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Article

On Inner Derivations of Leibniz Algebras

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Abstract: Leibniz algebras are generalizations of Lie algebras. Similar to Lie algebras, inner derivations play a crucial role in characterizing complete Leibniz algebras. In this work, we demonstrate that the algebra of inner derivations of a Leibniz algebra can be decomposed into the sum of the algebra of left multiplications and a certain ideal. Furthermore, we show that the quotient of the algebra of derivations of the Leibniz algebra by this ideal yields a complete Lie algebra. Our results independently establish that any derivation of a semisimple Leibniz algebra can be expressed as a combination of three derivations. Additionally, we compare the properties of the algebra of inner derivations of Leibniz algebras with the algebra of central derivations.

Keywords: Leibniz algebra; Lie algebra; derivation; inner derivation; central derivation; completeness; semisimple

MSC: 17A32; 17A60

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1. Introduction

Leibniz algebras, which generalize Lie algebras, have been a subject of interest since their introduction in 1965 by Bloh [1] and further development by Loday [2] in 1993. These algebras are vector spaces equipped with a binary operation which has the property of being a derivation for the algebra itself. Since Leibniz algebras extend Lie algebras, many research studies have focused on extending the properties of Lie algebras to Leibniz algebras. Similar to Lie algebras, the study of inner derivations is fundamental in understanding the structure of Leibniz algebras and their properties. Ancochea and Campoamor introduced the concept of inner derivations in 2013 [3], defining them as derivations of a Leibniz algebra $\mathbf{A}$ that can be expressed as L_a for some $a \in \mathbf{A}$, where $L_a(x) = [a, x]$ for all $x \in \mathbf{A}$. However, as noted in [4], there exists a simple Leibniz algebra containing an outer derivation based on this definition. Subsequently, Kristen, Misra, and Stitzinger, in 2020 [4], defined a derivation d of a Leibniz algebra $\mathbf{A}$ as inner if $\operatorname{im}(d - L_a) \subseteq \operatorname{Leib}(\mathbf{A})$ for some $a \in \mathbf{A}$, where $\operatorname{Leib}(\mathbf{A})$ is the Leibniz kernel of $\mathbf{A}$. They showed that under this definition, the semisimple Leibniz algebra does not contain an outer derivation resulting in its completeness, the same property as for the semisimple Lie algebra [5]. In our work, we aim to deepen the understanding of derivations in Leibniz algebras by following the definition of inner derivations as defined in [4]. Let I be the set of all derivations of a Leibniz algebra $\mathbf{A}$ whose image is a subset of $\operatorname{Leib}(\mathbf{A})$. We show that the algebra of inner derivations of a Leibniz algebra can be decomposed into the sum of the algebra of left multiplications and the ideal I. By using this result, we independently prove (see [6]) that any derivation of a semisimple Leibniz algebra can be written as a combination of three derivations.

A Lie algebra is said to be complete [7] if all of its derivations are inner and it has trivial center. A Leibniz algebra $\mathbf{A}$ is said to be complete [4] if all of its derivations are inner and the center of $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$, the liezation of $\mathbf{A}$, is trivial. In [5], Meng showed that the Lie algebra of derivations of any complete Lie algebra is complete. However, in [8], Kongsomprach et al. showed that this result does not hold for complete Leibniz algebras. We focus on a Leibniz algebra with complete liezation and prove that the quotient of the Lie algebra of derivations of these Leibniz algebras by the ideal I is complete, and this quotient algebra is isomorphic to the Lie algebra of derivations of the liezation. The definition of central derivations of Leibniz algebras is the same as that of Lie algebras. In [9], Tôgô studied the properties of inner derivations of Lie algebras by comparing them with the set of central derivations. In Section 4, we investigate some analogues of those properties for Leibniz algebras. Throughout this paper, all algebras are assumed to be finite dimensional over an algebraically closed field $\mathbb{F}$ with characteristic zero.

2. Preliminaries

Following Barnes [10], in this paper, Leibniz algebras always refer to left Leibniz algebras.

A *(left) Leibniz algebra* [11] $\mathbf{A}$ is a vector space over $\mathbb{F}$ with a bilinear map $[\ ,\ ] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ satisfying the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all $a, b, c \in \mathbf{A}$.

A Leibniz algebra $\mathbf{A}$ is called *abelian* if $[\mathbf{A}, \mathbf{A}] = \{0\}$. A subspace I of a Leibniz algebra $\mathbf{A}$ is said to be a *subalgebra* if $[I, I] \subseteq I$, and a *left* (resp. *right*) *ideal* of $\mathbf{A}$ if $[\mathbf{A}, I] \subseteq I$ (resp. $[I, \mathbf{A}] \subseteq I$). If I is both a left ideal and a right ideal, then I is called an *ideal* of $\mathbf{A}$. A Leibniz algebra $\mathbf{A}$ has an abelian ideal $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{[x, x] \mid x \in \mathbf{A}\}$ [11] called the *Leibniz kernel* of $\mathbf{A}$. The ideal $\mathrm{Leib}(\mathbf{A}) = \{0\}$ if and only if $\mathbf{A}$ is a Lie algebra. For any ideal I of $\mathbf{A}$, we define the quotient Leibniz algebra in the usual way. In fact, $\mathrm{Leib}(\mathbf{A})$ is the minimal ideal such that $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a Lie algebra [11]. For a Leibniz algebra $\mathbf{A}$, we define the ideals $\mathbf{A}^{(1)} = \mathbf{A} = \mathbf{A}^1, \mathbf{A}^{(i)} = [\mathbf{A}^{(i-1)}, \mathbf{A}^{(i-1)}]$, and $\mathbf{A}^i = [\mathbf{A}, \mathbf{A}^{i-1}]$ for $i \in \mathbb{Z}_{\geq 2}$. The Leibniz algebra is said to be *solvable* (resp. *nilpotent*) if $\mathbf{A}^{(m)} = \{0\}$ (resp. $\mathbf{A}^m = \{0\}$) for some positive integer m. The *maximal solvable* (resp. *nilpotent*) ideal of $\mathbf{A}$ is called the *radical* (resp. *nilradical*), denoted by $\mathrm{rad}(\mathbf{A})$ (resp. $\mathrm{nilrad}(\mathbf{A})$). A Leibniz algebra $\mathbf{A}$ is called *simple* if $[\mathbf{A}, \mathbf{A}] \neq \mathrm{Leib}(\mathbf{A})$ and its ideals are only $\{0\}$, $\mathrm{Leib}(\mathbf{A})$, and $\mathbf{A}$. A Leibniz algebra $\mathbf{A}$ is *semisimple* if $\mathrm{rad}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$. We recall an analog of Levi's theorem for Leibniz algebras which will be used in this paper.

