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Boundedness of Solutions for an Attraction–Repulsion Model with Indirect Signal Production

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Abstract: In this paper, we consider the following two-dimensional chemotaxis system of attraction–repulsion with indirect signal production

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \left(\chi_1 u \nabla v_1\right) + \nabla \cdot (\chi_2 u \nabla v_2), & x \in \mathbb{R}^2,\ t > 0, \\ 0 = \Delta v_j - \lambda_j v_j + w, & x \in \mathbb{R}^2,\ t > 0, \quad (j = 1,2), \\ \partial_t w + \delta w = u, & x \in \mathbb{R}^2,\ t > 0, \\ u(0,x) = u_0(x),\ \ w(0,x) = w_0(x), & x \in \mathbb{R}^2, \end{cases}$$

where the parameters $\chi_i \geq 0, \lambda_i > 0\ (i = 1,2)$ and non-negative initial data $(u_0(x), w_0(x)) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. We prove the global bounded solution exists when the attraction is more dominant than the repulsion in the case of $\chi_1 \geq \chi_2$. At the same time, we propose that when the radial solution satisfies $\chi_1 - \chi_2 \leq \frac{2\pi\delta}{\|u_0\|_{L^1(\mathbb{R}^2)} + \|w_0\|_{L^1(\mathbb{R}^2)}}$, the global solution is bounded. During the proof process, we found that adding indirect signals can constrict the blow-up of the global solution.

Keywords: Keller–Segel; attraction–repulsion model; indirect signal production; radial solution; boundedness

MSC: 35B65; 35Q35; 35Q92; 92C17

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1. Introduction

The chemotactic model was proposed by Keller and Segel in [1] to describe the movement mechanism of organisms, cells or bacteria under the action of chemicals. Classified based on the direction of movement, we have chemotactic attraction and chemotactic repulsion. These forces play a crucial role in many development systems. In recent years, the issue of chemotaxis has been extensively studied. For example, global solvability has been studied in [2–24], large time behavior in [6,25–29], finite time blow-up in [6,30–36], nontrivial stationary solutions in [18,37–40], nonlinear diffusion in [41,42], indirect signaling in [43–47], etc. These studies provide important assistance for us to gain a deeper understanding of chemotactic phenomena.

A classical chemotaxis model is described as follows:

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot \left(\chi u \nabla v\right), & x \in \Omega,\ t > 0, \\ \tau v_t - \Delta v - u + \lambda v = 0, & x \in \Omega,\ t > 0, \end{cases}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote cell density and chemical concentration, respectively. $\Omega \in \mathbb{R}^n$ is a domain and $\tau \in \{0,1\}$. When Ω is bounded, we propose the homogeneous Neumann initial boundary value conditions:

$$\begin{cases} \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

When $\Omega = \mathbb{R}^n$, we give the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n.$$

These equations are used to describe the phenomenon of chemotactic aggregation. However, many biological processes also involve chemotactic repulsion. In [48], Luca proposed a more general model to describe attraction and repulsion phenomena as follows:

$$\begin{cases} \partial_t u - \Delta u = -\chi_1 \nabla \cdot \left(u \nabla v_1\right) + \chi_2 \nabla \cdot \left(u \nabla v_2\right), & x \in \Omega,\ t > 0, \\ \tau \partial_t v_1 - \Delta v_1 - u + \lambda_1 v_1 = 0, & x \in \Omega,\ t > 0, \\ \tau \partial_t v_2 - \Delta v_2 - u + \lambda_2 v_2 = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{1}$$

If the chemicals diffuse much more rapidly than the movement of cells, the case where $\tau = 0$ can be considered as an approximate version of the case where $\tau = 1$. Rigorous proof of this limiting process can be found in [49]. For $\tau = 1$, Jiu and Liu in [50] considered a balanced case. For $\tau = 0$, Shi and Wang in [6] found that the system (1) has unique non-negative solutions locally in time for initial data u_0 satisfying

$$u_0 \geq 0 \text{ on } \mathbb{R}^2,\ u_0 \not\equiv 0,\ u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

The non-negative solutions exist globally in time and are bounded in the repulsion-dominant $\chi_1 < \chi_2$. Nagai and Yamada investigated the case $\chi_1 > \chi_2$ in [5].

Based on the motivation of indirect signal influence, we hope to see that the addition of an indirect signal does not damage the solution of the original system. In the real world, systems can be influenced by other signals at any time. Therefore, we consider the following indirect signal model:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot \left(\chi_1 u \nabla v_1\right) + \nabla \cdot (\chi_2 u \nabla v_2), & x \in \mathbb{R}^2,\ t > 0, \\ 0 = \Delta v_j - \lambda_j v_j + w, & x \in \mathbb{R}^2,\ t > 0, \quad (j = 1, 2), \\ \partial_t w + \delta w = u, & x \in \mathbb{R}^2,\ t > 0, \\ u(0,x) = u_0(x),\ \ w(0,x) = w_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{2}$$

We suppose that the initial data satisfy

$$(u_0, w_0) \geq 0 \text{ on } \mathbb{R}^2,\ u_0 \not\equiv 0,\ (u_0, w_0) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2). \tag{3}$$

Our main result is stated as follows:

Theorem 1. *Let $\chi_1 \geq \chi_2$ and assume that $\|u_0\|_{L^1(\mathbb{R}^2)} < \frac{8\pi\delta}{4 + (\chi_1 - \chi_2)^2}$. Then, we have*

$$\|u\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2} \|v_i\|_{W^{2,p}(\mathbb{R}^2)} + \|w\|_{L^p(\mathbb{R}^2)} \leq C,\ \ i = 1, 2$$

for all $1 \leq p \leq \infty$.

Theorem 2. *Let $M_0 := \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)} + \|w_0\|_{L^1(\mathbb{R}^2)}$ and assume that non-negative initial data u_0, w_0 are radial and $(\chi_1 - \chi_2)M_0 \leq 2\pi$. Then, we have*

$$\|u\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2} \|v_i\|_{W^{2,p}(\mathbb{R}^2)} + \|w\|_{L^p(\mathbb{R}^2)} \leq C,\ \ i = 1, 2$$

for all $1 \le p \le \infty$.

