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Risk Measures' Duality on Ordered Linear Spaces

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Abstract: The aim of this paper is to provide a dual representation of convex and coherent risk measures in partially ordered linear spaces with respect to the algebraic dual space. An algebraic robust representation is deduced by weak separation of convex sets by functionals, which are assumed to be only linear; thus, our framework does not require any topological structure of the underlying spaces, and our robust representations are found without any continuity requirement for the risk measures. We also use such extensions to the representation of acceptability indices.

Keywords: risk functionals; monetary risk measures; partially ordered linear spaces; internal points; positive cones

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1. Motivation of the Paper

In [1], it is showed that for convex and coherent risk measures defined on a rearrangement invariant and solid space, i.e. for every random variables X and Y with the same distribution $X \in \mathcal{X} \subset L^0(\Omega, \mathscr{F}, \mathsf{P})$ implies $Y \in \mathcal{X}$ and $|Y| \leqslant |X|$, the domain must actually be a subset of $L^1(\Omega, \mathscr{F}, \mathsf{P})$, written L^1, and the corresponding representation need no completeness assumption, in contrast to the continuity requirement for positive linear functionals employed in the Namioka-Klee theorem for ordered linear spaces. The motivation of [1] was inspired by the consideration of financial positions having an infinite mean, such as those modeled by Pareto-distributed random variables whose density has a tail index less than one (namely heavy-tailed random variables), which showed up in coping with operational risk, see [2]. Motivated by the above, we provide standing-alone structure theorems for convex and coherent risk measures that are not direct consequences of known results using neither topological structure for the dual spaces nor continuity from above for the risk measures involved.

In fact, another source of inspiration for using this approach stems from [3], where in establishing a one-to-one correspondence among the partial ordering of $\mathcal{X}$ (interpreted, from the decision theory perspective, as a preference relation), sets of acceptable positions, valuation bounds (actually good-deal bounds), coherent risk measures, and price systems, the authors assert that differently from typical works on arbitrage theory that use topological structures such as closeness and compactness with associated strong separation, it is possible to restate the theory in a purely algebraic manner. In particular, they noticed that a linear pricing functional should lie in the algebraic dual $\mathcal{X}'$ of $\mathcal{X}$, so in the light of their unified approach to valuation and risk measurement, it seems reasonable to attack the representation problem for monetary risk measures by conceiving cones $\mathcal{A} \subseteq \mathcal{X}$ containing $e \geqslant 0$ as their *internal point*. The latter should be interpreted as a *benchmark* position that can be different from the usual risk-free one given by a zero-coupon bond, and $\mathcal{A}$ also contains the nonnegative financial positions, and its convex structure typically induces the partial ordering on $\mathcal{X}$. Moreover, acceptability of a position $X \in \mathcal{X}$ can be understood as the minimal amount of capital $m\,e$, for $m \in \mathbb{R}$, such that $X + m\,e \in \mathcal{A}$. We can also think of e as a risk-free bank account in which a liquid amount of money m is reserved to make the position acceptable; see [4,5] for the case of uncertainty affecting e, that can lead to cash subadditivity). This leads us to propose an entirely algebraic dual representation of both convex and coherent risk measures defined on ordered linear spaces, where, in the spirit of [Th.2] in [3], we use a linear functional interpreted as a pricing functional without worrying about its continuity. To obtain a dual representation in the general framework of ordered linear spaces, we resort to Edelheit's separation theorem, see [6]. We used it in order to extend the Fenchel-Moreau theorem for conjugate representation of convex functions ρ, namely its Fenchel-Legendre transform. It turns out that the notion of internal points of cones is crucial for this hyperplane separation theorem to apply. Armed with these algebraic tools, we obtain the dual representation of monetary risk measures with respect to the algebraic dual space $\mathcal{X}'$ of all the financial positions in $\mathcal{X}$, which is different from the topological dual $\mathcal{X}^*$.

In our Theorems 2 and 3, the domain of ρ is indeed a general Riesz space $\mathcal{X}$ with a partial order induced by some cone with a non-empty quasi-interior. On the other hand, we are aware of the relevant special case $\mathcal{X} = L^1$. In [7], the authors showed that this is the right ambient space for representing law-invariant convex risk measures provided the underlying probability space is standard; see also [8] for the extension to atomless probability spaces, which is actually a supplementary note to [9]. We must care that the topological (or even norm) interior of the positive cone $\mathcal{X}_+ \subseteq \mathcal{A}$ is empty, and since topological interior points are internal points, there is no sufficient condition to guarantee the direct determination of such latter points. We solve this problem by using the notion of quasi-interior points, since any of such points is also an internal point of the positive cone L^1_+, see [10]. Our contribution to the theory of financial risk measures is in the direction of a convex duality theory without topology: we do not need the underlying spaces to be topological; we do not require continuity of the linear functionals involved in the dual-robust representation of convex/coherent risk measures; we do not impose any axiom concerning continuity of the risk measures ourselves, in part to recover an order-type continuity in the special case $\mathcal{X} = L^1$. This enables us to extend previous results on partially ordered linear spaces towards weak separation of convex sets by functional without further requiring any topological assumption. Since some recent literature has investigated the relationship between risk measures and performance measures, we also employ our algebraic framework to extend the notions of coherent acceptability indices and quasi-concave acceptability indices of performance; see [4,11,12] and the references therein.

The paper is organized as follows: Section 2 sets out some definitions and preliminary results to be used in the sequel. Section 3 contains our first result concerning the dual representation of coherent risk measures on partially ordered linear spaces. Section 4 is on our second result concerning the dual representation of convex (not necessarily coherent) risk measures on partially ordered linear spaces. Section 5 provides an application of our convex duality theory to the case of financial position with finite mean, discussing some issues concerning interior and internal points and introduces the special 'order continuity' of the corresponding convex risk measures. In Section 6, a further application of our algebraic framework is presented: acceptability indices based on either coherent risk measures or convex risk measures on partially ordered linear spaces are treated. Section 7 contains some concluding remarks.

2. Preliminaries

Given a vector space $\mathcal{X}$ containing financial positions, including the constant ones, $\mathcal{X}'$ denote the algebraic dual of $\mathcal{X}$, which is the vector space of all linear functionals (real-valued) on $\mathcal{X}$, then

$$\mathcal{A}^\circ = \{f \in \mathcal{X}' \mid f(X) \geqslant 0, X \in \mathcal{A}\}$$

is called the *polar wedge* of $\mathcal{A}$. For $\mathcal{A} \subseteq \mathcal{X}$, a vector $A \in \mathcal{A}$ is an internal point of $\mathcal{A}$ if given some $X \in \mathcal{X}$, there exists a real number $\delta > 0$ such that $A + \lambda X \in \mathcal{A}$ for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \delta$. Recall that, assuming convexity, the set of internal points is convex; see, for example, [Th.1, Ch.4] in [13] or [Th.0.2.1] in [14]. In fact, internal and interior points coincide for convex sets in topological vector spaces; see, for example, [Lemma 1.4] in [15] in the case of normed spaces. It is worth noting that in finite-dimensional spaces (the Euclidean ones), all locally convex topologies are equivalent, then internal and interior points coincide; see [Th.5.60] in [16]. We let $(\mathcal{X}, \geqslant)$ be partially ordered by

$$X \geqslant Y \text{ if and only if } X - Y \in \mathcal{A},$$

i.e., the partial order is induced by a cone $\mathcal{A} \subseteq \mathcal{X}$. In the case $\mathcal{A} \neq \{0\}$, the partial ordering is reflexive, antisymmetric, transitive, and compatible with the linear structure of $\mathcal{X}$:

(i) $X \geqslant X$ for any $X \in L$;
(ii) If $X \geqslant Y$ and $Y \geqslant X$, then $X = Y$;
(iii) If $X \geqslant Y$ and $Y \geqslant Z$, then $X \geqslant Z$;
(iv) If $X \geqslant Y$, then $\lambda X \geqslant \lambda Y$ for any $\lambda \in \mathbb{R}_+$;
(v) If $X \geqslant Y$, then $X + Z \geqslant Y + Z$, for any $Z \in \mathcal{X}$.