Theorem 1 ([12])**.** *Let $\mathbf{A}$ be a Leibniz algebra. Then there exists a subalgebra S (which is a semisimple Lie algebra) of $\mathbf{A}$ such that $\mathbf{A} = S + \mathrm{rad}(\mathbf{A})$ and $S \cap \mathrm{rad}(\mathbf{A}) = \{0\}$.*

The *left center* of $\mathbf{A}$ is defined by $Z^l(\mathbf{A}) = \{x \in \mathbf{A} \mid [x, a] = 0 \text{ for all } a \in \mathbf{A}\}$, and the *right center* of $\mathbf{A}$ is defined by $Z^r(\mathbf{A}) = \{x \in \mathbf{A} \mid [a, x] = 0 \text{ for all } a \in \mathbf{A}\}$. The *center* of $\mathbf{A}$ is $Z(\mathbf{A}) = Z^l(\mathbf{A}) \cap Z^r(\mathbf{A})$. It is easy to see that the center $Z(\mathbf{A})$ and the left center $Z^l(\mathbf{A})$ are ideals of $\mathbf{A}$, but the right center $Z^r(\mathbf{A})$ does not necessarily have to be an ideal of $\mathbf{A}$. A linear map $d : \mathbf{A} \to \mathbf{A}$ is called a *derivation* if $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in \mathbf{A}$. Let $\mathrm{Der}(\mathbf{A})$ be the Lie algebra of all derivations of $\mathbf{A}$ under the commutator bracket $[d_1, d_2] := d_1 d_2 - d_2 d_1$ for all $d_1, d_2 \in \mathrm{Der}(\mathbf{A})$. For $a \in \mathbf{A}$, the *left multiplication operator* $L_a : \mathbf{A} \to \mathbf{A}$ is defined by $L_a(x) = [a, x]$ for all $x \in \mathbf{A}$. Clearly, $L_a \in \mathrm{Der}(\mathbf{A})$ for all $a \in \mathbf{A}$.

3. On Inner Derivations

Let $\mathbf{A}$ be a Leibniz algebra. An ideal I of $\mathbf{A}$ is a *characteristic ideal* if $d(I) \subseteq I$ for all $d \in \mathrm{Der}(\mathbf{A})$. It is known that $\mathrm{Leib}(\mathbf{A})$ is a characteristic ideal of $\mathbf{A}$ (e.g., see [4]). Let $I_{\mathbf{A}} = \{x \in \mathbf{A} \mid \mathrm{im}(L_x) \subseteq \mathrm{Leib}(\mathbf{A})\}$. It is clear that $\mathrm{Leib}(\mathbf{A}) \subseteq I_{\mathbf{A}}$. The followings are easy but important observations.

Proposition 1. *$I_{\mathbf{A}}$ is a characteristic ideal of $\mathbf{A}$.*

Proof. To show that $I_{\mathbf{A}}$ is an ideal of $\mathbf{A}$, let $x \in I_{\mathbf{A}}$ and $a \in \mathbf{A}$. Then for all $y \in \mathbf{A}$, $L_{[x,a]}(y) = [[x,a],y] \in \mathrm{Leib}(\mathbf{A})$ and $L_{[a,x]}(y) = [[a,x],y] \in \mathrm{Leib}(\mathbf{A})$, hence, $[x,a],[a,x] \in I_{\mathbf{A}}$. To show that $I_{\mathbf{A}}$ is a characteristic ideal, let $x \in I_{\mathbf{A}}$ and $d \in \mathrm{Der}(\mathbf{A})$. Then for all $y \in \mathbf{A}$, $L_{d(x)}(y) = [d(x),y] = d([x,y]) - [x,d(y)] = d(L_x(y)) - L_x(d(y)) \in \mathrm{Leib}(\mathbf{A})$, and so, $d(x) \in I_{\mathbf{A}}$. This proves that $I_{\mathbf{A}}$ is a characteristic ideal of $\mathbf{A}$. □

Proposition 2. $Z^l(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \cong I_{\mathbf{A}}/\mathrm{Leib}(\mathbf{A})$.

Proof. Clearly, $\mathrm{Leib}(\mathbf{A})$ is an ideal of $I_{\mathbf{A}}$. Then $Z^l(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{x + \mathrm{Leib}(\mathbf{A}) \mid [x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})] = \mathrm{Leib}(\mathbf{A}) \text{ for all } y \in \mathbf{A}\} = \{x + \mathrm{Leib}(\mathbf{A}) \mid [x,y] \in \mathrm{Leib}(\mathbf{A}) \text{ for all } y \in \mathbf{A}\}$. According to the trivial isomorphism φ defined by $\varphi(x + \mathrm{Leib}(\mathbf{A})) = x + \mathrm{Leib}(\mathbf{A})$ for all $x + \mathrm{Leib}(\mathbf{A}) \in Z^l(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$, we have $Z^l(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \cong I_{\mathbf{A}}/\mathrm{Leib}(\mathbf{A})$. □

Example 1. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{w,x,y,z\}$ *with non-zero multiplications defined by* $[w,w] = z, [w,x] = y, [x,w] = -y$*, and* $[x,x] = z$*. We determine that* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$ *and* $I_{\mathbf{A}} = \mathrm{span}\{y,z\}$*. Thus,* $I_{\mathbf{A}}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{y + \mathrm{Leib}(\mathbf{A})\} = Z^l(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

We denote using $L(\mathbf{A})$ the vector space of left multiplication operators $\{L_a \mid a \in \mathbf{A}\}$. It is known that $L(\mathbf{A})$ forms a Lie algebra under the commutator bracket. The following result is easily derived.