Remark 1. *Because* $4 + (\chi_1 - \chi_2)^2 \ge 4(\chi_1 - \chi_2)$, *we have*

$$\frac{8\pi\delta}{4 + (\chi_1 - \chi_2)^2} \le \frac{2\pi}{\chi_1 - \chi_2}.$$

Thus, the constriction of initial data for $\|u_0\|_{L^1(\mathbb{R}^2)}$ *becomes weaker.*

Remark 2. *Using the Duhamel's principle in* (2), *we denote the solution as*

$$u(x,t) = G(\cdot,t) * u_0 + \int_0^t \nabla G(t-\tau,\cdot) * (uV)(\tau)d\tau,$$

where $G(x,t) = (4\pi t)^{-1} e^{-\frac{|x|^2}{4t}}$ *is the heat kernel, and*

$$V = \chi_1 \nabla v_1 - \chi_2 \nabla v_2 = \Big[\chi_1 \nabla (\lambda_1 - \Delta)^{-1} - \chi_2 \nabla (\lambda_2 - \Delta)^{-1}\Big] u.$$

Here, $(\lambda_i - \Delta)^{-1}$ $(i = 1,2)$ *denotes the inverse pseudo-differential operator of* $\lambda_i - \Delta$, *and we represent it using Fourier and inverse Fourier transformations. That is,*

$$\begin{aligned}(\lambda - \Delta)^{-1} u :&= \mathcal{F}^{-1}\Big[(\lambda - \big|\sqrt{-1}\xi\big|^2)^{-1} \mathcal{F}[u](\xi)\Big] \\ &= \mathcal{F}^{-1}\Big[(\lambda + |\xi|^2)^{-1}\Big] * u \\ &= K_\lambda * u,\end{aligned}$$

where $K_\lambda(x) = \int_0^\infty e^{-\lambda t} G(x,t)dt$ *is the Bessel kernel. We also denote the fractional-order differential operator* $\Lambda^k = (-\Delta)^{\frac{k}{2}}$ $(k > 0)$ *by*

$$\Lambda^k u = \mathcal{F}^{-1}\Big[|\xi|^k \mathcal{F}[u](\xi)\Big].$$

For more detailed related information, please refer to [51,52].

Applying the following Young's inequality of convolution

$$\|(\lambda - \Delta)^{-1} f\|_{L^p(\mathbb{R}^2)} = \|K_\lambda * f\|_{L^p(\mathbb{R}^2)} \le \|K_\lambda\|_{L^r(\mathbb{R}^2)} \|f\|_{L^q(\mathbb{R}^2)} \quad \text{for all } 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$$

and the estimates

$$\|K_\lambda\|_{L^p(\mathbb{R}^2)} < \infty, \;\; 1 \le p < \infty \quad \text{and} \quad \|\partial_x K_\lambda\|_{L^p(\mathbb{R}^2)} < \infty, \;\; 1 \le p < 2,$$

we can obtain the Lemma 2 below.

2. Preliminaries

Before giving an energy estimate, we need to utilize the following important mass conservation properties.

Lemma 1. *Let* (u, v_i, w) $(i = 1,2)$ *be the non-negative solution to the Cauchy problem* (2) *with non-negative initial data* u_0, w_0 *satisfying* (3). *Then, we have*

$$\|u\|_{L^1(\mathbb{R}^2)} = \|u_0\|_{L^1(\mathbb{R}^2)} \tag{4}$$

and

$$\|w\|_{L^1(\mathbb{R}^2)} \le \|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)} \tag{5}$$

as well as

$$\|w\|_{L^1(\mathbb{R}^2)} = \lambda_i \|v_i\|_{L^1(\mathbb{R}^2)} \ , i = 1, 2. \tag{6}$$

Proof. We integrate the first equation of (2) to obtain (4). Integrating the third equation of (2) and using (4), we see that

$$\frac{d}{dt}\int_{\mathbb{R}^2} w dx + \delta \int_{\mathbb{R}^2} w dx = \int_{\mathbb{R}^2} u_0 dx.$$

For $t > 0$, we solve the above ODE to obtain

$$\begin{aligned}\|w\|_{L^1(\mathbb{R}^2)} &= \|w_0\|_{L^1(\mathbb{R}^2)} e^{-\delta t} + \frac{\|u_0\|_{L^1(\mathbb{R}^2)}}{\delta}(1 - e^{-\delta t}) \\ &\le \|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Integrating the second equation of (2), we complete the proof. □

Lemma 2. *For $\lambda > 0$, we have*

$$\|v_j(t)\|_{L^p(\mathbb{R}^2)} \le C(p,q)\|w(t)\|_{L^q(\mathbb{R}^2)}, \quad 1 \le q \le p < \infty, \tag{7}$$

$$\|v_j(t)\|_{L^\infty(\mathbb{R}^2)} \le C(q)\|w(t)\|_{L^q(\mathbb{R}^2)}, \quad 1 < q \le \infty, \tag{8}$$

$$\|\nabla v_j(t)\|_{L^\infty(\mathbb{R}^2)} \le C(q)\|w(t)\|_{L^q(\mathbb{R}^2)}, \quad 2 < q \le \infty. \tag{9}$$

In particular, the following holds

$$\|v_j(t)\|_{L^p(\mathbb{R}^2)} \le C(p)\|w(t)\|_{L^1(\mathbb{R}^2)}, \quad 1 \le p < \infty. \tag{10}$$

According to Lemma 2.3 in [5], we have

Lemma 3. *If the non-negative function $g \in L^1(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$, it holds that*

$$\int_{\mathbb{R}^2} g^2 dx \le \frac{1+\varepsilon}{4\pi}\left(\int_{\mathbb{R}^2} g dx\right)\left(\int_{\mathbb{R}^2} \frac{|\nabla g|^2}{1+g} dx\right) + \frac{2}{\varepsilon}\int_{\mathbb{R}^2} g dx \quad \text{for all} \ \ \varepsilon > 0.$$

According to Lemma 2.1 of [53], we have the following lemma.

Lemma 4. *If the non-negative function $g \in L^1(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$, it holds that*

$$\int_{\mathbb{R}^2} g^3 dx \le \varepsilon\left(\int_{\mathbb{R}^2} (1+g)\log(1+g) dx\right)\left(\int_{\mathbb{R}^2} |\nabla g|^2 dx\right) + C(\varepsilon)\int_{\mathbb{R}^2} g dx,$$

where ε is any positive number and $C(\varepsilon)$ tends to infinity as $\varepsilon \to 0$.