The set $\mathcal{X}_+ := \{X \in \mathcal{X} \mid X \geqslant 0\}$ is a positive cone provided that $(\mathcal{X}, \geqslant)$ satisfies properties (i) together with (iii) to (v); by adding antisymmetry (ii), it is also pointed. The following statement provides separation of convex sets in (ordered) linear spaces; see [Th.0.2.4] in [14].

Theorem 1 (Eidelheit's Separation Theorem)**.** *Suppose that $\mathcal{A}, \mathcal{B}$ are convex subsets of $\mathcal{X}$ such that $int(\mathcal{A}) \neq \varnothing$ and $int(\mathcal{A}) \cap \mathcal{B} = \varnothing$. Then there is a non-zero functional f of $\mathcal{X}'$, such that*

$$\inf_{X \in \mathcal{A}} f(X) \geqslant \sup_{X \in \mathcal{B}} f(X).$$

Theorem 1 was established in [6]. This is the separating hyperplane theorem we mainly used in our representation results. Equivalently, there exists some $z \in \mathbb{R}$ such that $f(X) \geqslant z$ if $X \in \mathcal{A}$ and $f(X) \leqslant z$ if $X \in \mathcal{B}$, or in short $\mathcal{A} \subseteq \{f \leqslant z\}$ and $\mathcal{B} \subseteq \{f \geqslant z\}$, which is weak separation. Observe that the same statement is [Th.3.8] in [17]. Compare also with [Th.5.6.1] in [16]. A convex subset $\mathcal{C}$ of a cone $\mathcal{A}$ is a *base* if for any $X \in \mathcal{A} \setminus \{0\}$ there exists some real number $t(X) > 0$, depending on X, such that $t(X)X \in \mathcal{C}$. If $f \in \mathcal{X}' \setminus \{0\}$, the set $\mathcal{C}_f = \{X \in \mathcal{A} \mid f(X) = 1\}$ is a base of $\mathcal{A}$. The solid subspace generated by $X \in \mathcal{X} \setminus \{0\}$ is defined as $I_X := \cup_{n=1}^{\infty} [-nX, nX]$, where $n \in \mathbb{N}$. If $\mathcal{X}$ is a normed linear space and I_X is dense in $\mathcal{X}$, then X is called quasi-interior point. For more details about partially ordered linear spaces, see [14].

The properties of coherent/convex risk measures and acceptance sets are established in [18] and [19] or [20], respectively, see also [Ch.4] in [21]. For the sake of completeness, we list the relevant properties of risk measures. We interpret $X \in \mathcal{X}$ as profit and loss from the holding of a portfolio over a fixed time horizon (negative values are losses). Then, a mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a *monetary risk measure* if:

- it is decreasing monotone, i.e., every $X, Y \in \mathcal{X}$ such that $X \leqslant Y$ implies $\rho(X) \geqslant \rho(Y)$;
- it is cash additive, i.e., for every $m \in \mathbb{R}$ we have $\rho(X + m\,e) = \rho(X) - m$, where $e \neq 0$ is an internal point of the cone $\mathcal{A} := \{X \in \mathcal{X} \mid \rho(X) \leqslant 0\}$ of acceptable positions.

The internal point e can be interpreted as a proxy for the riskless bond, or in the terminology of [3] relatively secure cash stream. If, in addition, ρ is

- (first degree) positive homogeneous, $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda \geqslant 0$ and,
- subadditive, $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$, for all $X, Y \in \mathcal{X}$,

then it is a coherent risk measure, which is also a convex mapping. On the other hand, ρ is a convex risk measure if satisfies decreasing monotonicity, cash additivity, and

- convexity, $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$, for any $\lambda \in [0,1]$ and for all $X, Y \in \mathcal{X}$.

Clearly, ρ convex is not necessarily coherent. Subadditivity becomes additivity for comonotone X, Y; see [Sec.4.7] in [21].

3. Coherent Risk Measures on Ordered Linear Spaces

The dual representation of risk measures on a partially ordered linear space $\mathcal{X}$ of financial positions can be traced back to [18] and then [22], in the case $\mathcal{X} = L^0(\Omega, \mathscr{F}, \mathrm{P})$. The underlying probability space is assumed to be complete and atomless. We keep this assumption for the rest of the paper. We start with a restatement of the classical construction of a cash additive risk measure, given an acceptance set $\mathcal{A} \subset \mathcal{X}$ of financial positions. Recall that $\mathcal{X}$ is only assumed to be a partially ordered linear space, and $\mathcal{X}'$ is the algebraic dual of $\mathcal{X}$, and $\mathcal{A}^\circ$ is the polar wedge of $\mathcal{A}$.

Theorem 2. *A coherent risk measure $\rho : \mathcal{X} \to \mathbb{R}$, whose acceptance set is the cone $\mathcal{A} \subseteq \mathcal{X}$, such that $e \in \mathcal{A} \setminus \{0\}$ and e is an internal point of $\mathcal{A}$, admits the following dual representation:*

$$\rho(X) = \sup_{\pi \in \mathcal{B}} \pi(-X), \tag{1}$$

where $\mathcal{B} := \{\pi \in \mathcal{X}' \mid \pi \in \mathcal{A}^\circ, \pi(e) = 1\}$.

