency of the communication channels. This key observation distinguishes the classically defined diffusion equation describing spontaneous diffusion between "mindless" molecules of matter from social interactions between intelligent agents. Taking this key observation into account, the impact of prioritization of actions and awareness level has been extensively discussed in [105–107] for KD and in [56,79] for OD.

4.3. Application in Business

Applications of KD in business can also be found in the literature [108,109]. For example, in [108], knowledge diffusion among Indian companies (agents) in a weighted directed network is realized due to the presence of interlocking directors, i.e., individuals who are simultaneously members of different directors' boards of different companies. Social interaction essentially takes place in board meetings, with affiliated directors acting as channels of knowledge transfer. Links are directed, with their direction indicating the affiliation of the corresponding director, and weighted, with weight $w_{i\to j}^{p_j}$ representing a separate common interlocking director p_j belonging to the boards of both companies i and j and affiliated towards company j. Two connected agents i and j may exchange knowledge via two sets of multiple weighted directed links, denoted by $e_{i\to j}$ and $e_{j\to i}$ indicating the set of weighted directed links $w_{i\to j}^{p_j}$ and $w_{j\to i}^{p_i}$ respectively.

As in [99], each agent (company) i is assigned a knowledge vector containing several knowledge items $s = 1, 2, \ldots N$, and a corresponding absorptive capacity a_i. The knowledge level of agent i in item s at time t is denoted by $k_i^s(t)$. For each agent i, knowledge gain in an item s consists of two parts, namely: (a) a free knowledge gain G_i^s derived from "free" preliminary contributions from interlocking directors, and (b) a non-free knowledge gain F_i^s which is based on the "quid pro quo" exchange principle ("win–win" interaction). For simplicity, it is assumed that agents possess knowledge in only two items. Suppose that there is a relative knowledge deficit for agent i in item $s = 1$ and for agent j in item $s = 2$, i.e., $k_i^1(t) < k_j^1(t)$ and $k_j^2(t) < k_i^2(t)$ correspondingly. Similar to (36), knowledge diffusion is formulated as follows:

$$\begin{aligned} &\textit{For item } s = 1: \left\{ \begin{matrix} k_i^1(t+1) = k_i^1(t) + G_i^1 + F_i^1 \\ k_j^1(t+1) = k_j^1(t) \end{matrix} \right\} \textit{because } k_i^1(t) < k_j^1(t). \\ &\textit{For item } s = 2: \left\{ \begin{matrix} k_i^2(t+1) = k_i^2(t) \\ k_j^2(t+1) = k_j^2(t) + G_j^2 + F_j^2 \end{matrix} \right\} \textit{because } k_j^2(t) < k_i^2(t), \end{aligned} \tag{38}$$

where the free knowledge gain components are formulated as:

$$G_i^1 = \sum_{p_i=1}^{|e_{j\to i}|} w_{j\to i}^{p_i} a_i \left(k_j^1(t) - k_i^1(t) \right), \; G_j^2 = \sum_{p_j=1}^{|e_{i\to j}|} w_{i\to j}^{p_j} a_j \left(k_i^2(t) - k_j^2(t) \right), \tag{39}$$

and the non-free knowledge gain components are formulated as:

$$F_i^1 = a_i \min\left\{k_j^1(t) - k_i^1(t),\ k_i^2(t) - k_j^2(t)\right\}, F_j^2 = a_j \min\left\{k_j^1(t) - k_i^1(t),\ k_i^2(t) - k_j^2(t)\right\} \quad (40)$$

Note that in the case of barter knowledge exchange (38), knowledge gain (incorporating the difference in absorption rates) has to be equal in terms of quantity for both agents, i.e., $\frac{F_i^1}{a_i} = \frac{F_j^2}{a_j}$. In contrast, in [99], equality is only in terms of the number of the knowledge items to be shared.