3. A Prior Estimate

Lemma 5. *Let $\chi_1 \ge \chi_2$ and assume that $\|u_0\|_{L^1(\mathbb{R}^2)} < \frac{8\pi\delta}{4+(\chi_1-\chi_2)^2}$. Then,*

$$K := \sup_{0<t<T} \|\big(1+u(t)\big)\log\big(1+u(t)\big)\|_1 < \infty \quad \text{for all} \ \ T > 0.$$

Proof. Multiplying the first equation of (2) with $\log(1+u)$ and noting $\int_{\mathbb{R}^2} \partial_t u = 0$, $\log(1+u) \le u$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)dx=\int_{\mathbb{R}^2}\log(1+u)\,\partial_t u dx\\
&=\int_{\mathbb{R}^2}\Delta u\log(1+u)dx-\int_{\mathbb{R}^2}\nabla\cdot\Big(u\nabla\big(\chi_1 v_1-\chi_2 v_2\big)\Big)\log(1+u)dx\\
&=-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+\int_{\mathbb{R}^2}\frac{u}{1+u}\nabla u\cdot\nabla\big(\chi_1 v_1-\chi_2 v_2\big)dx\\
&=-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+\int_{\mathbb{R}^2}\nabla u\cdot\nabla\big(\chi_1 v_1-\chi_2 v_2\big)dx-\int_{\mathbb{R}^2}\frac{1}{1+u}\nabla u\cdot\nabla\big(\chi_1 v_1-\chi_2 v_2\big)\\
&=-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx-\int_{\mathbb{R}^2}u\big(\chi_1\Delta v_1-\chi_2\Delta v_2\big)dx+\int_{\mathbb{R}^2}\log(1+u)\big(\chi_1\Delta v_1-\chi_2\Delta v_2\big)dx\\
&=-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+(\chi_1-\chi_2)\int_{\mathbb{R}^2}uwdx-\int_{\mathbb{R}^2}u(\chi_1\lambda_1 v_1-\chi_2\lambda_2 v_2)dx\\
&\quad-(\chi_1-\chi_2)\int_{\mathbb{R}^2}w\log(1+u)dx+\int_{\mathbb{R}}(\chi_1\lambda_1 v_1-\chi_2\lambda_2 v_2)\log(1+u)dx\\
&\le-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+(\chi_1-\chi_2)\int_{\mathbb{R}^2}uwdx+\varepsilon_1\int_{\mathbb{R}^2}u^2dx+2C(\varepsilon_1)\sum_{i=1}^{2}\int_{\mathbb{R}^2}\chi_i^2\lambda_i^2 v_i^2dx\\
&\le-\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+\frac{\delta}{2}\int_{\mathbb{R}^2}w^2dx+\left[\frac{(\chi_1-\chi_2)^2}{2\delta}+\varepsilon_1\right]\int_{\mathbb{R}^2}u^2dx+2C(\varepsilon_1)\sum_{i=1}^{2}\int_{\mathbb{R}^2}\chi_i^2\lambda_i^2 v_i^2dx,
\end{aligned}
\tag{11}
$$

where $0<\varepsilon_1<1$ is small enough to be determined.

On the other hand, we multiply the third equation of (2) with $2w$ and use the Young's inequality to obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2}w^2dx+2\delta\int_{\mathbb{R}^2}w^2dx&=2\int_{\mathbb{R}^2}uwdx\\
&\le\frac{\delta}{2}\int_{\mathbb{R}^2}w^2dx+\frac{2}{\delta}\int_{\mathbb{R}^2}u^2dx.
\end{aligned}
\tag{12}
$$

Adding the Equations (11) and (12), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^2}(1+u)\log(1+u)dx+\int_{\mathbb{R}^2}w^2dx\right)+\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+\delta\int_{\mathbb{R}^2}w^2dx\\
&\le\left[\frac{(\chi_1-\chi_2)^2+4}{2\delta}+\varepsilon_1\right]\int_{\mathbb{R}^2}u^2dx+2C(\varepsilon_1)\sum_{i=1}^{2}\int_{\mathbb{R}^2}\chi_i^2\lambda_i^2v_i^2dx.
\end{aligned}
\tag{13}
$$

Let $\delta_\star:=\min\Big\{\varepsilon_1,\delta\Big\}$. Thus, we add the two sides of the Equation (13) with $\delta_\star\cdot\int_{\mathbb{R}^2}(1+u)\log(1+u)dx$ and apply the inequality $(1+u)\log(1+u)\le u+u^2$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^2}(1+u)\log(1+u)dx+\int_{\mathbb{R}^2}w^2dx\right)+\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx+\delta_\star\cdot\left(\int_{\mathbb{R}^2}w^2dx+\int_{\mathbb{R}^2}(1+u)\log(1+u)dx\right)\\
&\le\left[\frac{(\chi_1-\chi_2)^2+4}{2\delta}+2\varepsilon_1\right]\int_{\mathbb{R}^2}u^2dx+2C(\varepsilon_1)\sum_{i=1}^{2}\int_{\mathbb{R}^2}\chi_i^2\lambda_i^2v_i^2dx+\varepsilon_1\|u_0\|_{L^1(\mathbb{R}^2)}.
\end{aligned}
\tag{14}
$$

Applying Lemmas 2 and 3 as $g=u(t)$ to (14) yields the following:

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^2}(1+u)\log(1+u)dx+\int_{\mathbb{R}^2}w^2dx\right)+\left\{1-\left[\frac{(\chi_1-\chi_2)^2+4}{2\delta}+2\varepsilon_1\right]\cdot\frac{1+\varepsilon_1}{4\pi}\|u_0\|_{L^1(\mathbb{R}^2)}\right\}\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}dx\\
&\quad+\delta_\star\cdot\left(\int_{\mathbb{R}^2}w^2dx+\int_{\mathbb{R}^2}(1+u)\log(1+u)dx\right)\\
&\leq 2C(\varepsilon_1)\sum_{i=1}^{2}\int_{\mathbb{R}^2}\chi_i^2\lambda_i^2v_i^2dx+\left[\frac{(\chi_1-\chi_2)^2+4}{\delta\varepsilon_1}+4\right]\|u_0\|_{L^1(\mathbb{R}^2)}\leq C(\|u_0\|_{L^1(\mathbb{R}^2)},\varepsilon_1,\chi_i,\lambda_i,\delta).
\end{aligned}\tag{15}$$

Thanks to $\|u_0\|_{L^1(\mathbb{R}^2)}<\frac{8\pi\delta}{4+(\chi_1-\chi_2)^2}$, we can take a small enough value of ε_1 such that

$$1-\left[\frac{(\chi_1-\chi_2)^2+4}{2\delta}+2\varepsilon_1\right]\cdot\frac{1+\varepsilon_1}{4\pi}\|u_0\|_{L^1(\mathbb{R}^2)}\geq 0.$$

Using Gronwall's inequality, we have

$$\|\big(1+u(t)\big)\log\big(1+u(t)\big)\|_1\leq\Big(\|\big(1+u_0\big)\log\big(1+u_0\big)\|_1+\|w_0\|^2_{L^2(\mathbb{R}^2)}\Big)e^{-\delta_\star t}+C(\|u_0\|_{L^1(\mathbb{R}^2)},\chi_i,\lambda_i,\delta),\ i=1,2.$$

Therefore, we complete the proof of Lemma 5. □

Next, we will give the gradient estimates of v_i.