Proof. Let $X \in \mathcal{A}$, i.e., X is acceptable. Thus, $\rho(X + \rho(X)e) = 0$ from cash additivity with respect to e. Then $\pi(X + \rho(X)e) \geqslant 0$ for any $\pi \in \mathcal{B}$. Hence, $\rho(X) \geqslant \pi(-X)$ and $\rho(X) \geqslant \sup_{\pi \in \mathcal{B}} \pi(-X)$. Applying again cash additivity with respect to e, $\rho(X + (\rho(X) - \varepsilon)e) = \varepsilon > 0$ for any $\varepsilon > 0$. Since $\mathcal{A}$ and $\{X + (\rho(X) - \varepsilon)e\}$ are both convex sets, from Eidelheit's separation theorem, we have $\mathcal{A} \cap \{X + (\rho(X) - \varepsilon)e\} = \varnothing$, hence there exists some $\pi \neq 0$ with $\pi \in \mathcal{X}'$ and $\inf_{X \in \mathcal{A}} \pi(X) \geqslant \pi(X + (\rho(X) - \varepsilon)e) = \pi(X) + \rho(X) - \varepsilon \leqslant 0$. The latter inequality is especially true for any $\pi \in \mathcal{B}$ and $\varepsilon > 0$. Moreover, this is true for any $X \in \mathcal{X}$. This implies that $\rho(X) - \varepsilon \leqslant \sup_{\pi \in \mathcal{B}} \pi(-X)$, thus $\rho(X) = \sup_{\pi \in \mathcal{B}} \pi(-X)$. □

Observe that, equivalently, $\rho(X) = -\inf_{\pi \in \mathcal{B}} \pi(X)$. In fact, we have that the mapping $\rho_{\mathcal{A},e} : \mathcal{X} \to \mathbb{R}$ with respect to the cone $\mathcal{A}$ and the position $e \in \mathcal{A} \setminus \{0\}$ defined as

$$\rho_{\mathcal{A},e}(X) := \inf\{m \in \mathbb{R} \mid X + m\, e \in \mathcal{A}\}, \tag{2}$$

is a coherent risk measure. The proof can be easily deduced in the current algebraic setting by using cash additivity; see [22] or [Ch.4] in [21] for the case $\mathcal{X} = L^\infty$.

4. Convex Risk Measures on Ordered Linear Spaces

In this section, we characterize convex risk measures apart from the representation of coherent risk measures provided in the last section. First, we assume that $\mathcal{C}$ is a cone of $\mathcal{X}$, such that $\mathcal{C} \subseteq \mathcal{A}$, where as usual $\mathcal{A} = \{X \in \mathcal{X} \mid \rho(X) \leqslant 0\}$ is the acceptance set of a convex risk measure ρ satisfying cash additivity with respect to e, which is supposed to be an internal point of $\mathcal{C}$.

Theorem 3. *Given a cone $\mathcal{C} \subseteq \mathcal{A}$ of acceptable positions, containing e as an internal point, a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ admits the following dual representation:*

$$\rho(X) = \sup_{\pi \in \mathcal{B}} \{\pi(-X) - a(\pi)\},$$

where $\mathcal{B} := \{\pi \in \mathcal{X}' \mid \pi \in \mathcal{C}^\circ, \pi(e) = 1\}$, $\mathcal{C}^\circ$ is the polar wedge of the acceptance cone $\mathcal{C}$ and $a(\pi) = \sup_{X \in \mathcal{X}} \{\pi(-X) - \rho(X)\}$.

We refer to the mapping $a : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ as a *penalty* function.

Proof. First, we have that

$$\begin{aligned} a(\pi) &= \sup_{X\in\mathcal{X}}\{\pi(-X) - \rho(X)\} \geqslant \sup_{X\in\mathcal{A}}\{\pi(-X) - \rho(X)\} \\ &\geqslant \hat{a}(\pi) = \sup_{X\in\mathcal{A}} \pi(-X), \end{aligned}$$

alike in [Th.5] in [19]. If $X \in \mathcal{A}$, then $a(\pi) \geqslant \hat{a}(\pi)$ for any $\pi \in \mathcal{B}$. To show the reverse inequality, we consider $X' = X + \rho(X)e$. Thus, $X' \in \mathcal{A}$ and

$$\hat{a}(\pi) \geqslant \pi(-X') = \pi(-X) - \rho(X),$$

for every $X \in \mathcal{X}$. This implies $\hat{a}(\pi) \geqslant a(\pi)$ for any $\pi \in \mathcal{B}$. From the definition of the penalty function a, we obtain that

$$\rho(Y) \geqslant \sup_{\pi\in\mathcal{B}}\{\pi(-Y) - a(\pi)\},$$

for any $Y \in \mathcal{X}$. To show that an equality holds for any $Y \in \mathcal{X}$, we suppose that there exists some $Y_0 \in \mathcal{X}$ such that

$$\rho(Y_0) > \sup_{\pi\in\mathcal{B}}\{\pi(-Y_0) - a(\pi)\}.$$

This inequality implies there exists some $m \in \mathbb{R}$ for which

$$\rho(Y_0) > m > \sup_{\pi\in\mathcal{B}}\{\pi(-Y_0) - a(\pi)\}.$$

Hence, $\rho(Y_0 + m\ e) > 0$, implying $\{Y_0 + m\ e\}$ is not a subset of $\mathcal{A}$. Now, e is an internal point of $\mathcal{C}$, thus it is also an internal point of $\mathcal{A}$, which is a convex subset of $\mathcal{X}$. The singleton $\{Y_0 + m\,e\}$ is also a convex subset of $\mathcal{X}$. The Eidelheit's separation theorem implies the existence of some $g \neq 0$, lying in the algebraic dual $\mathcal{X}'$ of $\mathcal{X}$, such that $g(Y_0 + m\,e) \geqslant g(X)$, for every $X \in \mathcal{A}$. Hence $g'(Y_0 + m\,e) \leqslant g'(X)$, for any $X \in \mathcal{A}$, where $g' := -g$. Then:

$$g'(-Y_0 - m\,e) \leqslant a(g') = \sup_{X\in\mathcal{A}} g'(-X).$$

The values of g' are positive on $\mathcal{A}$ and consequently on $\mathcal{C}$. Then, $g' \in \mathcal{C}^\circ$, and by normalization, we get $g' \in \mathcal{B}$, provided that $g'(e) = 1$. Now, assuming that there exists some R such that $g'(R) < 0$, where $R \in \mathcal{A}$, we have that $g'(tR) = tg'(R)$, for any $t \in \mathbb{R}$, and if $t \to +\infty$, the above separation inequality is violated. Hence, such an R does not exist. By the separation argument with $g'(e) = 1$, we obtain

$$g(-Y_0 - m\,e) = g'(-Y_0) - m \leqslant a(g') = \sup_{X\in\mathcal{A}} g'(-X),$$

which is a contradiction since $g'(-Y_0) - a(g') \leqslant m$. Hence,

$$\sup_{\pi\in\mathcal{B}}\{\pi(-Y_0) - a(\pi)\} \leqslant m < m,$$

which is again a contradiction for any $m \in \mathbb{R}$, as a consequence of the assumption on the existence of Y_0. Eventually, the dual representation is valid for any $X \in \mathcal{X}$, and the proof is complete. □

Incidentally, applying Theorem 3, we get another interesting result. Recall that any real-valued convex function f on a convex subset $\mathcal{S}$ of a linear space $\mathcal{X}$ may be defined on the whole $\mathcal{X}$ by setting it to $+\infty$ on $\mathcal{X} \setminus \mathcal{S}$. Thus, if the effective domain $\{X \in \mathcal{X} \mid f(X) < +\infty\}$ is nonempty and f nowhere takes the value $-\infty$, then it is called *proper*.

Theorem 4 (Fenchel-Moreau Type)**.** *Let $\mathcal{X} \neq \varnothing$ be a partially ordered linear space. Define a mapping $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ being proper convex, the level set $\mathcal{A}_f := \{X \in \mathcal{X} \mid f(X) \leqslant 0\}$ being convex and containing a cone $\mathcal{C}$, the vector $e \in \mathcal{X}$ being an internal point of $\mathcal{C}$. Then f admits the following representation:*

$$f(X) = \sup_{\pi \in \mathcal{B}} \{\pi(-X) - a(\pi)\},$$

where $\mathcal{B}$ and the penalty term a are defined as in Theorem 3.