5. A Critique of Social Diffusion Dynamics in Networks

5.1. Gaps and Issues in the Application of Sociophysics in Real-World Socioeconomic Phenomena

As things stand today, Sociophysics makes possible the quantitative description and analysis of complex social phenomena, by means of mathematical modelling in social networks. The age-old dream for a positive social science expressed by thinkers like August Comte, Émile Durkheim (1858–1917) and Herbert Spencer (1820–1902) might not have been fully realized but its distant echo can be heard in the modern social diffusion dynamics research. The value of Sociophysics has been already demonstrated in past cases of successful simulations of the social environment, as evidenced by reportedly accurate predictions of Sociophysics models on the results of important social processes, e.g., national elections, referenda, etc. [110]. Of similar value is the applications of Sociophysics in real-world social processes in the domains of Behavioral Finance, Social Trading and Business, taking into account the crucial sociological insight that the opinions of financially oriented agents are socially dependent [67,76,77]. Nonetheless, in order for the Sociophysics paradigm to offer a substantial contribution to the alleviation of already existing and emerging societal problems (e.g., disinformation/misinformation, spread of pseudo-scientific theories, lack of consensus among international actors like states, organizations, etc.) certain evident boundaries of current research have to be overcome.

Firstly, it should be noted that while Sociophysics applications in the field of behavioral finance/social trading/business abound in the case of OD, they seem to be scarce in the cases of GDM and KD. This is partly to be expected since the modern free market environment contains primarily self-interested agents. Thus, collaboration is not expected to be the norm. Knowledge sharing between financial/business agents is also uncommon, save for the case when there is a “quid pro quo” benefit to be gained. Financial/business actors generally prefer to keep their innovations secret [111]. Nonetheless, existing research suggests that there is certainly room to further investigate collaboration processes in the financial/business domain. Instances do exist where collaboration and knowledge sharing between multiple financial/business agents takes place, e.g., the common practice of co-branding [112] or the existence of partnerships between start-ups and large established companies [113].

Secondly, there is a distinctive lack of empirical validation for most proposed models. Despite the fact that Sociophysics models are built on the basis of common-sense assumptions (e.g., the existence of purposeful opinion-spreaders, the desire to collaborate and find a commonly acceptable solution or the existence of agents with a strong tendency for innovation), model calibration and validation on the basis of real-world data is sorely missing, except for certain empirically verified statistical observations on behavioral patterns, e.g., the fact that existing interpersonal ties tend to influence buying and selling practices of traders [63,64]. The lack of empirical validation is partly justified by the complexity of the social world and the expected inability to independently observe social interactions in their totality, contrary to physical phenomena that can be reproduced in a lab setting. However, the proliferation of Big Data and Machine Learning/Artificial Intelligence technologies and the tendency for increasing data accumulation in the corporate and state sectors indicate that Sociophysics research might hopefully be able to overcome, at least in part, the above-mentioned limitations [12]. In that case, it would be preferable that model development be conducted not a priori but in conjunction to the available data.

5.2. Misalignment between Sociophysics Modelling Assumptions and Social Reality

As regards the extent that Sociophysics modelling assumptions accurately capture, interpret and represent social reality (at least in accordance with current sociological understandings) we wish to underscore some important and rather overlooked issues and their implications. Firstly, it must be noted in most proposed models, simulations converge to a steady state considered to be the end of the social diffusion process. Although much focus is placed in formulating how social structure is built from the bottom (micro- level) to the top (macro- level), no particular emphasis is placed on the opposite, i.e., how an existing social structure is altered by micro-interactions in turn influenced by it. Contrary to structural functionalist expectations who expect general social stability and order [114], the contemporary social world is characterized by temporality, as agents interact with and transform existing social structures through their actions [115]. Continuous social flux seems to be the norm, per the well-known aphorism of the Greek philosopher Heraclitus ("Everything flows"—τὰ πάντα ῥεῖ). Thus, most social structures (e.g., opinion uniformity, opinion polarization, etc.), are bound to be eventually all but be discarded as agents will react to the existing "status quo".

The presence of a co-evolution mechanism between structure and agents' attributes, indicating a continuous micro-macro interaction, has been already highlighted in certain studies [76,77,79,107,116]. Nevertheless, further exploration would be more than welcome in order to more accurately describe various co-evolutionary processes of social diffusion. For example, micro-macro interaction could be modeled not only via taking into account the influence of neighbors on agents' attributes and actions but of broader structural network properties as well. See, for example, the probabilistic opinion shift for contrarian agents in [69] where global distribution of opinion is taken into account and, in particular, the operation of the GDM model in [90], where opinion update for an agent is simultaneously dependent on all levels of interaction, micro (opinion of neighbors), meso (cluster opinion) and macro (collective opinion) (Table 2). We, however, note that what is missing from the last two formulations is the incorporation of modifications in network structure (i.e., activation/deactivation or strengthening/weakening of specific links) as agent attributes, such as opinion, reputation, authority, etc., shift in time due to the influence of micro, meso and macro network parameters. In short, simultaneous rejection of link staticity and incorporation of the direct influence of higher-level properties on agent attributes are paramount for a proper depiction of agent-structure interaction. To the best of our knowledge, such contribution is sorely missing from current Sociophysics research.