Lemma 6. *Let $0<T\leq\infty$ and assume $\chi_1\geq\chi_2$ holds. We have the following estimate*

$$M:=\sum_{i=1}^{2}\left(\sup_{0<t<T}\|v_j(t)\|_{L^\infty(\mathbb{R}^2)}+\sup_{0<t<T}\|\nabla v_j(t)\|_{L^\infty(\mathbb{R}^2)}\right)<\infty.$$

Proof. Multiplying the first equation of (2) with u^{p-1} $(p>1)$ and integrating by parts, we can deduce that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\mathbb{R}^2)}+\frac{4(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_2^2\\
&=\chi_2\int_{\mathbb{R}^2}u^{p-1}(\nabla u\cdot\nabla v_2+u\Delta v_2)-\chi_1\int_{\mathbb{R}^2}u^{p-1}(\nabla u\cdot\nabla v_1+u\Delta v_1)\\
&=\left(1-\frac{1}{p}\right)\int_{\mathbb{R}}u^p(\chi_2\Delta v_2-\chi_1\Delta v_1)dx\\
&=\left(1-\frac{1}{p}\right)\int_{\mathbb{R}}u^p\big[\chi_2\lambda_2v_2-\chi_1\lambda_1v_1+(\chi_1-\chi_2)w\big]dx\\
&\leq\left(1-\frac{1}{p}\right)\chi_2\lambda_2\int_{\mathbb{R}^2}u^pv_2+\left(1-\frac{1}{p}\right)(\chi_1-\chi_2)\int_{\mathbb{R}^2}u^pwdx\\
&\leq\frac{\delta}{4}\int_{\mathbb{R}^2}w^{p+1}dx+\left[\frac{4^p(p-1)^p(\chi_1-\chi_2)^p}{p^p\delta^p}+\frac{p}{p+1}\right]\int_{\mathbb{R}^2}u^{p+1}dx+\frac{1}{p+1}\int_{\mathbb{R}^2}v_2^{p+1}dx.
\end{aligned}\tag{16}$$

We multiply the third equation of (2) by w^p and apply the Young's inequality yielding

$$\begin{aligned}
\frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{R}^2}w^{p+1}dx+\delta\int_{\mathbb{R}^2}w^{p+1}dx&=\int_{\mathbb{R}^2}uw^pdx\\
&\leq\frac{\delta}{4}\int_{\mathbb{R}^2}w^{p+1}dx+\frac{4^p}{\delta^p}\int_{\mathbb{R}^2}u^{p+1}dx.
\end{aligned}\tag{17}$$

Combining (16) and (17), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{p}\|u\|^p_{L^p(\mathbb{R}^2)}+\frac{1}{p+1}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)}\right)+\frac{4(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^2)}+\frac{\delta}{2}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)}\\&\le\left[\frac{4^p(p-1)^p(\chi_1-\chi_2)^p+4^pp^p}{p^p\delta^p}+\frac{p}{p+1}\right]\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^2)}+\frac{1}{p+1}\|v_2\|^{p+1}_{L^{p+1}(\mathbb{R}^2)}.\end{aligned}\tag{18}$$

Next, we take $p=2$ in (18) to obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{3}\|w\|^3_{L^3(\mathbb{R}^2)}\right)+\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\frac{\delta}{2}\|w\|^3_{L^3(\mathbb{R}^2)}\le\left(\frac{4(\chi_1-\chi_2)^2+16}{\delta^2}+\frac{2}{3}\right)\|u\|^3_{L^3(\mathbb{R}^2)}+\frac{1}{3}\|v_2\|^3_{L^3(\mathbb{R}^2)}.\tag{19}$$

Applying the estimate (10), we have

$$\|v_2\|_{L^3(\mathbb{R}^2)}\le\sqrt[3]{C_1}\|w(t)\|_{L^1(\mathbb{R}^2)}\le\sqrt[3]{C_1}\left(\|w_0\|_{L^1(\mathbb{R}^2)}+\frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)}\right),\tag{20}$$

where C_1 is a positive constant.

For the term $\|u\|_{L^3(\mathbb{R}^2)}$, we use the Lemma 4 to obtain

$$\begin{aligned}\|u\|^3_{L^3(\mathbb{R}^2)}&\le\varepsilon_2\|(1+u)\log(1+u)\|_{L^1(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+C(\varepsilon_2)\|u\|_{L^1(\mathbb{R}^2)}\\&\le\varepsilon_2K\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+C(\varepsilon_2)\|u_0\|_{L^1(\mathbb{R}^2)}\quad\text{for all } 0<\varepsilon_2<1.\end{aligned}\tag{21}$$

Thus, substituting (20) and (21) into (19) yields

$$\begin{aligned}&\frac{d}{dt}\left(3\|u\|^2_{L^2(\mathbb{R}^2)}+2\|w\|^3_{L^3(\mathbb{R}^2)}\right)+\left[6-\varepsilon_2K\left(4+\frac{24(\chi_1-\chi_2)^2+96}{\delta^2}\right)\right]\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+3\delta\|w\|^3_{L^3(\mathbb{R}^2)}\\&\le\left(4+\frac{24(\chi_1-\chi_2)^2+96}{\delta^2}\right)C(\varepsilon_2)\|u_0\|_{L^1(\mathbb{R}^2)}+C_1\left(\|u_0\|^3_{L^1(\mathbb{R}^2)}+\frac{1}{\delta^3}\|u_0\|^3_{L^1(\mathbb{R}^2)}\right).\end{aligned}\tag{22}$$

We can use the Gagliardo–Nirenberg inequality and Young's inequality to obtain

$$\|u\|^2_{L^2(\mathbb{R}^2)}\le C_{GN}\|\nabla u\|_{L^2(\mathbb{R}^2)}\|u_0\|_{L^1(\mathbb{R}^2)}\le\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\frac{C^2_{GN}}{4}\|u_0\|^2_{L^1(\mathbb{R}^2)}.\tag{23}$$

Taking a suitable value of ε_2 such that $6-\varepsilon_2K\left(4+\frac{24(\chi_1-\chi_2)^2+96}{\delta^2}\right)=\min\left\{\frac{9}{2}\delta,6-\varepsilon_2^*\right\}>0$, then substituting (23) into (22), we have

$$\frac{d}{dt}\left(3\|u\|^2_{L^2(\mathbb{R}^2)}+2\|w\|^3_{L^3(\mathbb{R}^2)}\right)+\min\left\{\frac{3}{2}\delta,2-\frac{\varepsilon_2^*}{3}\right\}\left(3\|u\|^2_{L^2(\mathbb{R}^2)}+2\|w\|^3_{L^3(\mathbb{R}^2)}\right)\le C(\varepsilon_2,\delta,\chi_1,\chi_2,C_{GN},\|u_0\|_{L^1(\mathbb{R}^2)},\|w_0\|_{L^1(\mathbb{R}^2)})\tag{24}$$

for all $3<\varepsilon_2^*<6$.