Proof. Follows from Theorem 3. □

Theorem 4 is an algebraic version of the classical Fenchel-Moreau theorem for the dual representation of convex risk measures, where the above domain $\mathcal{X}$ should be turned into a locally convex topological ordered linear space with topologically dual $\mathcal{X}^*$. Moreover, in applying the classical Fenchel-Moreau Theorem to the dual representation of convex risk measures, the well-known extended Namioka theorem is useful; see, for example, [Th.2.2] in [23]. But in the present paper, we do not require that $\mathcal{X}$ be endowed with a locally convex topology; hence, Theorem 4 above is sufficient for the dual characterization of convex risk measures in our algebraic setting and, in addition, does not require any assumption of monotonicity on ρ. We also notice that our result of the Fenchel-Moreau type provides an extension of [Th.2.4] in [23] on L^p spaces without the assumption of $\|\cdot\|_p$-lower semicontinuity. We end this section by verifying the properties satisfied by the proposed convex risk measure. For the converse of Theorem 3, we have the following proposition, whose proof is omitted.

Proposition 1. *A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ that admits the representation*

$$\rho(X) = \sup_{\pi \in \mathcal{B}} \{\pi(-X) - a(\pi)\}, \tag{3}$$

is a convex risk measure, where $\mathcal{C} \subseteq \mathcal{A}$ is a cone of acceptable positions, containing e as an internal point, $\mathcal{B} := \{\pi \in \mathcal{X}' \mid \pi \in \mathcal{C}^\circ, \pi(e) = 1\}$ $\mathcal{C}^\circ$ is the polar wedge of the acceptance cone $\mathcal{C}$, and the penalty function $a : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ is given by $a(\pi) = \sup_{X \in \mathcal{X}} \{\pi(-X) - \rho(X)\}$.

As in the special cases $\mathcal{X} = L^p$ for $p \in [1, +\infty)$ developed in the literature, coherent and convex risk measures on general ordered linear spaces differ by a penalty function. The financial meaning of the penalty function is that an additional capital loading is needed in order to *secure* some financial position X. Namely, the convex, non-coherent risk measures correspond to more conservative economic capital requirements for any financial position. Nevertheless, a monetary risk measure ρ defined on a more general ordered linear space $\mathcal{X}$, which is cash additive, decreasing monotone, and convex, is represented as $\rho(X) = \sup_{\pi \in \mathcal{B}} \{\pi(-X) - a(\pi)\}$. Hence, assuming a zero penalty function, $a(\pi) = 0$, implies ρ becomes positively homogeneous and thus a coherent risk measure. To see why this is true, observe that $\pi(-\lambda X) - a(\pi) = \lambda \pi(-X)$, for every $\lambda \geqslant 0$, so taking the supremum over the linear functionals $\pi \in \mathcal{B}$ does the job. Moreover, as for the classical Lebesgue spaces, we can show that the addition of positive homogeneity making ρ coherent would imply $a(\pi) = 0$ or $a(\pi) = +\infty$. First, we note that by the convexity and the positive homogeneity of ρ, together with decreasing monotonicity, it follows normalization $\rho(0) = 0$. Next, consider those linear functionals $\pi \in \mathcal{B}$ for which $\rho(X) \geqslant \pi(-X)$, each $X \in \mathcal{X}$. Evaluating the penalty function over them yields

$$a(\pi) = \sup_{X \in \mathcal{X}} \{\pi(-X) - \rho(X)\} \geqslant \pi(0) - \rho(0) = 0$$

if and only if $a(\pi) = 0$, since obviously $0 \leqslant \pi(-X) - \rho(X) \leqslant 0$. On the other hand, pick any linear functional $\pi \in \mathcal{B}$ such that $\rho(X) \not\geqslant \pi(-X)$ for some $X \in \mathcal{X}$. It follows from the positive homogeneity of ρ that the mapping $X \mapsto \pi(-X) - \rho(X)$ is positive homogeneous. As a byproduct, for those strictly positive real numbers λ, we get:

$$a(\pi) \geqslant \sup_{\lambda>0}\{\pi(-\lambda X) - \rho(\lambda X)\} = \sup_{\lambda>0} \lambda\{\pi(-X) - \rho(X)\} = +\infty.$$

This confirms how the penalty function equals the indicator $\imath$ of the convex subset

$$\mathcal{B}' \subseteq \{\pi \in \mathcal{X}' \mid \rho(X) \geqslant \pi(-X), \text{ for every } X \in \mathcal{X}\},$$

where in the current algebraic setting no closeness is required, that is, $\imath_{\mathcal{B}'}(\pi) = 1$ if $\pi \in \mathcal{B}'$ and $\imath_{\mathcal{B}'}(\pi) = +\infty$ otherwise. It is worth noting that the penalty function is also convex and proper, since $\mathrm{dom}(\rho) \neq \varnothing$ and also $\mathcal{B}$ is non-empty. The dual representation of coherent risk measures given by Theorem 2 together with representation (2) clearly show that we must restrict $\mathcal{B}'$ to be a subset of $\mathcal{B}$ instead. This is due to cash additivity and decreasing monotonicity. Explicitly, picking $\pi \in \mathcal{B}'$ entails

$$\begin{aligned} -1 &= \rho(e) \geqslant \pi(-e) \\ 1 &= \rho(-e) \geqslant \pi(e), \end{aligned}$$

implying $\pi(e) = 1$, where we used the cash additivity of ρ. If $X \geqslant 0$ then by decreasing monotonicity $\rho(X) \leqslant 0$, and again picking $\pi \in \mathcal{B}'$ yields $\pi(-X) \leqslant 0$, leading to imposing $\pi \in \mathcal{C}^\circ$ over all the financial positions $X \in \mathcal{X}$.

Example 1. *Assume $\mathcal{X} = L^1$ with algebraic dual $\mathcal{X}'$ and let $e = \mathbf{1} := \mathbf{I}_\Omega$ be the unit order interpreted as a risk-free position. Then, let the set of linear functionals in Theorems 2 and 3 be given by $\mathcal{B} := \left\{\pi \in \mathcal{X}^* \,\middle|\, \pi = \frac{\mathrm{dQ}}{\mathrm{dP}},\ \mathsf{Q} \sim \mathsf{P}\right\}$, where as usual Q are probability measures equivalent to the original P, and consider the mapping $X \mapsto \mathsf{E}^{\mathsf{P}}(XZ)$ for any fixed Radon-Nikodým derivatives $Z = \frac{\mathrm{dQ}}{\mathrm{dP}}$ where $\pi(X) = \mathsf{E}(XZ)$ together with $\pi(\mathbf{1}) \equiv \mathsf{E}^{\mathsf{P}}(\mathbf{1}Z) = 1$. Next, remove the $\|\cdot\|_1$-norm topology or the weak topology $\sigma(L^1, L^\infty)$ and instead assume the topology induced by the Ky Fan metric*

$$\alpha(X,Y) := \inf\{\epsilon \geqslant 0 \mid \mathsf{P}(|X-Y| > \epsilon) \leqslant \epsilon\}, \quad \text{for every } X, Y \in L^1.$$

Recall that this metric is not induced by a norm and that it metrizes convergence in probability in the larger space L^0. Recall also how convergence in probability does not imply $\|\cdot\|_1$-convergence. Hence, the dual representation $\rho(X) = \sup_Z\{\mathsf{E}^{\mathsf{P}}(-XZ) - a(Z)\}$ for convex risk measures, or coherent risk measures in the case $a(Z) = 0$, still holds thanks to our representation results even if the Radon-Nikodým derivatives $Z = \frac{\mathrm{dQ}}{\mathrm{dP}}$ in the dual set $\mathcal{B}$ no longer produce a continuous linear functional $\pi(-X) = \mathsf{E}^{\mathsf{P}}(-XZ)$. In fact, $XY \in L^1$, but the expectation functional is not α-continuous, but Theorems 2 and 3 do not require any continuity.