Theorem 2. $\mathbf{A}/Z^l(\mathbf{A}) \cong L(\mathbf{A})$.

Proof. Define $\varphi : \mathbf{A} \to L(\mathbf{A})$ using $\varphi(x) = L_x$ for all $x \in \mathbf{A}$. Then for any $x,y,z \in \mathbf{A}$, we have $\varphi([x,y])(z) = L_{[x,y]}(z) = [[x,y],z]$ and $[\varphi(x),\varphi(y)](z) = [L_x, L_y](z) = L_x L_y(z) - L_y L_x(z) = [x,[y,z]] - [y,[x,z]] = [[x,y],z] + [y,[x,z]] - [y,[x,z]] = [[x,y],z]$. Therefore, $\varphi([x,y]) = [\varphi(x),\varphi(y)]$. Clearly, φ is onto and $\ker(\varphi) = \{x \in \mathbf{A} \mid L_x = 0\} = \{x \in \mathbf{A} \mid [x,y] = 0 \text{ for all } y \in \mathbf{A}\} = Z^l(\mathbf{A})$. Hence, $\mathbf{A}/Z^l(\mathbf{A}) \cong L(\mathbf{A})$. □

The following is immediately obtained from Proposition 2 and Theorem 2.

Corollary 1. $\mathbf{A}/I_{\mathbf{A}} \cong L(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Remark 1. *For a Lie algebra* $\mathbf{L}$*, a derivation* $d : \mathbf{L} \to \mathbf{L}$ *is inner if there exists* $x \in \mathbf{L}$ *such that* $d = \mathrm{ad}_x$*, where* $\mathrm{ad}_x : \mathbf{L} \to \mathbf{L}$ *is defined by* $\mathrm{ad}_x(y) = [x,y]$ *for all* $y \in \mathbf{L}$*. Several authors have adopted the same definition for inner derivations of Leibniz algebras. It is known that all derivations of simple Lie algebras are inner. However, as shown in [4] with this definition, there is a simple Leibniz algebra that contains an outer derivation. Moreover, Tôgô [13] proved that a derivation d of a Lie algebra* $\mathbf{L}$ *is inner if and only if there exists* $x \in \mathbf{L}$ *such that* $d|_{\mathrm{rad}(\mathbf{L})} = \mathrm{ad}_x|_{\mathrm{rad}(\mathbf{L})}$*. Hence, we use the analogous definition to this well-known result for the inner derivations in Lie algebras for Leibniz algebras given in [4].*

Definition 1 ([4])**.** *Let* $\mathbf{A}$ *be a Leibniz algebra. A derivation* $d : \mathbf{A} \to \mathbf{A}$ *is said to be inner if there exists* $x \in \mathbf{A}$ *such that* $\mathrm{im}(d - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$.

We denote using $\mathrm{IDer}(\mathbf{A})$ the set of all inner derivations of a Leibniz algebra $\mathbf{A}$ and $I = \{d \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$. Clearly, $L(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A}) \subseteq \mathrm{Der}(\mathbf{A})$ and $\mathrm{IDer}(\mathbf{A})$ is a subspace of $\mathrm{Der}(\mathbf{A})$. It is known that $L(\mathbf{A})$ is an ideal of $\mathrm{Der}(\mathbf{A})$. Then it is also an ideal of $\mathrm{IDer}(\mathbf{A})$. Since $\mathrm{Leib}(\mathbf{A})$ is a characteristic ideal of $\mathbf{A}$, I is an ideal of $\mathrm{Der}(\mathbf{A})$, and hence, an ideal of $\mathrm{IDer}(\mathbf{A})$.

Theorem 3. *Let* $\mathbf{A}$ *be a Leibniz algebra. Then* $\mathrm{IDer}(\mathbf{A})$ *is an ideal of* $\mathrm{Der}(\mathbf{A})$ *and* $\mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$*. Moreover, if* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ *is trivial, then* $L(\mathbf{A}) \cap I = \{0\}$.

Proof. Let $d \in \mathrm{IDer}(\mathbf{A})$. Then there exists $x \in \mathbf{A}$ such that $\mathrm{im}(d - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$. Then $d - L_x \in I$, and hence, $d \in L(\mathbf{A}) + I$. This implies that $\mathrm{IDer}(\mathbf{A}) \subseteq L(\mathbf{A}) + I$. Since the reverse inclusion is clear, we have $\mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$. Consequently, $\mathrm{IDer}(\mathbf{A})$ is an ideal of $\mathrm{Der}(\mathbf{A})$. Note that $L(\mathrm{Leib}(\mathbf{A})) = \{L_a \mid a \in \mathrm{Leib}(\mathbf{A})\} = \{0\}$ because $\mathrm{Leib}(\mathbf{A}) \subseteq Z^l(\mathbf{A})$. Suppose that $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ is trivial. Let $L_x \in L(\mathbf{A}) \cap I$. Then $[x, a] \in \mathrm{Leib}(\mathbf{A})$ for all $a \in \mathbf{A}$. Thus, $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$, which implies that $x \in \mathrm{Leib}(\mathbf{A})$. Therefore, $L(\mathbf{A}) \cap I \subseteq L(\mathrm{Leib}(\mathbf{A})) = \{0\}$. □