Applying Gronwall's inequality in (24), we show that

$$\|u\|_{L^2(\mathbb{R}^2)}+\|w\|_{L^3(\mathbb{R}^2)}\le C,$$

where $C>0$ is a constant depending on $\varepsilon_2,\delta,\chi_1,\chi_2,C_{GN},\|u_0\|_{L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)},\|w_0\|_{L^1(\mathbb{R}^2)\cap L^3(\mathbb{R}^2)}$.

Taking $p=4$ in (18), we can obtain

$$\frac{d}{dt}\left(\frac{1}{4}\|u\|^4_{L^4(\mathbb{R}^2)}+\frac{1}{5}\|w\|^5_{L^5(\mathbb{R}^2)}\right)+\frac{3}{4}\|\nabla u^2\|^2_{L^2(\mathbb{R}^2)}+\frac{\delta}{2}\|w\|^5_{L^5(\mathbb{R}^2)}\le\left[\frac{81(\chi_1-\chi_2)^4+256}{\delta^4}+\frac{4}{5}\right]\|u\|^5_{L^5(\mathbb{R}^2)}+\frac{1}{5}\|v_2\|^5_{L^5(\mathbb{R}^2)}.\tag{25}$$

To control $\|u\|_{L^5(\mathbb{R}^2)}$, we use the Gagliardo–Nirenberg inequality and Young's inequality to deduce

$$\begin{aligned}\|u\|^5_{L^5(\mathbb{R}^2)} = \|u^2\|^{\frac{5}{2}}_{L^{\frac{5}{2}}(\mathbb{R}^2)} &\le C_{GN}\|\nabla u^2\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}\|u^2\|_{L^1(\mathbb{R}^2)} \\ &\le \varepsilon_3\|\nabla u^2\|^2_{L^2(\mathbb{R}^2)} + C(\varepsilon_3, C_{GN})\|u\|^8_{L^2(\mathbb{R}^2)},\end{aligned} \tag{26}$$

where $0 < \varepsilon_3 < 1$ has yet to be determined. Using the estimate (10) again, there exist a constant $C_2 > 0$ such that

$$\|v_2\|_{L^5(\mathbb{R}^2)} \le \sqrt[5]{C_2}\Big(\|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)}\Big). \tag{27}$$

Taking the appropriate ε_3 such that $\frac{3}{4} - \Big[\frac{81(\chi_1-\chi_2)^4+256}{\delta^4} + \frac{4}{5}\Big]\varepsilon_3 = \min\Big\{\frac{5\delta}{8}, \frac{3}{4} - \varepsilon_3^*\Big\} > 0$, we can obtain

$$\frac{d}{dt}\Big(\frac{1}{4}\|u^2\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{5}\|w\|^5_{L^5(\mathbb{R}^2)}\Big) + 4\min\Big\{\frac{5\delta}{8}, \frac{3}{4} - \varepsilon_3^*\Big\}\Big(\frac{1}{4}\|\nabla u^2\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{5}\|w\|^5_{L^5(\mathbb{R}^2)}\Big) \le C_3 \tag{28}$$

for all $0 < \varepsilon_3^* < \frac{3}{4}$, where $C_3 > 0$ depends on $\|u_0\|_{L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)}, \|w_0\|_{L^1(\mathbb{R}^2)\cap L^3(\mathbb{R}^2)}$ and C_2. Similarly to (23), there is a constant $C_4 > 0$ such that

$$\|u^2\|^2_{L^2(\mathbb{R}^2)} \le \|\nabla u^2\|^2_{L^2(\mathbb{R}^2)} + C_4\|u_0\|^4_{L^2(\mathbb{R}^2)}. \tag{29}$$

Combining (28) with (29) and applying Gronwall's inequality yields

$$\|u\|_{L^4(\mathbb{R}^2)} + \|w\|_{L^5(\mathbb{R}^2)} \le C, \tag{30}$$

where $C > 0$ depends on $\varepsilon_3, \delta, \chi_1, \chi_2, C_{GN}, \|u_0\|_{L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)\cap L^4(\mathbb{R}^2)}, \|w_0\|_{L^1(\mathbb{R}^2)\cap L^3(\mathbb{R}^2)\cap L^5(\mathbb{R}^2)}$. Using (8) and (9) in Lemma 2 and (30), we complete the proof of Lemma 6. □

In what follows, we will give the boundedness of u, v_i and w.

Lemma 7. *Assume $\chi_1 \ge \chi_2$ holds. Then, for any $p \in [1, \infty]$, we have*

$$\|u\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2}\|v_i\|_{W^{2,p}(\mathbb{R}^2)} + \|w\|_{L^p(\mathbb{R}^2)} \le C, \quad i = 1, 2,$$

where the constant C depends on $\chi_i, \lambda_i, \|u_0\|_{L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}$ and $\|w_0\|_{L^\infty(\mathbb{R}^2)}$.

Proof. To obtain the L^∞ estimate of u, we first employ an energy inequality and Moser's iteration technique. Next, we utilize the ODE comparison principle and classical elliptic theory to derive the estimates of w and v_j.

Recalling (16) and (2), we integrate by parts for the right-hand side and use Young's inequality to obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\mathbb{R}^2)} + \frac{4(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_2^2 \\
&= \int_{\mathbb{R}^2} u^{p-1}\Big[-\nabla\cdot\big(\chi_1 u\nabla v_1\big) + \nabla\cdot(\chi_2 u \nabla v_2)\Big]dx \\
&= (p-1)\int_{\mathbb{R}^2} u^{p-1}(\chi_1\nabla v_1 - \chi_2\nabla v_2)\cdot\nabla u\, dx \\
&\le (p-1)(\chi_1+\chi_2)M\int_{\mathbb{R}^2} u^{p-1}|\nabla u| dx \\
&= 2(\chi_1+\chi_2)\frac{(p-1)M}{p}\int_{\mathbb{R}^2} u^{\frac{p}{2}}|\nabla u^{\frac{p}{2}}| dx \\
&\le 2(\chi_1+\chi_2)\frac{(p-1)M}{p}\|u^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}\|\nabla u^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{2(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_2^2 + 2(p-1)(\chi_1+\chi_2)^2M^2\|u\|^p_{L^p(\mathbb{R}^2)}.
\end{aligned}
\tag{31}
$$