Example 2. *We call incomplete market a subspace $\mathcal{M} \subseteq \mathcal{X}$ which is not $\{0\}$ or the entire space of financial positions, where $\mathcal{M}'$ denotes its algebraic dual. We want to define coherent and convex risk measures in incomplete markets. Thanks to Theorems 2 and 3 and supposing that e is an internal point of the cone $\mathcal{A}$, which is the acceptance set of a risk measure $\rho : \mathcal{M} \to \mathrm{I\!R}$, namely $\mathcal{A} \equiv \mathcal{A}_\rho$, and that $\mathcal{C} \subseteq \mathcal{A}$ is a cone, we obtain the following:*

- *If ρ is coherent, then $\rho(X) = \sup_{\pi\in\mathcal{B}} \pi(-X)$, where $\mathcal{B} = \{\pi \in \mathcal{M}' \mid \pi \in \mathcal{A}^\circ, \pi(e) = 1\}$.*
- *If ρ is convex, then $\rho(X) = \sup_{\pi\in\tilde{\mathcal{B}}}\{\pi(-X) - a(\pi)\}$, where $\tilde{\mathcal{B}} = \{\pi \in \mathcal{M}' \mid \pi \in \mathcal{C}^\circ, \pi(e) = 1\}$, and $a(\pi) = \sup_{X\in\mathcal{X}}\{\pi(-X) - \rho(X)\}$.*

Observe that $\mathcal{C}$ is a cone, either in the case where ρ is coherent or in the case where ρ is a convex risk measure. In the case of a coherent risk measure, $\mathcal{C} = \mathcal{A}_\rho$ is the acceptance set of ρ. In the case of a convex risk measure, $\mathcal{C}$ is a cone, which is a subset of the acceptance set $\mathcal{A}_\rho$. In both of the cases, $e \in \mathcal{X}$ is an internal point of $\mathcal{C}$.

5. Financial Positions as Interior and Internal points

Referring to the case of financial positions $\mathcal{X} = L^1$ is mainly based on the insights in [7] characterized by a one-to-one correspondence between law-invariant convex risk measures on L^∞ and L^1; see also [8]. Furthermore, the choice $\mathcal{X} = L^1$ is also supported by empirical evidence underlying the stylized fact that portfolio losses have a finite expectation. On the other hand, there is well-understood empirical evidence that many traded positions have a distribution with fat tails and statistical models for profits and losses such as stable distributions with a suitable (typically greater than one) value of the stability index have a finite mean; see [Ch.3, Sec.1a] in [24]. See in addition [Ch.3] in [25] for further details on stylized facts about profits and losses.

Our algebraic approach calls for $(L^1, \geqslant)$ considered as a vector lattice or a *Riesz space*: The real linear space L^1 is equipped with the usual partial ordering $Y(\omega) \geqslant X(\omega)$, for pointwise P-a.s. every scenario $\omega \in \Omega$, which in addition has a supremum $\sup\{X, Y\}$ for each pair of financial positions. Observe that the partial ordering $(L^1, \geqslant)$ is compatible with the algebraic structure of L^1. Moreover, $\sup\{X, Y\}(\omega) := \sup\{X(\omega), Y(\omega)\}$ for any scenario $\omega \in \Omega$ and similarly $|X|(\omega) := |X(\omega)|$. Eventually, $\inf\{X, Y\} = -\sup\{-X, -Y\}$. The absolute value $|X|$ is defined as $X \vee (-X) := \max\{-X, X\}$. Equivalently, the absolute value in a vector lattice is defined as $X^+ - X^-$, where $X^+ = X \vee 0$ and $X^- = (-X) \vee 0$. This algebraic structure is more general than that of a topological vector lattice, namely L^1 endowed with the norm $\|X\|_1 := \mathsf{E}(|X|)$, for every position $X \in L^1$. In fact, we do not use the Banach lattice feature of L^1, where the $\|\cdot\|_1$-norm is also complete. (This norm is monotone with respect to $|X|$ and entails L^1 as a locally convex-solid space, see [Ch.9] in [16]). The positive cone of the positions is $L^1_+ := \{X \in L^1 \,|\, X \geqslant 0\}$. The following negative result is important.

Proposition 2. *Assuming $(\Omega, \mathscr{F}, \mathsf{P})$ is a complete, atomless probability space, then the set of $\|\cdot\|_1$-interior points of L^1_+ is empty.*

Proof. Since in this case $L^1(\Omega, \mathscr{F}, \mathsf{P})$ is infinite-dimensional, the proof is a consequence of [Th.4.4.4] in [14]. This is true since L^1_+ is a well-based cone. $\square$

With the above negative result in mind, we next recall that the order interval $I_X = \cup_{n=1}^{\infty}[-nX, nX]$ equals the solid subspace generated by $X \in L^1 \setminus \{0\}$, which is L^1-dense, then yielding X as a quasi-interior point. We define the *principal ideal*

$$E_X := \{Y \in L^1 \,|\, \exists\, t > 0 \text{ such that } |Y| \leqslant t|X|\},$$

generated by X. The former definition requires topological arguments, while the latter is purely algebraic and depends upon the vector lattice structure of L^1 with respect to the pointwise P-a.s. partial order. The following results are remedies to the issue given in Proposition 2.

Proposition 3. *Let the dual pair $\langle L^1, L^\infty \rangle$ be given. Then, the set of quasi-interior points of L^1_+ is nonempty.*

Theorem 5. *Any quasi-interior point X of L^1_+ is an internal point of L^1_+.*

Remark 1. *Any point in L^1_+ valued strictly positively by a non-zero continuous linear functional in L^∞ (consider the weak topology $\sigma(L^1, L^\infty)$) is a quasi-interior point in L^1_+, and Proposition 3 yields that the order interval I_X and the principal ideal E_X do coincide for any non-zero $X \in L^1_+$.*

The two results above are proved in [10] and are reported here to highlight the importance of using internal points in vector optimization problems as the ones underlying our algebraic robust representation of coherent and convex risk measures.