Example 2. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{w, x, y, z\}$ *with non-zero multiplications defined by* $[w, w] = y$ *and* $[x, w] = z$. *Clearly,* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{y, z\}$. *Through direct calculation, we determine that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$, *where*

$$\begin{array}{llll}
d_1(w) = w, & d_1(x) = 0, & d_1(y) = 2y, & d_1(z) = z, \\
d_2(w) = x, & d_2(x) = 0, & d_2(y) = z, & d_2(z) = 0, \\
d_3(w) = y, & d_3(x) = 0, & d_3(y) = 0, & d_3(z) = 0, \\
d_4(w) = z, & d_4(x) = 0, & d_4(y) = 0, & d_4(z) = 0, \\
d_5(w) = 0, & d_5(x) = x, & d_5(y) = 0, & d_5(z) = z, \\
d_6(w) = 0, & d_6(x) = y, & d_6(y) = 0, & d_6(z) = 0, \\
d_7(w) = 0, & d_7(x) = z, & d_7(y) = 0, & d_7(z) = 0.
\end{array}$$

Then we have $L(\mathbf{A}) = \mathrm{span}\{d_3 = L_w, d_4 = L_x\}$ *and* $I = \mathrm{span}\{d_3, d_4, d_6, d_7\}$. *Hence,* $\mathrm{IDer}(\mathbf{A}) = \mathrm{span}\{d_3, d_4, d_6, d_7\} = L(\mathbf{A}) + I$. *Note that* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \mathrm{span}\{w + \mathrm{Leib}(\mathbf{A}), x + \mathrm{Leib}(\mathbf{A})\}$, *and* $L(\mathbf{A}) \cap I = \mathrm{span}\{d_3, d_4\}$ *in this case.*

Example 3. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{x, y, z\}$ *with non-zero multiplications defined by* $[x, y] = y, [y, x] = -y$ *and* $[x, x] = z$. *In this case, we have* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\} = Z(\mathbf{A})$, *and* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ *is trivial. Through direct calculation, we determine that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\} = \mathrm{IDer}(\mathbf{A})$, *where*

$$\begin{array}{lll}
d_1(x) = y, & d_1(y) = 0, & d_1(z) = 0, \\
d_2(x) = z, & d_2(y) = 0, & d_2(z) = 0, \\
d_3(x) = 0, & d_3(y) = y, & d_3(z) = 0.
\end{array}$$

Then we have $L(\mathbf{A}) = \mathrm{span}\{d_1 = L_{-y}, d_2 + d_3 = L_x\}$ *and* $I = \mathrm{span}\{d_2\}$. *Hence,* $\mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$ *and* $L(\mathbf{A}) \cap I = \{0\}$.

Example 4. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{x, y, z\}$ *with non-zero multiplications defined by* $[x, y] = y, [y, x] = -y$ *and* $[x, z] = z$. *Clearly,* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}, Z(\mathbf{A}) = \{0\}$, *and* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ *is trivial. Through direct calculation, we determine that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\} = \mathrm{IDer}(\mathbf{A})$, *where*

$$\begin{array}{lll}
d_1(x) = y, & d_1(y) = 0, & d_1(z) = 0, \\
d_2(x) = 0, & d_2(y) = 0, & d_2(z) = z, \\
d_3(x) = 0, & d_3(y) = y, & d_3(z) = 0.
\end{array}$$

Then we have $L(\mathbf{A}) = \mathrm{span}\{d_1 = L_{-y}, d_2 + d_3 = L_x\}$ *and* $I = \mathrm{span}\{d_2\}$. *Hence,* $\mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$ *and* $L(\mathbf{A}) \cap I = \{0\}$ *in this case.*

Definition 2 ([4], Definition 3.1)**.** *A Leibniz algebra* $\mathbf{A}$ *is said to be complete if*

(i) $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$, *and*
(ii) all derivations of $\mathbf{A}$ *are inner.*

In ([6], Theorem 3.2), it is proven that any derivation of a simple Leibniz algebra can be represented as a combination of three derivations. Here, we present a distinct approach to this proof specifically tailored to semisimple Leibniz algebras.

Theorem 4. *Let* $\mathbf{A}$ *be a semisimple Leibniz algebra. Then any derivation* d *of* $\mathbf{A}$ *can be written as* $d = L_a + \alpha + \delta$, *where* $a \in S, \alpha : \mathrm{Leib}(\mathbf{A}) \to \mathrm{Leib}(\mathbf{A}), \delta : S \to \mathrm{Leib}(\mathbf{A})$, *where* S *is a semisimple Lie algebra and* $\alpha([x,y]) = [x, \alpha(y)]$ *for all* $x, y \in \mathbf{A}$. *Moreover, if* $\mathbf{A}$ *is simple, then* α *is either zero or* $\alpha(\mathrm{Leib}(\mathbf{A})) = \mathrm{Leib}(\mathbf{A})$.

Proof. Let $\mathbf{A}$ be a semisimple Leibniz algebra. According to Theorem 1, $\mathbf{A} = S + \mathrm{Leib}(\mathbf{A})$, where S is a semisimple Lie algebra. Then $L(\mathbf{A}) = L(S)$. According to ([4], Theorem 3.3), $\mathbf{A}$ is complete, and so $\mathrm{Der}(\mathbf{A}) = \mathrm{IDer}(\mathbf{A})$. Let $d \in \mathrm{Der}(\mathbf{A})$. According to Theorem 3, $d = L_a + k$ for some $a \in S$ and $k \in I$. Set $\alpha = k|_{\mathrm{Leib}(\mathbf{A})}$ and $\delta = k|_S$. Then we can extend α to be a derivation on $\mathbf{A}$ by defining $\alpha(x+y) = \alpha(y)$ for any $x \in S$ and $y \in \mathrm{Leib}(\mathbf{A})$. Similarly, we can extend δ to be a derivation of $\mathbf{A}$ by defining $\delta(x+y) = \delta(x)$ for any $x \in S$ and $y \in \mathrm{Leib}(\mathbf{A})$. Thus, $d = L_a + \alpha + \delta$, $\alpha(\mathrm{Leib}(\mathbf{A})) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\delta(S) \subseteq \mathrm{Leib}(\mathbf{A})$ as $\mathrm{Leib}(\mathbf{A})$ is a characteristic ideal of $\mathbf{A}$. Since $\mathrm{Leib}(\mathbf{A}) \subseteq Z^l(\mathbf{A})$, $\alpha([x,y]) = [\alpha(x), y] - [x, \alpha(y)] = [x, \alpha(y)]$ for any $x, y \in \mathbf{A}$. If $\mathbf{A}$ is simple, then $\alpha(\mathrm{Leib}(\mathbf{A}))$ is either $\{0\}$ or $\mathrm{Leib}(\mathbf{A})$, which implies that α is either zero or $\alpha(\mathrm{Leib}(\mathbf{A})) = \mathrm{Leib}(\mathbf{A})$. □