Secondly, to further elaborate our point, we note that the above-mentioned misalignment between Sociophysics and social reality can also be attributed to the fact that there is a tendency in many eminent Sociophysics models (e.g., [28,31,32,99]) to treat social interaction as the product of chance encounters inducing change in social properties (opinion, knowledge, etc.), much like how random collisions between air molecules induce changes in their physical properties (direction, speed, mass, etc.). Evidently, this is not how the social world primarily operates. We are aware of only a handful of modelling attempts [67,105–107], where social agents are implicitly or explicitly considered purposeful individuals, consciously deciding, on the basis of individual preferences and previous experience, with whom they will interact and what their position will be. Therefore, socially influenced as the agents may be, they still operate on the basis of certain rational criteria, whichever these may be (e.g., previous beneficial interaction). We think that this is in line with how existing social structures are formed and re-formed under the influence of individual decisions and actions, notwithstanding the fact that actors usually do not possess perfect knowledge as regards potential consequences of their actions [66] due to the immanent social complexity and the resulting interdependence between individual decisions, selections and actions. This observation has great political, beyond socioeconomic, relevance and it is probably of key importance for a successful incorporation of the element of social complexity in the domains of political science in general and international politics in particular [117–119].

Accordingly, the bottom line of our argument is that Sociophysics should treat social phenomena not as the social equivalent of stochastic or law-based neutral physical processes but as aspects of emergent and shifting social structures. These structures are formed, preserved and updated via the deliberate choices of conscious—or, at the very least, partially conscious due to their potentially biased dispositions—social actors/agents. Each agent is equipped with (a) a set of individual selection criteria, (b) limited knowledge of their surroundings and (c) a distinct position in the social structure, implying a certain degree of influence in the micro, meso and macro levels. The finite structural "resources" at his disposal permit him to implement—or prevent him from fully implementing—his own personal goals (e.g., transfer or extraction of knowledge, promotion of opinion, etc.). Thus, social structure exerts enabling but also constraining effects on conscious individual action [120].

We believe that the Network Science framework, already employed in Sociophysics, is ideal for further mathematical elaboration of social structure and its co-evolutionary interaction with individual agent behavior. The element of deliberate inter-agent communication in the process of mutual constitution between agency and structure is a perhaps overlooked but nonetheless extremely important point that an aspiring "computational social science" [12] like Sociophysics should fully take into account in order to soothe the already noted ontological tension between individual freedom of action and the placement of agent behavior in a set of mathematical constraints, taking the form of social diffusion equations.

The incorporation of these omissions (neglect of the agent-structure co-evolutionary interdependence and non-random, conscious communication), as well as the evident limitations outlined in Section 5.1. (dearth of applications in behavioral finance and business in the cases of GDM and KD and general lack of empirical validation), is crucial in order for the distinct advantages and added value of Sociophysics to become apparent. We suggest that the issues described herein (see Table 3) be tackled in short order for the discipline of Sociophysics to become accepted as the bleeding edge of applied mathematics in social sciences (economics, political science, etc.).

Table 3. Key issues, recommendations and indicative approaches for future tackling of important research gaps/issues in the field of Sociophysics.

Issue	Recommendation	Preliminary or Indicative Approaches
Dearth of socioeconomic applications in the GDM and KD categories.	Exploration of collaborative aspects of socioeconomic reality.	Chao et al. (2021a) [95] Chao et al. (2021b) [96] Schweitzer at al. (2022) [121]
	Exploration of knowledge transfer phenomena between financially oriented agents.	Vaccario et al. (2018) [122] Sankar et al. (2020) [108] Shi et al. (2020) [109]
Lack of empirical validation.	Incorporation of available datasets for the calibration and testing of ABM sociophysical models.	Schweitzer et al. (2022) [121] Vaccario et al. (2018) [122]
Misalignment between Sociophysics and social reality.	Emphasis on agent-structure co-evolution.	DeLellis et al. (2017) [77] DeLellis et al. (2018) [76] Ioannidis et al. (2020) [79] Ioannidis et al. (2021) [107] Antoniou et al. (2022) [116]
	Depiction of micro, meso and macro-level influence on agents' actions and vice-versa.	Zubillaga et al. (2022) [69] Li et al. (2022) [90]
	Incorporation of the element of conscious selection and interaction.	Panchenko et al. (2013) [67] Ioannidis et al. (2018a) [105] Ioannidis et al. (2018b) [106] Ioannidis et al. (2021) [107]