We close this section with a result on continuity of ρ when $\mathcal{X} = L^p$ for $p \in [1, +\infty)$, which is different from the usual strong, weak, or weak* continuity requirements for convex risk measures; see [Def.3.1] in [23] and [26]. Recall that a sequence $(X_n)_{n \in \mathbb{N}}$ in a Riesz space $\mathcal{X}$ is order convergent to a vector $X \in \mathcal{X}$, written $X_n \xrightarrow{o} X$, if there exits another sequence $(Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$ that is decreasing to the zero vector $Y_n \geqslant Y_{n+1}$ for each $n \in \mathbb{N}$, also written $Y_n \downarrow$, such that $0 = \inf\{Y_n\} := \bigvee_{n \in \mathbb{N}} Y_n$, written $Y_n \downarrow 0$, and with $|X_n - X| \leqslant Y_n$ for all $n \in \mathbb{N}$.

Theorem 6 (Order Continuity)**.** *Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ be a convex risk measure, for $p \in [1, +\infty)$, and let $(X_n)_{n \in \mathbb{N}} \subseteq L^p$ be a sequence such that $|X_n| \leqslant Y$ in the* P*-a.s. sense for every $n \in \mathbb{N}$, where $Y \in L^p_+$. If $X_n \xrightarrow{o} X$, then $\rho(X_n) \to \rho(X)$.*

Proof. The sequence $(X_n)_{n \in \mathbb{N}}$ is order-bounded with respect to the partial ordering, which makes L^p a vector lattice. By [Lemma 8.17] in [16], we have $X_n(f) = f(X_n) \to X(f) = f(X)$ for any f lying in the algebraic dual of $(L^p)'$. The convergence is valid for any $\pi \in \mathcal{B}$ as defined in Theorem 3. Hence, $\pi(-X_n) \to \pi(-X)$ for any $\pi \in \mathcal{B}$. From the dual representation of a convex risk measure in Theorem 3, we get $\rho(X_n) \to \rho(X)$ since $\pi(-X_n) - a(\pi) \to \pi(-X) - a(\pi)$ for any $\pi \in \mathcal{B}$. $\square$

6. Acceptability Indices on Ordered Linear Spaces

In this section, we provide another representation of acceptability indices of performance using our algebraic framework. First, we consider the following:

Definition 1. *A mapping $\mathsf{a}\iota : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ is an acceptability index of performance if it satisfies the following properties.*

- *Quasi-concavity: given a pair $X, Y \in \mathcal{X}$ and for every $\lambda \in [0,1]$ such that $\mathsf{a}\iota(X) \geqslant x$ and $\mathsf{a}\iota(Y) \geqslant x$ one has*
$$\mathsf{a}\iota(\lambda X + (1-\lambda)Y) \geqslant x. \tag{4}$$
- *Monotonicity: for any $X, Y \in \mathcal{X}$*
$$X \leqslant Y \ \ \mathsf{P}\text{-}a.s. \Longrightarrow \mathsf{a}\iota(X) \leqslant \mathsf{a}\iota(Y). \tag{5}$$
- *Scale invariance: for every $\lambda > 0$ and $X \in \mathcal{X}$*
$$\mathsf{a}\iota(\lambda X) = \mathsf{a}\iota(X). \tag{6}$$

Differently from [11], we do not use a fourth axiom by calling for the upper semicontinuity of $\mathsf{a}\iota$ in the appropriate topology because $\mathrm{dom}(\mathsf{a}\iota) = \mathcal{X}$ is only a partially ordered linear space.

Remark 2. *Acceptable positions at level $x \in \mathbb{R}_+$ form a convex superlevel set*
$$\mathcal{A}_x := \left\{ X \in \mathcal{X} \,\middle|\, \mathsf{a}\iota(X) \geqslant x \right\}, \tag{7}$$
which is a natural requirement for any performance measure.

Quasi-concavity is equivalent to the convexity of $\mathcal{A}_x$ for each $x \in \mathbb{R}_+$: any diversified position performs at least as well as its components. Moreover, $\mathsf{a}\iota$ is increasing monotone, and Y is at least as acceptable as X, provided that the latter is dominated by the former. By scale invariance, $\mathcal{A}_x$ is actually a cone for every $x \in \mathbb{R}_+$; the level of acceptance remains the same whenever we scale positions.

Now, with a little abuse of notation for each $x \in \mathbb{R}_+$, we let $\mathcal{A}_x = \{X \in \mathcal{X} \mid \rho_x(X) \leqslant 0\}$ be the cone of acceptability for the coherent risk measure ρ_x such that $\rho_x(X) = \inf\{m \in \mathbb{R} \mid X + m\, e \in \mathcal{A}_x\}$, where e is an internal point of $\mathcal{A}_x$. By Theorem 2 we have

$$\rho_x(X) = \sup_{\pi \in \mathcal{B}_x} \pi(-X) = - \inf_{\pi \in \mathcal{B}_x} \pi(X), \tag{8}$$

with

$$\mathcal{B}_x := \{\pi \in \mathcal{X}' \mid \pi \in \mathcal{A}_x^\circ,\ \pi(e) = 1\}, \text{ and } \mathcal{A}_x^\circ := \{\pi \in \mathcal{X}' \mid \pi(X) \geqslant 0, \text{ for all } X \in \mathcal{X}\}$$

the latter being the polar wedge of $\mathcal{A}_x$. Thus, by construction, we have a family of sets of linear functionals $(\mathcal{B}_x)_{x \in \mathbb{R}_+}$ supporting the representation of each coherent risk measure ρ_x, with corresponding acceptance set $\mathcal{A}_x$. Observe that every acceptance set $\mathcal{A}_x$ in (7) is clearly decreasing in x. Moreover, if ρ_x is increasing in x, then $\mathcal{B}_x$ is obviously increasing in x too, by definition (8), as well as $\mathcal{A}_x^\circ$. In fact, defining

$$\rho_x(X) := \inf\{m \in \mathbb{R} \mid \mathsf{a}\iota(X + m\, e) \geqslant x\}, \text{ for every } x \in \mathbb{R}_+, X \in \mathcal{X} \tag{9}$$

yields ρ_x increasing in x, since

$$\{m \in \mathbb{R} \mid \mathsf{a}\iota(X + m\, e) \geqslant y\} \subset \{m \in \mathbb{R} \mid \mathsf{a}\iota(X + m\, e) \geqslant x\}, \text{ for all } 0 \leqslant x \leqslant y,$$

and as usual, e is an internal point of the cone $\mathcal{A}_x$. The equivalence of the two definitions of $\mathcal{A}_x$, in terms of $\mathsf{a}\iota(X)$ and in terms of ρ_x, now follows from the lemma below, which shows the equivalence $\mathsf{a}\iota(X) \geqslant x \Longleftrightarrow \rho_x(X) \leqslant 0$.