Example 5. *Let* $S = \mathrm{span}\{e, f, h\} \oplus \mathrm{span}\{a, b, c\}$ *and* $V = \mathrm{span}\{x, y\}$. *Define* $\mathbf{A} = S \oplus V$ *with the multiplications in* $\mathbf{A}$ *given by* $[e, f] = h, [f, e] = -h, [h, e] = 2e, [e, h] = -2e, [h, f] = -2f, [f, h] = 2f, [e, y] = x, [f, x] = y, [h, x] = x, [h, y] = -y, [a, b] = c, [b, a] = -c, [c, a] = 2a, [a, c] = -2a, [c, b] = -2b, [b, c] = 2b$. *Then* $\mathbf{A}$ *is a semisimple Leibniz algebra with* $\mathrm{Leib}(\mathbf{A}) = V$. *Through direct calculation, we determine that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4, d_5, d_6, d_7\} = \mathrm{IDer}(\mathbf{A})$, *where*

$$
\begin{array}{llllllll}
d_1(e) = e, & d_1(f) = -f, & d_1(h) = 0, & d_1(x) = x, & d_1(y) = 0, & d_1(a) = 0, & d_1(b) = 0, & d_1(c) = 0, \\
d_2(e) = -e, & d_2(f) = f, & d_2(h) = 0, & d_2(x) = 0, & d_2(y) = y, & d_2(a) = 0, & d_2(b) = 0, & d_2(c) = 0, \\
d_3(e) = 0, & d_3(f) = h, & d_3(h) = -2e, & d_3(x) = 0, & d_3(y) = x, & d_3(a) = 0, & d_3(b) = 0, & d_3(c) = 0, \\
d_4(e) = -h, & d_4(f) = 0, & d_4(h) = 2f, & d_4(x) = y, & d_4(y) = 0, & d_4(a) = 0, & d_4(b) = 0, & d_4(c) = 0, \\
d_5(e) = 0, & d_5(f) = 0, & d_5(h) = 0, & d_5(x) = 0, & d_5(y) = 0, & d_5(a) = a, & d_5(b) = -b, & d_5(c) = 0, \\
d_6(e) = 0, & d_6(f) = 0, & d_6(h) = 0, & d_6(x) = 0, & d_6(y) = 0, & d_6(a) = 0, & d_6(b) = c, & d_6(c) = -2a, \\
d_7(e) = 0, & d_7(f) = 0, & d_7(h) = 0, & d_7(x) = 0, & d_7(y) = 0, & d_7(a) = c, & d_7(b) = 0, & d_7(c) = -2b.
\end{array}
$$

Then $L(\mathbf{A}) = \mathrm{span}\{d_1 - d_2, d_3, d_4, d_5, d_6, d_7\} = L(S)$. *Let* $k = d_1 + d_2$. *Then* $k \in I$ *and* $d_1 = L_{h/2} + k|_V + k|_S$ *and* $d_2 = L_{-h/2} + k|_V + k|_S$.

Recall that a Lie algebra $\mathbf{L}$ is called complete if it has a trivial center and all of its derivations are inner. In ([5], Theorem 4.3), Meng proved that for a Lie algebra $\mathbf{L}$ with a trivial center, if $\mathrm{ad}(\mathbf{L})$ is a characteristic ideal of $\mathrm{Der}(\mathbf{L})$, then $\mathrm{Der}(\mathbf{L})$ is a complete Lie algebra. This implies that for a complete Lie algebra $\mathbf{L}$, $\mathrm{Der}(\mathbf{L})$ is a complete Lie algebra. However, as shown in ([8], Example 3.11–3.12), there exists a complete Leibniz algebra $\mathbf{A}$ such that $\mathrm{Der}(\mathbf{A})$ is not complete. We examine the Leibniz algebras with complete liezation and obtain the following results.

Theorem 5. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ *is a complete Lie algebra. Then*