That is,

$$
\frac{d}{dt}\|u\|^p_{L^p(\mathbb{R}^2)} + p(p-1)\|u\|^p_{L^p(\mathbb{R}^2)} + \frac{2(p-1)}{p}\|\nabla u^{\frac{p}{2}}\|_2^2 \le p(p-1)\Big[2(\chi_1+\chi_2)^2M^2+1\Big]\|u\|^p_{L^p(\mathbb{R}^2)}. \tag{32}
$$

On the other hand, we can use the Gagliardo–Nirenberg inequality to deduce

$$
\begin{aligned}
\|u\|^p_{L^p(\mathbb{R}^2)} = \|u^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^2)} &\le C_{GN}\|\nabla u^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}\|u^{\frac{p}{2}}\|_{L^1(\mathbb{R}^2)} \\
&\le \frac{2}{p^2\Big[2(\chi_1+\chi_2)^2M^2+1\Big]}\|\nabla u^{\frac{p}{2}}\|_2^2 + \frac{p^2\Big[2(\chi_1+\chi_2)^2M^2+1\Big]C_{GN}^2}{2}\|u^{\frac{p}{2}}\|^2_{L^1(\mathbb{R}^2)}.
\end{aligned}
\tag{33}
$$

Substituting (33) into (32) gives

$$
\frac{d}{dt}\|u\|^p_{L^p(\mathbb{R}^2)} + p(p-1)\|u\|^p_{L^p(\mathbb{R}^2)} \le \frac{p^3(p-1)\Big[2(\chi_1+\chi_2)^2M^2+1\Big]C_{GN}^2}{2}\|u\|^p_{L^{\frac{p}{2}}(\mathbb{R}^2)}.
$$

Using Gronwall's inequality, we have

$$
\|u\|^p_{L^p(\mathbb{R}^2)} \le e^{-p(p-1)t}\|u_0\|^p_{L^p(\mathbb{R}^2)} + \frac{p^2\Big[2(\chi_1+\chi_2)^2M^2+1\Big]C_{GN}^2}{2}\sup_{0\le t<\infty}\|u(t)\|^p_{L^{\frac{p}{2}}(\mathbb{R}^2)}.
$$

Let $B(p) = \max\Big\{\|u_0\|_{L^1(\mathbb{R}^2)}, \|u_0\|_{L^\infty(\mathbb{R}^2)}, \sup\limits_{0\le t<\infty}\|u(t)\|_{L^p(\mathbb{R}^2)}\Big\}$. This yields

$$
B(p) \le C_5^{\frac{1}{p}} p^{\frac{2}{p}} B(\frac{p}{2}) \quad \text{for all } p \ge 2,
$$

where $C_5 = 1 + \frac{\Big[2(\chi_1+\chi_2)^2M^2+1\Big]C_{GN}^2}{2}$ is a constant.

Taking $p = 2^j \ (j = 1, 2, \cdots)$ and applying the above iterative inequality, we see that

$$
\begin{aligned}
B(2^j) &\le C_5^{2^{-j}} 2^{j\cdot 2^{1-j}} B(2^{j-1}) \\
&\le C_5^{2^{-j}} 2^{j\cdot 2^{1-j}} \cdot C_5^{2^{-j+1}} 2^{(j-1)\cdot 2^{2-j}} B(2^{j-2}) \cdots \\
&\le C_5^{(2^{-j}+2^{-j+1}+\cdots+2^{-2}+2^{-1})} \times 2^{j\cdot 2^{1-j}+(j-1)2^{2-j}+\cdots+2\times 2^{-1}+1\times 2^0} B(1) \\
&\le 4C_5 B(1).
\end{aligned}
\tag{34}
$$

By virtue of (34) and the boundedness of $B(1)$, we have

$$\|u\|_{L^\infty(\mathbb{R}^2)} = \lim_{j\to+\infty} B(2^j) \le 4C_5 B(1) \le 4C_5 \max\left\{\|u_0\|_{L^1(\mathbb{R}^2)}, \|u_0\|_{L^\infty(\mathbb{R}^2)}\right\}. \tag{35}$$

Recalling (17) and applying interpolation inequality and Young's inequality, we further have

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} + \delta\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} uw^p dx \\ &\le \frac{\delta}{2}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} + \frac{2^p}{\delta^p}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} \\ &\le \frac{\delta}{2}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} + \frac{2^p}{\delta^p}\|u\|_{L^1(\mathbb{R}^2)}\|u\|^p_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

This means that

$$\frac{d}{dt}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} + \frac{\delta(p+1)}{2}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} \le \frac{(p+1)(2C_6)^p}{\delta^p}, \tag{36}$$

where C_6 is a constant independent of p. Applying Gronwall's inequality and the Lemma 2.1 in [29], we obtain

$$\|w\|_{L^{p+1}(\mathbb{R}^2)} \le \left(\|w_0\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} + \frac{2^{p+1}C_6^p}{\delta^{p+1}}\right)^{\frac{1}{p+1}} \le \|w_0\|_{L^{p+1}(\mathbb{R}^2)} + \frac{2}{\delta}C_6^{\frac{p}{p+1}}. \tag{37}$$

Letting $p \to +\infty$, we can obtain the boundedness of $\|w\|_{L^\infty(\mathbb{R}^2)}$. Finally, we use the classical elliptic estimate to obtain

$$\|D^2 v_i\|_{L^p(\mathbb{R}^2)} \le C_7\|w\|_{L^p(\mathbb{R}^2)}.$$

Applying the boundedness of w in (37), the proof is complete. □

Proof of Theorem 1. By virtue of Lemma 6 and Lemma 7 being complete, we complete the proof of the Theorem 1. □

4. Boundedness of Radial Solutions

In this section, we focus on the case of radial solutions. We assume that the non-negative initial data u_0, w_0 are radially symmetric with respect to the spatial variable x and satisfy (3). We redefine the function $\tilde{u}(t,s), \tilde{v}_j(t,s)$ $(j = 1,2)$ and $\tilde{w}(t,s)$ as

$$u(t,x) = \tilde{u}(t,s),\ \ v_j(t,x) = \tilde{v}_j(t,s),\ \ w(t,x) = \tilde{w}(t,s), \quad s = \pi|x|^2$$

and the initial data as $u_0(x) = \tilde{u}(s), w_0(x) = \tilde{w}(s)$. We denote the following:

$$U(t,s) = \int_0^s \tilde{u}(t,\sigma)d\sigma, \quad V_j(t,s) = \int_0^s \tilde{u}(t,\sigma)d\sigma, \quad W(t,s) = \int_0^s \tilde{w}(t,\sigma)d\sigma \quad (j = 1,2) \tag{38}$$

and $U_0(s) = \int_0^s \tilde{u}_0(\sigma)d\sigma$, $W_0(s) = \int_0^s \tilde{u}_0(\sigma)d\sigma$.