Proposition 4. *Let ρ_x be defined as in (9) through an acceptability index $\mathsf{a}\iota$. Then, ρ_x is a coherent risk measure on the ordered linear space $\mathcal{X}$, for every $x \in \mathbb{R}_+$. Vice-versa, let $(\rho_x)_{x \in \mathbb{R}_+}$ be a family of coherent risk measures on the ordered linear space $\mathcal{X}$, which is increasing in x. This is an acceptability index of performance (we take $\sup \varnothing = 0$) defined on the same space.*

Proof. (First part) For the $m \in \mathbb{R}$ and $x \in \mathbb{R}_+$ conditions, $\mathsf{a}\iota(X + m\, e) \geqslant x$ is equivalent to $X + m\, e \in \mathcal{A}_x \subset \mathcal{X}$, and $X \leqslant Y$ P-a.s. together with $X \in \mathcal{A}_x$ do imply $Y \in \mathcal{A}_x$. To check decreasing monotonicity of ρ_x, take $x \in \mathbb{R}_+$ and pick $X, Y \in \mathcal{X}$ such that $X \geqslant Y$ P-a.s. By increasing monotonicity of $\mathsf{a}\iota$, we have

$$\mathsf{a}\iota(Y + m\, e) \leqslant \mathsf{a}\iota(X + m\, e), \quad \text{for every } m \in \mathbb{R}.$$

Thus we deduce $\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + m\, e) \geqslant x\} \supset \{m \in \mathbb{R} \,|\, \mathsf{a}\iota(Y + m\, e) \geqslant x\}$, and taking the infimum of both sets, we get

$$\rho_x(X) := \inf\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + m\, e) \geqslant x\} \leqslant \{m \in \mathbb{R} \,|\, \mathsf{a}\iota(Y + m\, e) \geqslant x\} := \rho_x(Y).$$

To show cash additivity, for every $x \in \mathbb{R}_+$ and $X \in \mathcal{X}$, we note that

$$\begin{aligned} \rho_x(X + c\, e) &:= \inf\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + c\, e + m\, e) \geqslant x\} \\ &= \inf\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + (c + m)\, e) \geqslant x\} \\ &= \inf\{c + m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + (c + m)\, e) \geqslant x\} - c \\ &= \inf\{r \in \mathbb{R} \,|\, \mathsf{a}\iota(X + r\, e) \geqslant x\} - c \\ &=: \rho_x(X) - c. \end{aligned}$$

To show positive homogeneity, it suffices to call for the scale invariance of $\mathsf{a}\iota$. To check for subadditivity, pick $m_1, m_2 \in \mathbb{R}$ such that $\mathsf{a}\iota(X + m_1\, e) \geqslant x$ and $\mathsf{a}\iota(Y + m_2\, e) \geqslant x$, for every $X, Y \in \mathcal{X}$ and $x \in \mathbb{R}_+$. By the quasi-concavity of $\mathsf{a}\iota$, for every $\lambda \in [0, 1]$, we have

$$\mathsf{a}\iota(\lambda X + \lambda m_1\, e + (1-\lambda) Y + (1-\lambda) m_2\, e) \geqslant x,$$

at the same acceptability level x. Choosing $\lambda = \frac{1}{2}$ and using the scale invariance of the acceptability index again entails

$$\mathsf{a}\iota(X + Y + (m_1 + m_2)\, e) \geqslant x.$$

Therefore, the scalar $m_1 + m_2$ belongs to the set $\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + Y + m\, e) \geqslant x\}$, and it is greater than or equal to the infimum over the same set, which in turn is just $\rho_x(X+Y)$. This inequality holds true for all m_1 and all m_2 belonging to $\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(X + m\, e) \geqslant x\}$ and to $\{m \in \mathbb{R} \,|\, \mathsf{a}\iota(Y + m\, e) \geqslant x\}$, respectively. As a consequence, taking the infimum with respect to m_1 and then with respect to m_2, we get $\rho_x(X) + \rho_x(Y) \geqslant \rho_x(X+Y)$.
(Second part) Let $x \geqslant 0$, then by decreasing the monotonicity of ρ_x in x; we have

$$\rho_x(X) \leqslant \rho_x(Y), \quad \text{for all } \; X, Y \in \mathcal{X} \; \text{ such that } \; X \geqslant Y, \; \text{P-a.s.}$$

For any $x_0 \in \{x \in \mathbb{R}_+ \,|\, \rho_x(X) \leqslant 0\}$, we also have $\rho_{x_0}(X) \leqslant 0$,, which, together with the monotonicity, entails $\rho_{x_0}(X) \leqslant \rho_{x_0}(Y) \leqslant 0$, for all $X \geqslant Y$ P-a.s. As a consequence, we have the set inclusion

$$\{x \in \mathbb{R}_+ \,|\, \rho_x(X) \leqslant 0\} \supset \{x \in \mathbb{R}_+ \,|\, \rho_x(Y) \leqslant 0\},$$

and, taking the supremum of both sets, the increasing monotonicity of $\mathsf{a}\iota$ is proved. To check quasi-concavity of $\mathsf{a}\iota$, we first choose a pair $X, Y \in \mathcal{X}$ such that $\mathsf{a}\iota(X) \geqslant x_0$ and $\mathsf{a}\iota(Y) \geqslant x_0$ whenever $x_0 \in (0, +\infty)$. By decreasing monotonicity of ρ_x, we have $\rho_x(X) \leqslant \rho_{x_0}(X) \leqslant 0$ and $\rho_x(Y) \leqslant \rho_{x_0}(Y) \leqslant 0$, for all $x < x_0$. This combined with the positive homogeneity of ρ_x entails

$$\rho_x(\lambda X) = \lambda \rho_x(X) \leqslant 0, \quad \rho_x((1-\lambda)Y) = (1-\lambda)\rho_x(Y) \leqslant 0,$$

for every $\lambda \in [0,1]$. Moreover, by subadditivity of ρ_x and again for every $x < x_0$, we additionally have

$$\rho_x(\lambda X + (1-\lambda)Y) \leqslant 0,$$

which entails $\sup\{x \in \mathbb{R}_+ \,|\, \rho_x(\lambda X + (1-\lambda)Y) \leqslant 0\} \geqslant x_0$. Eventually, this implies $\mathsf{a}\iota(\lambda X + (1-\lambda)Y) \geqslant x_0$ and quasi-concavity easily follows. The scale invariance of $\mathsf{a}\iota$ follows immediately from the positive homogeneity of ρ_x. $\square$

With all this in mind, we are ready to state:

Theorem 7. *A mapping $\mathsf{a}\iota : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ is an acceptability index of performance if and only if there exists a family $(\mathcal{B}_x)_{x \in \mathbb{R}_+}$ of subsets $\mathcal{B}_x \subset \mathcal{X}'$ increasing in x with*

$$\mathsf{a}\iota(X) = \sup\{x \in \mathbb{R}_+ \mid X \in \mathcal{A}_x\} \tag{10}$$

and $\mathcal{A}_x = \{X \in \mathcal{X} \mid \inf_{\pi \in \mathcal{B}_x} \pi(X) \geqslant 0\}$, where $\inf \varnothing = +\infty$ and $\sup \varnothing = 0$.

The proof of Theorem 7 does not present any serious difficulty with respect to that of the original representation in [Th.1] in [11], but we provide it for the sake of comparison with our algebraic setting. However, since the special choice $\mathcal{X} = L^\infty$ in [Th 1] in [11] is referred to as a Riesz space, Theorem 7 extends it to general partially ordered linear space.