(i) $I_{\mathbf{A}} = \mathrm{Leib}(\mathbf{A})$,

(ii) $\mathrm{Der}(\mathbf{A})/I$ *is a complete Lie algebra.*

Proof. (i) Let $\mathbf{A}$ be a Leibniz algebra such that $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a complete Lie algebra. According to ([4], Proposition 3.2), $\mathbf{A}$ is complete. Then according to Corollary 1, $\mathbf{A}/I_{\mathbf{A}} \cong \mathrm{ad}(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \cong \mathbf{A}/\mathrm{Leib}(\mathbf{A})$. Hence, $I_{\mathbf{A}} = \mathrm{Leib}(\mathbf{A})$. (ii) Let $\mathbf{A}$ be a Leibniz algebra such that $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a complete Lie algebra. Then $\mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ is complete. Define a linear map $\varphi : \mathrm{Der}(\mathbf{A}) \to \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ using $\varphi(d) = d'$, where $d'(x + \mathrm{Leib}(\mathbf{A})) = d(x) + \mathrm{Leib}(\mathbf{A})$ for all $d \in \mathrm{Der}(\mathbf{A})$ and $x \in \mathbf{A}$. Let $d_1, d_2 \in \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$. Then for all $x \in \mathbf{A}$, $\varphi([d_1, d_2])(x + \mathrm{Leib}(\mathbf{A})) = d_1'(d_2'(x) + \mathrm{Leib}(\mathbf{A})) - d_2'(d_1'(x) + \mathrm{Leib}(\mathbf{A})) = [\varphi(d_1), \varphi(d_2)](x + \mathrm{Leib}(\mathbf{A}))$. Hence, $\varphi([d_1, d_2]) = [\varphi(d_1), \varphi(d_2)]$. Clearly, $I = \{d \in \mathrm{Der}(\mathbf{A}) | \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\} \subseteq \ker(\varphi)$. Let $d \in \ker(\varphi)$. Then $d(x) + \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$ for all $x \in \mathbf{A}$, which implies that $d \in I$, and hence, $\ker(\varphi) \subseteq I$. Thus, $\ker(\varphi) = I$. To show that φ is onto, let $d' \in \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$. Since $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is complete, there exists $a + \mathrm{Leib}(\mathbf{A}) \in \mathbf{A}/\mathrm{Leib}(\mathbf{A})$ such that $d' = \mathrm{ad}_{a+\mathrm{Leib}(\mathbf{A})}$. Thus, for all $a + \mathrm{Leib}(\mathbf{A}) \in \mathbf{A}/\mathrm{Leib}(\mathbf{A})$, we have $L_a(x) + \mathrm{Leib}(\mathbf{A}) = [a, x] + \mathrm{Leib}(\mathbf{A}) = [a + \mathrm{Leib}(\mathbf{A}), x + \mathrm{Leib}(\mathbf{A})] = \mathrm{ad}_{a+\mathrm{Leib}(\mathbf{A})}(x + \mathrm{Leib}(\mathbf{A}))$. This implies that $\varphi(L_a) = d'$. Hence, φ is onto and $\mathrm{im}(\varphi) = \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$. Therefore, $\mathrm{Der}(\mathbf{A})/I \cong \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$. This proves that $\mathrm{Der}(\mathbf{A})/I$ is complete. $\square$

The following is an immediate result from the above theorem.

Corollary 2. *Let $\mathbf{A}$ be a Leibniz algebra such that $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a complete Lie algebra. Then $\mathbf{A}/I_{\mathbf{A}}$ is a complete Lie algebra and* $\dim(\mathrm{Der}(\mathbf{A})) = \dim(\mathbf{A}) - \dim(\mathrm{Leib}(\mathbf{A})) + \dim(I)$.

4. On Central Derivations

In [9], Tôgô studied the properties of inner derivations of Lie algebras by comparing them with the set of central derivations. In this section, we investigate analogous results for left Leibniz algebras. Note that Shermatova and Khudoyberdiyev, in [14], also studied central derivations by comparing them with inner derivations. However, their works are on the right Leibniz algebras, using the definition of inner derivations in [3].

Definition 3. *Let $\mathbf{A}$ be a Leibniz algebra. A derivation $d \in \mathrm{Der}(\mathbf{A})$ is called a central derivation if* $\mathrm{im}(d) \subseteq Z(\mathbf{A})$.

We denote $\mathrm{CDer}(\mathbf{A})$ to be the set of all central derivations of $\mathbf{A}$. It should be noted that $\mathrm{CDer}(\mathbf{A})$ is a subalgebra of $\mathrm{Der}(\mathbf{A})$. We start by examining derivations of Leibniz algebras that are both inner and central. Let $\mathbf{A}$ be a Leibniz algebra. According to Theorem 3, $\mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$ where $I = \{d \in Der(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$. The following proposition is the Leibniz algebra analogue of the result in ([9], Lemma 2).

Proposition 3. *Let $\mathbf{A}$ be a Leibniz algebra and $J = I \cap \mathrm{CDer}(\mathbf{A})$. Then the following hold.*

(i) $\mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) = L(Z_1) + J$, *where* $Z_1 = \{x \in \mathbf{A} \mid [x, \mathbf{A}] \subseteq Z(\mathbf{A})\}$.

(ii) $\mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) \subseteq L(Z_2) + J$, *where* $Z_2 = \{r \in \mathrm{rad}(\mathbf{A}) \mid [r, \mathrm{rad}(\mathbf{A}^2)] = 0\}$.

Proof. (i) $\mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) = L(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) + I \cap \mathrm{CDer}(\mathbf{A}) = \{L_x \mid \mathrm{im}(L_x) \subseteq Z(\mathbf{A})\} + J = L(Z_1) + J$, where $Z_1 = \{x \in \mathbf{A} \mid [x, \mathbf{A}] \subseteq Z(\mathbf{A})\}$. (ii) Let $d \in \mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A})$. According to (i), there exist $z \in Z_1$ and $h \in J$ such that $d = L_z + h$. According to Theorem 1, there exists a semisimple Lie algebra S such that $\mathbf{A} = S + \mathrm{rad}(\mathbf{A})$ and $S \cap \mathrm{rad}(\mathbf{A}) = \{0\}$. Thus, $\mathbf{A}^2 = S + \mathrm{rad}(\mathbf{A}^2)$ and there exist $s \in S$ and $r \in \mathrm{rad}(\mathbf{A})$ such that $z = s + r$. Since $\mathrm{im}(h) \subseteq Z(\mathbf{A})$, we have $h(S) = h([S, S]) = 0$, and hence, $h(\mathrm{rad}(\mathbf{A}^2)) = h(\mathbf{A}^2) = 0$. Since $\mathrm{im}(d) \subseteq Z(\mathbf{A})$, we also have $d(S) = 0$ and $d(\mathbf{A}^2) = 0$, which implies that $d(\mathrm{rad}(\mathbf{A}^2)) = 0$. It follows that $0 = L_{s+r}(S) = [s + r, S] = [s, S] + [r, S]$. Hence, $[s, S] = 0$, and therefore, $s = 0$. Thus, $d = L_r + h$ and $[r, \mathrm{rad}(\mathbf{A}^2)] = 0$. $\square$