Similarly to Lemma 1, we have

$$U(t,\infty) = \int_0^\infty \tilde{u}(t,s)ds = 2\pi\int_0^\infty \tilde{u}(t,\pi r^2)rdr = \int_{\mathbb{R}^2} u(t,x)dx = \|u_0\|_{L^1(\mathbb{R}^2)} \tag{39}$$

and

$$W(t,\infty) = \int_0^\infty \tilde{w}(t,s)ds = \int_{\mathbb{R}^2} w(t,x)dx \le \|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)} \tag{40}$$

as well as

$$V_j(t,\infty) = \int_0^\infty \tilde{v}_j(t,s)ds = \int_{\mathbb{R}^2} v_j(t,x)dx \le \frac{1}{\lambda_j}\|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\lambda_j\delta}\|u_0\|_{L^1(\mathbb{R}^2)} \quad (j=1,2). \tag{41}$$

Lemma 8. *If the spatial variable x is a radial region and U is defined by (38), it holds that*

$$\partial_t U \le 4\pi s\partial_s^2 U + \Big(\chi_2\lambda_2 V_2 + (\chi_1-\chi_2)M_0\Big)\partial_s U. \tag{42}$$

Proof. By straightforward calculations, we have

$$\partial_{x_j} u = \partial_s\tilde{u}\,\partial_{x_j}s = 2\pi x_j\partial_s\tilde{u}$$

and

$$\partial^2_{x_ix_j}u = \partial_{x_j}(2\pi x_i\partial_s\tilde{u}) = 2\pi\delta_{ij}\partial_s\tilde{u} + 4\pi^2x_ix_j\partial_s^2\tilde{u}.$$

Therefore, we obtain

$$\Delta u = \sum_{j=1}^{2}\partial^2_{x_j}u = 4\pi\partial_s\tilde{u} + 4\pi^2|x|^2\partial_s^2\tilde{u} = 4\pi\partial_s\tilde{u} + 4\pi s\partial_s^2\tilde{u} = 4\pi\partial_s(s\partial_s\tilde{u}) \tag{43}$$

and

$$\begin{aligned}\nabla\cdot(u\nabla v_j) &= \sum_{i=1}^{2}\partial_i(u\partial_i v_j) = \sum_{i=1}^{2}\partial_i u\partial_i v_j + u\sum_{i=1}^{2}\partial_i^2 v_j\\ &= 4\pi^2|x|^2\partial_s\tilde{u}\partial_s\tilde{v}_j + 4\pi\tilde{u}\partial_s(s\partial_s\tilde{v}_j)\\ &= 4\pi s\partial_s\tilde{u}\partial_s\tilde{v}_j + 4\pi\tilde{u}\partial_s(s\partial_s\tilde{v}_j)\\ &= 4\pi\partial_s(s\tilde{u}\partial_s\tilde{v}_j)\end{aligned} \tag{44}$$

as well as

$$\begin{aligned}\partial_t\tilde{u} = \partial_t u &= \Delta u - \nabla\cdot(\chi_1 u\nabla v_1) + \nabla\cdot(\chi_2 u\nabla v_2)\\ &= 4\pi\partial_s(s\partial_s\tilde{u}) - 4\pi\chi_1\partial_s(s\tilde{u}\partial_s\tilde{v}_1) + 4\pi\chi_2\partial_s(s\tilde{u}\partial_s\tilde{v}_2)\\ &= 4\pi\partial_s(s\partial_s\tilde{u}) - 4\pi\partial_s\Big(s\tilde{u}\partial_s(\chi_1\tilde{v}_1 - \chi_2\tilde{v}_2)\Big).\end{aligned} \tag{45}$$

Integrating both sides of (2) from 0 to s and combining (43)–(45) yields

$$\begin{aligned}\partial_t U &= 4\pi(s\partial_s\tilde{u}) - 4\pi\Big(s\tilde{u}\partial_s(\chi_1\tilde{v}_1 - \chi_2\tilde{v}_2)\Big)\\ &= 4\pi s\partial_s^2 U - 4\pi s\partial_s U\partial_s(\chi_1\tilde{v}_1 - \chi_2\tilde{v}_2)\\ &= 4\pi s\partial_s^2 U - \partial_s U\Big(4\pi\chi_1 s\partial_s\tilde{v}_1 - 4\pi\chi_2 s\partial_s\tilde{v}_2\Big).\end{aligned} \tag{46}$$

Applying the second equations of (2) and (43), we can deduce

$$\Delta v_j = 4\pi\partial_s(s\partial_s\tilde{v}_j) = \lambda_j v_j - w \quad (j=1,2). \tag{47}$$

Similarly to (46), integrating both sides of (47), we see that

$$4\pi s\partial_s\tilde{v}_j = \lambda_j V_j - W. \tag{48}$$

Substituting (48) into (46), we have

$$\begin{aligned}\partial_t U &= 4\pi s\partial_s^2 U - \partial_s U\Big[\chi_1(\lambda_1 V_1 - W) - \chi_2(\lambda_2 V_2 - W)\Big]\\ &= 4\pi s\partial_s^2 U + (\chi_1-\chi_2)W\partial_s U - (\chi_1\lambda_1 V_1 - \chi_2\lambda_2 V_2)\partial_s U.\end{aligned} \tag{49}$$

Using the third equation of (2) again, we can solve the ODE to obtain

$$w(t,x) = w_0(x)e^{-\delta t} + \int_0^t e^{\delta(\tau-t)} u(\tau,x)d\tau. \tag{50}$$

So, we can integrate the both sides of (50) to obtain

$$\begin{aligned} W(t,s) &= e^{-\delta t} \cdot \int_0^s \tilde{w}_0(\sigma)d\sigma + \int_0^t e^{\delta(\tau-t)} \int_0^s \tilde{u}(\tau,\sigma)d\sigma d\tau \\ &= e^{-\delta t} W_0(s) + \int_0^t e^{\delta(\tau-t)} U(\tau,s)d\tau. \end{aligned} \tag{51}$$