6. Conclusions—Future Prospects

Through a meticulous presentation of the most notable examples of the Sociophysics discipline in diffusion dynamics modelling, we were able to offer an "eagle's eye" view of the existing literature. We provide a much-needed critical appraisal of the current standing of the discipline, hoping that our contribution will prove to be a beneficial one for the future of Sociophysics/Econophysics and computational social science in general. In particular, we showcase how the social world through the use of a mathematics-based quantitative approach can be fruitfully studied and we present key general approaches and applications of a more specified socioeconomic character, focused on interactions between financially oriented agents. We additionally highlight the promises and shortcomings of current Sociophysics research as we additionally aim to offer a critical view on the Sociophysics discipline, bringing to the foreground crucial aspects of its often-problematic correspondence with social reality.

Still, further elaboration is possible, i.e., through indication of specific advantages and disadvantages of the analyzed methods, an insight that is beyond the scope of this work. For now, suffice it to say that the appropriateness of each model is dependent on the specific context in which it is applied and the related well-defined and often conflicting assessment criteria. More specifically, with respect to the criterion of computational complexity, linear models, such as the ones found, for example in [18,21] have a comparative advantage. On the contrary, with respect to the criterion of closeness to socioeconomic reality, nonlinear models like the ones found, for instance, in [28,31] are rather more preferable, since they more effectively capture social complexity. Thus, a balance between conflicting requirements often needs to be achieved. The appearance of more data-enhanced real world applications of Sociophysics [121,122] will, hopefully, make apparent which models are more suitable for modelling which aspect of social reality, according to the criteria mentioned.

From the analysis conducted above, we conclude that current and future approaches on opinion, consensus and knowledge diffusion dynamics can be beneficially applied in the context of the modern complex information society [13], especially if the suggestions presented in Section 5 are to be taken into account through the application of Sociophysics principles into rather unexplored areas of socioeconomic interaction such as inter-firm collaboration [95,96,121], inter-firm patent diffusion [108,109,122], etc. Certain crucial questions include—but not are not limited to—the following: To what extent and manner firms and entrepreneurs exchange ideas, opinions and information? How can a company fully benefit from its collaborative interactions with other companies? How decision-making on the implementation of tasks with important social impact can be achieved among affected stakeholders, including firms? All the above can be explored through the incorporation of an increasing abundance of available time series data, pertaining to, for example, financial transactions, instances of collaboration or common participation and endorsement of social events. However, we believe that a more detailed mathematical depiction of agent–structure co-evolution is what is required the most.

We believe that the above proposals are of great relevance for all categories of social diffusion processes analyzed herein, OD, GDM and KD. In the case of OD, the need for firms, organizations and governmental agencies to adopt current norms and prevent the spread of disruptive disinformation/misinformation and fake news in relevant social networks [123,124] suggests that opinion diffusion modelling will be considered a must-have tool in the near future. The same is true for KD, since the existence of a rapidly shifting global social structure has intensified the need for the development of efficient mechanisms for knowledge transfer [125] and knowledge management [126]. The effective diffusion of knowledge and innovations in various important domains (e.g., energy efficiency technologies [109]) is crucial for the sustainable competitiveness and orderly operation of enterprises, financial organizations or innovation networks. Lastly, efficient GDM procedures ensure the accommodation of crucial financial/economic issues in areas such as earthquake sheltering [127] and urban resettlement [96].

Considering, in toto, the benefits and shortcomings of current Sociophysics research, it is important to underscore the apparent potential of the Sociophysics/Econophysics discipline to accurately describe the dynamics of social and economic reality. We believe that a Sociophysics-based approach might limit the intervention of subjectivity on the part of the social researcher, particularly as regards the way interaction between different structural social levels (micro, meso and macro) takes place.

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