Example 6. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{w, x, y, z\}$ *with non-zero multiplications defined by* $[w, x] = y, [x, w] = z, [w, y] = z$ *and* $[x, x] = z$*. Then we have* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{y, z\}$*, and* $Z(\mathbf{A}) = \mathrm{span}\{z\}$*. Through direct calculation, we determine that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\} = \mathrm{IDer}(\mathbf{A}) = I$*, where*

$$\begin{array}{llll} d_1(w) = z, & d_1(x) = 0, & d_1(y) = 0, & d_1(z) = 0, \\ d_2(w) = 0, & d_2(x) = y, & d_2(y) = z, & d_2(z) = 0, \\ d_3(w) = 0, & d_3(x) = z, & d_3(y) = 0, & d_3(z) = 0. \end{array}$$

Then $\mathrm{CDer}(\mathbf{A}) = \mathrm{span}\{d_1, d_3\} = J$ *and* $Z_1 = \mathrm{span}\{x, y, z\}$*. Thus,* $\mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) = L(Z_1) + J$*. Moreover, we determine that* $\mathbf{A} = \mathrm{rad}(\mathbf{A})$ *and* $\mathrm{rad}(\mathbf{A}^2) = \mathrm{span}\{y, z\}$*. Since* $Z_2 = \mathrm{span}\{x, y, z\}$*,* $\mathrm{IDer}(\mathbf{A}) \cap \mathrm{CDer}(\mathbf{A}) \subseteq L(Z_2) + J$*.*

Next, we investigate Leibniz algebras where all central derivations are inner, yielding the Leibniz algebra analogue of ([9], Lemma 3).

Theorem 6. *Let* $\mathbf{A}$ *be a Leibniz algebra satisfying* $\mathrm{CDer}(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A})$*. If* $\mathrm{rad}(\mathbf{A})$ *is abelian, then either* $Z(\mathbf{A}) = \{0\}$ *or* $\mathbf{A} = \mathbf{A}^2$*.*

Proof. Let $\mathbf{A}$ be a Leibniz algebra satisfying $\mathrm{CDer}(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A})$. According to Theorem 1, there exists a semisimple Lie algebra S such that $\mathbf{A} = S + \mathrm{rad}(\mathbf{A})$ and $S \cap \mathrm{rad}(\mathbf{A}) = \{0\}$. Suppose that $Z(\mathbf{A}) \neq \{0\}$ and $\mathbf{A} \neq \mathbf{A}^2$. Since $\mathbf{A}^2 = S + [S, \mathrm{rad}(\mathbf{A})] + [\mathrm{rad}(\mathbf{A}), S]$, we have $[S, \mathrm{rad}(\mathbf{A})] + [\mathrm{rad}(\mathbf{A}), S] \subsetneq \mathrm{rad}(\mathbf{A})$. Choose a subspace U of $\mathrm{rad}(\mathbf{A})$ such that $\mathrm{rad}(\mathbf{A}) = U + [S, \mathrm{rad}(\mathbf{A})] + [\mathrm{rad}(\mathbf{A}), S]$ and $U \cap ([S, \mathrm{rad}(\mathbf{A})] + [\mathrm{rad}(\mathbf{A}), S]) = \{0\}$. Define a nonzero linear map $d : \mathbf{A} \to \mathbf{A}$ such that $d(U) \subseteq Z(\mathbf{A})$ and $d(S + [S, \mathrm{rad}(\mathbf{A})] + [\mathrm{rad}(\mathbf{A}), S]) = 0$. Then d is a central derivation of $\mathbf{A}$. Since $\mathrm{CDer}(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A}) = L(\mathbf{A}) + I$ and $\mathbf{A} = S + \mathrm{rad}(\mathbf{A})$, there exist $s \in S, r \in \mathrm{rad}(\mathbf{A})$ and $h \in I$ such that $d = L_{s+r} + h$. Since $d(S) = 0$ and $[r, S] + h(S) \subseteq \mathrm{rad}(\mathbf{A})$, we have $[s, S] = 0$, and hence, $s = 0$. Therefore, $d(U) = [r, U] + h(U) \subseteq \mathrm{Leib}(\mathbf{A})$ as $[r, U] \subseteq [\mathrm{rad}(\mathbf{A}), \mathrm{rad}(\mathbf{A})] = \{0\}$. Let $0 \neq u \in U$. Then $d(u) = \alpha[x, x]$ for some $\alpha \in \mathbb{F}$ and $x \in \mathbf{A}$. Since S is a subalgebra, $x \notin S$, which implies that $x \in \mathrm{rad}(\mathbf{A})$. Thus, $d(u) = \alpha[x, x] \in [\mathrm{rad}(\mathbf{A}), \mathrm{rad}(\mathbf{A})] = \{0\}$, which contradicts our definition of d. Hence, we have either $Z(\mathbf{A}) = \{0\}$ or $\mathbf{A} = \mathbf{A}^2$. □

Corollary 3. *Let* $\mathbf{A}$ *be a Leibniz algebra satisfying* $\mathrm{CDer}(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A})$*. If* $Z(\mathbf{A}) \neq \{0\}$ *and* $\mathrm{CDer}(\mathbf{A}) \neq \{0\}$*, then* $\mathrm{rad}(\mathbf{A})$ *is not abelian.*

Proof. Let $\mathbf{A}$ be a Leibniz algebra satisfying $\mathrm{CDer}(\mathbf{A}) \subseteq \mathrm{IDer}(\mathbf{A})$. Suppose that $Z(\mathbf{A}) \neq \{0\}$ and $\mathrm{CDer}(\mathbf{A}) \neq \{0\}$. If $\mathrm{rad}(\mathbf{A})$ is abelian, then according to Theorem 6, $\mathbf{A} = \mathbf{A}^2$. Hence for all $d \in \mathrm{CDer}(\mathbf{A})$, $d(\mathbf{A}) = d([\mathbf{A}, \mathbf{A}]) = \{0\}$, which implies that $d = 0$. It follows that $\mathrm{CDer}(\mathbf{A}) = \{0\}$, a contradiction. Therefore, $\mathrm{rad}(\mathbf{A})$ is not abelian. □

Finally, we explore Leibniz algebras where all inner derivations are central, establishing the Leibniz algebra analogue of ([9], Theorem 3).