Combining (49) and (51) and using the first mean-value theorem and $\partial_s U \geq 0$, we have

$$\begin{aligned} \partial_t U &= 4\pi s \partial_s^2 U + (\chi_1 - \chi_2) W \partial_s U - (\chi_1 \lambda_1 V_1 - \chi_2 \lambda_2 V_2)\partial_s U \\ &= 4\pi s \partial_s^2 U + (\chi_1 - \chi_2)\Big(e^{-\delta t} W_0(s) + \int_0^t e^{\delta(\tau-t)} U(\tau,s)d\tau\Big)\partial_s U - (\chi_1 \lambda_1 V_1 - \chi_2 \lambda_2 V_2)\partial_s U \\ &= 4\pi s \partial_s^2 U + (\chi_1 - \chi_2)\Big(e^{-\delta t} W_0(s) + \frac{1}{\delta}\big(1 - e^{-\delta t}\big)U(\theta t,s)\Big)\partial_s U - (\chi_1 \lambda_1 V_1 - \chi_2 \lambda_2 V_2)\partial_s U \\ &= 4\pi s \partial_s^2 U + \frac{\chi_1 - \chi_2}{\delta}\big(1 - e^{-\alpha t}\big)U(\theta t,s)\partial_s U - \Big(\chi_1 \lambda_1 V_1 - \chi_2 \lambda_2 V_2 - (\chi_1 - \chi_2)e^{-\delta t} W_0(s)\Big)\partial_s U \\ &\leq 4\pi s \partial_s^2 U + \left[(\chi_1 - \chi_2)\Big(\frac{\|u_0\|_{L^1(\mathbb{R}^2)}}{\delta} + \|w_0\|_{L^1(\mathbb{R}^2)}\Big) + \chi_2 \lambda_2 V_2\right]\partial_s U. \end{aligned}$$

It means that

$$\partial_t U \leq 4\pi s \partial_s^2 U + \Big(\chi_2 \lambda_2 V_2 + (\chi_1 - \chi_2)M_0\Big)\partial_s U. \tag{52}$$

Therefore, we complete the proof of Lemma 8. □

Recalling (31), regarding the construction of iterative techniques, we need to $\sup_{t>0} \|\nabla v_j\|_{L^\infty} < \infty$ $(j = 1,2)$. By virtue of (48) and $s = \pi|x|^2$, we have

$$|\partial_i v_j(t,x)| = \Big|\frac{\partial s}{\partial x} \cdot \partial_s v_j(t,x)\Big| = 2\pi|x||\partial_s \tilde{v}_j(t,s)| = \frac{1}{2\sqrt{\pi s}}|\lambda_j V_j(t,s) - W(t,s)| \ \ (j = 1,2). \tag{53}$$

Lemma 9. *Suppose that the spatial variable is a radial region and $M_0 \leq \frac{2\pi}{\chi_1 - \chi_2}$ holds. Then, we have*

$$\sum_{i=1}^{2} \sup_{t>0} \|\nabla v_j\|_{L^\infty} < \infty.$$

Proof. First, for the term $V_j(t,s)$, applying the Hölder's inequality, the estimate of (10) and (41), we deduce that

$$\begin{aligned} 0 \leq V_j(t,s) &= \int_0^s \tilde{v}_j(t,\sigma)d\sigma \leq s^{\frac{1}{2}}\Big(\int_0^\infty v_j^2(t,\sigma)d\sigma\Big)^{\frac{1}{2}} \\ &\leq s^{\frac{1}{2}} C(2)\|w(t)\|_{L^1(\mathbb{R}^2)} \leq C(2)s^{\frac{1}{2}}\Big(\|w_0\|_{L^1(\mathbb{R}^2)} + \frac{1}{\delta}\|u_0\|_{L^1(\mathbb{R}^2)}\Big) \\ &= C(2)M_0\sqrt{s} \quad (j = 1,2). \end{aligned} \tag{54}$$

For convenience, taking the operator $\mathcal{N}g := 4\pi s\partial_s^2 g + \left[\left(C(2)\sqrt{s} + \chi_1 - \chi_2\right)M_0\right]\partial_s g$, we can obtain the equivalent form

$$\begin{cases} \partial_t U \le \mathcal{N}U, & t > 0, s > 0, \\ U(t,0) = 0, \ \ U(t,+\infty) = \|u_0\|_{L^1(\mathbb{R}^2)}, & t > 0, \\ U(0,s) = U_0(s), & s \ge 0. \end{cases} \tag{55}$$

Next, we define a comparison function G(s) as $G(s) = Rs^{1-\frac{(\chi_1-\chi_2)M_0}{4\pi}}e^{-\alpha\sqrt{s}}$, where $\alpha := \frac{C(2)M_0}{2\pi}$ and R are two positive constants. Let $\beta := 1 - \frac{(\chi_1-\chi_2)M_0}{4\pi} \ge \frac{1}{2}$. That is,

$$G(s) = Rs^{\beta}e^{-\alpha\sqrt{s}} \quad \text{for } \ \alpha > 0,\, \beta \ge \frac{1}{2}. \tag{56}$$

We can take a $R > 0$ value that is suitably large such that

$$U_0(s) \le \|u_0\|_{L^1(\mathbb{R}^2)} < Rs_0^{1-\frac{(\chi_1-\chi_2)M_0}{4\pi}}e^{-\alpha\sqrt{s_0}} \quad \text{for } \ 0 < s \le s_0. \tag{57}$$

Through a direct calculation, we can obtain

$$\mathcal{N}G(s) = 4\pi s\frac{d^2G(s)}{ds^2} + \left[\left(C(2)\sqrt{s} + \chi_1 - \chi_2\right)M_0\right]\frac{dG(s)}{ds} = 0. \tag{58}$$

Therefore, we have the following gradient flow

$$\begin{cases} \partial_t U \le \mathcal{N}U, \ \ \mathcal{N}G = 0, & 0 < t < \infty, 0 < s < s_0, \\ U(t,0) = G(0) = 0, U(t,s_0) \le \|u_0\|_{L^1(\mathbb{R}^2)} < G(s_0) & 0 \le t < \infty, \\ U(0,s) = U_0(s) < G(s), & 0 \le s \le s_0. \end{cases} \tag{59}$$

Applying the comparison principle, we have

$$U(t,s) \le G(s) \le Rs^{\beta}e^{-\alpha\sqrt{s}}.$$

Thus, applying (51), we see that

$$\begin{aligned} W(t,s) &= e^{-\delta t}\int_0^s w_0(\sigma)d\sigma + \int_0^t e^{\tau-t}U(\tau,s)d\tau \\ &\le \|w_0(x)\|_{L^2(\mathbb{R}^2)}\sqrt{s}e^{-\delta t} + Rs^{\beta}e^{-\alpha\sqrt{s}} \\ &\le \|w_0(x)\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}\|w_0(x)\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)}\sqrt{s} + Rs^{\beta}e^{-\alpha\sqrt{s}}. \end{aligned} \tag{60}$$

Noticing (53) and applying (54) and (60), we complete the proof of Lemma 9. □

Proof of Theorem 2. Thanks to the results obtained from Lemmas 8 and 9, and by employing the iterative technique described in Lemma 7, we complete the proof of Theorem 2. □

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