Theorem 7. *Let* $\mathbf{A}$ *be a Leibniz algebra. Then the following hold.*

(i) $\mathrm{IDer}(\mathbf{A}) \subseteq \mathrm{CDer}(\mathbf{A})$ *if and only if* $\mathbf{A}^2 \subseteq Z(\mathbf{A})$ *if and only if* $\mathbf{A}^3 = \{0\}$*.*

(ii) *If* $Z(\mathbf{A}) \neq \{0\}$ *and* $\mathrm{IDer}(\mathbf{A}) = \mathrm{CDer}(\mathbf{A})$*, then* $\mathbf{A}^2 = Z(\mathbf{A})$*.*

Proof. (i) Assume that $\mathrm{IDer}(\mathbf{A}) \subseteq \mathrm{CDer}(\mathbf{A})$. Then for all $x, y \in \mathbf{A}$, $L_x \in \mathrm{IDer}(\mathbf{A}) \subseteq \mathrm{CDer}(\mathbf{A})$ and $[x, y] = L_x(y) \in Z(\mathbf{A})$. Conversely, assume that $\mathbf{A}^2 \subseteq Z(\mathbf{A})$. Let $d \in \mathrm{IDer}(\mathbf{A})$. Then there exists $a \in \mathbf{A}$ such that $d(x) - L_a(x) \in \mathrm{Leib}(\mathbf{A})$ for any $x \in \mathbf{A}$. Thus, $d(x) \in \mathbf{A}^2 \subseteq Z(\mathbf{A})$, hence, $d \in \mathrm{CDer}(\mathbf{A})$. Clearly, $\mathbf{A}^2 \subseteq Z(\mathbf{A})$ if and only if $\mathbf{A}^3 = [\mathbf{A}, [\mathbf{A}, \mathbf{A}]] = 0$. (ii) Suppose $Z(\mathbf{A}) \neq \{0\}$ and $\mathrm{IDer}(\mathbf{A}) = \mathrm{CDer}(\mathbf{A})$. According to (i), $\mathbf{A}^2 \subseteq Z(\mathbf{A})$. If $\mathbf{A}^2 \neq Z(\mathbf{A})$, then according to ([15], Theorem 3.6), $\mathbf{A}$ has an outer central derivation which contradicts our assumption. Hence, $\mathbf{A}^2 = Z(\mathbf{A})$. □

Observe that ([9], Theorem 3 (iii)) is also valid in our case. In ([9], Theorem 3 (ii)), Tôgô proved that for a Lie algebra $\mathbf{L}$, if $Z(\mathbf{L}) \neq 0$, then $\mathrm{IDer}(\mathbf{L}) = \mathrm{CDer}(\mathbf{L})$ if and only if $\mathbf{L}^2 = Z(\mathbf{L})$ and $\dim(Z(\mathbf{L})) = 1$. However, as the following example demonstrates, there exists a Leibniz algebra $\mathbf{A}$ where $Z(\mathbf{A}) \neq \{0\}$ and $\mathrm{IDer}(\mathbf{A}) = \mathrm{CDer}(\mathbf{A})$, but $\dim(Z(\mathbf{A})) > 1$.

Example 7. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{w, x, y, z\}$ *with non-zero multiplications defined by* $[w, w] = z, [w, x] = y$, *and* $[x, w] = -y$. *We can see that* $Z(\mathbf{A}) = \mathbf{A}^2 = \mathrm{span}\{y, z\}$, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$, *and* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$, *where*

$$
\begin{array}{llll}
d_1(w) = w, & d_1(x) = 0, & d_1(y) = y, & d_1(z) = 2z, \\
d_2(w) = 0, & d_2(x) = x, & d_2(y) = y, & d_2(z) = 0, \\
d_3(w) = x, & d_3(x) = 0, & d_3(y) = 0, & d_3(z) = 0, \\
d_4(w) = y, & d_4(x) = 0, & d_4(y) = 0, & d_4(z) = 0, \\
d_5(w) = z, & d_5(x) = 0, & d_5(y) = 0, & d_5(z) = 0, \\
d_6(w) = 0, & d_6(x) = y, & d_6(y) = 0, & d_6(z) = 0, \\
d_7(w) = 0, & d_7(x) = z, & d_7(y) = 0, & d_7(z) = 0.
\end{array}
$$

Then $\mathrm{IDer}(\mathbf{A}) = \mathrm{span}\{d_4, d_5, d_6, d_7\} = \mathrm{CDer}(\mathbf{A})$.

5. Conclusions

In this paper, we utilize the recent definition of inner derivations for Leibniz algebras as given in [4] to describe the Lie algebras of inner derivations of Leibniz algebras. We also extend the result regarding the Lie algebra of derivations from [5] to Leibniz algebras with complete liezations. Using our result, we derive a similar description of derivations for semisimple Leibniz algebras in [6] through a different approach and establish relations between the inner derivation algebras of Leibniz algebras and the algebra of central derivations, analogous to the case in Lie algebras in [9]. Several open problems remain in understanding various aspects of inner derivations of Leibniz algebras and their relation with the structure of Leibniz algebras, mirroring the exploration of properties of inner derivations in Lie algebras.

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