Proof. (If part) Assuming the mapping $\mathsf{a}\iota$ is defined as in (10), checking it satisfies monotonicity and scale invariance appearing in Definition 1 is trivial. To check for quasi-concavity, pick $X, Y \in \mathcal{X}$ with $\mathsf{a}\iota(X), \mathsf{a}\iota(Y)$ both $\geqslant x$. Assuming $x_0 < x$ in $\mathbb{R}_+$ we have $\pi(X), \pi(Y)$ both $\geqslant 0$, for every linear functional $\pi \in \mathcal{B}_{x_0} \subset \mathcal{B}_x$. Since also $\pi(\lambda X + (1-\lambda)Y) \geqslant 0$, with $\lambda \in [0,1]$, then $\mathsf{a}\iota(\lambda X + (1-\lambda)Y) \geqslant x$.

(Only If part) Let the mapping $\mathsf{a}\iota$ be an acceptability index as in Definition 1, and fix $x \in \mathbb{R}_+$ such that $\mathcal{A}_x$ is the convex superlevel set given by (7), which is decreasing in x. Now, defining

$$\rho_x(X) = \inf\{m \in \mathbb{R} \mid X + m\, e \in \mathcal{A}_x\},$$

is equivalent to having $\mathsf{a}\iota(X + m\, e) \geqslant x$ as in (9). Hence, the mapping ρ_x is a coherent risk measure on $\mathcal{X}$, each $x \in \mathbb{R}_+$. It remains to show the existence of a family $(\mathcal{B}_x)_{x \in \mathbb{R}_+}$ of subsets $\mathcal{B}_x \subset \mathcal{X}'$ increasing in x, such that the representation

$$\mathsf{a}\iota(X) = \sup\left\{x \in \mathbb{R}_+ \middle| \inf_{\pi \in \mathcal{B}_x} \pi(X) \geqslant 0\right\}$$

holds. But as showed above, ρ_x is coherent, and thanks to Theorem 2, we can let $\mathcal{B}_x \subset \mathcal{X}'$ be the set of linear functionals supporting the representation of ρ_x given by (1), for $x \in \mathbb{R}_+$. If for such acceptability level x the superlevel set $\mathcal{A}_x$ given by (7) is equal to the entire ordered linear space $\mathcal{X}$, then $\mathcal{B}_x$ must be the empty set so that

$$\inf\{m \in \mathbb{R} \mid X + m\, e \in \mathcal{A}_x\} = -\infty = -\inf_{\pi \in \mathcal{B}_x} \pi(X) = -\inf \varnothing.$$

For the arbitrariness of x, we get $\rho_x(X) = -\inf_{\pi \in \mathcal{B}_x} \pi(X)$ for every $x \in \mathbb{R}_+$. Furthermore, $\mathcal{A}_x$ decreases in x so that $\rho_x(X)$ increases in x, which yields the sets of linear functionals $\mathcal{B}_x$ increase in x as required. Since Proposition 4 gives $\mathsf{a}\iota(X) \geqslant x$ if and only if $\rho_x(X) \leqslant 0$, we are done. □

The proof of Theorem 7 can be easily extended to the case of quasi-concave acceptability indices (see Definition 2 below) using an x-increasing family of convex risk measures $(\rho_x)_{x \in \mathbb{R}_+}$ by first recognizing that (10) can be written

$$\mathsf{a}\iota(X) = \sup\left\{x \in \mathbb{R}_+ \mid \rho_x(X) \leqslant 0\right\}, \tag{11}$$

then requiring additionally that $\mathsf{a}\iota(0) = +\infty$ (meaning that a zero position is always acceptable), $\mathsf{a}\iota(c) = 0$ for any $c < 0$, and $\rho_x(c) = -c$ for any $c \in \mathbb{R}$, see [Prop.3] in [27] for the case $\mathcal{X} = L^\infty$ where additional continuity properties for both $\mathsf{a}\iota$ and ρ_x are imposed.

In the special case $\mathcal{X} = L^p$, for $p \in [1, +\infty)$, the order-type continuity of coherent risk measures as stated above, is sufficient to have acceptability indices of performance as upper semicontinuous maps on these L^p spaces for order-bounded sequences of financial positions.

Corollary 1. *Let $\mathsf{a}\iota : L^p \to \mathbb{R}_+ \cup \{+\infty\}$ be a mapping given by (10). Assume $(\rho_x)_{x \in \mathbb{R}_+}$ is a family of coherent risk measures on L^p, increasing in x and continuous in the sense of Theorem 6. Then, for an order-bounded sequence $(X_n)_{n \in \mathbb{N}} \subset L^p$ such that $X_n \xrightarrow{o} X$. Then $\mathsf{a}\iota(X) \geqslant \limsup_{n \to +\infty} \mathsf{a}\iota(X_n)$.*

Proof. By Theorem 6 and its proof together with Theorem 7, since $\rho_x(X_n) \to \rho(X)$ by the hypotheses $X_n \xrightarrow{o} X$, we have $\pi(-X_n) \to \pi(-X)$ if and only if $\pi(X_n) \to \pi(X)$ by linearity, where $\pi \in \mathcal{B}_x$ for a fixed $x \in \mathbb{R}_+$. Assuming $\mathsf{a}\iota(X_n) \geqslant x$, we may pick any real number $x_0 < x$ and get $\pi(X_n) \geqslant 0$, for every $n \in \mathbb{N}$ and all linear functionals $\pi \in \mathcal{B}_{x_0} \subseteq \mathcal{B}_x$. This entails $\pi(X) \geqslant 0$ and thus $\mathsf{a}\iota(X) \geqslant x$, which is upper semicontinuity. □

For coherent risk measures, Definition 1 and Theorem 7 are the adequate framework for performance measurement in the sense of conic finance, see [28] and the reference therein for a more detailed study of this topic.

When convex, non-coherent risk measures come into play, we need a slightly different definition of an index of performance than Definition 1. Moreover, we need to drop the scale invariance from the minimal properties an acceptability index of performance must satisfy and try to represent it without the direct intervention of some monetary (actually quasi-convex) risk measures, as we did using coherent risk measures in (10).

Definition 2. *A mapping* $a\iota : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ *is a quasi-concave acceptability index of performance if it satisfies quasi-concavity and increasing monotonicity.*

For such indices of performance, we have the following:

Theorem 8. *A quasi-concave acceptability index of performance* $a\iota : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ *admits the following 'min-max' representation:*

$$a\iota(X) = \inf_{\pi \in \mathcal{B}} \sup_{Y \in \mathcal{X}} \{a\iota(Y) \mid \pi(Y) \leqslant \pi(X)\}, \tag{12}$$

for every $X \in \mathcal{X}$*, with* $\mathcal{B} := \{\pi \in \mathcal{X}' \mid \exists\, X \in \mathcal{X} \text{ with } \pi(X) \geqslant 0, \pi(e) = 1\}$*, where the risk-free position* e *is an internal point of the positive cone* $\mathcal{X}_+$*.*

Observe that we do not need any acceptability system of positions in Theorem 8.

Proof. The reasoning is the same as in the proof of the topological-version borrowed from [Th.40] in [12]. Define $G(\pi, t) := \sup_{Y \in \mathcal{X}}\{a\iota(Y) \mid \pi(Y) \leqslant t\}$, for every $t \in \mathbb{R}$ and $\pi \in \mathcal{X}'$. For a fixed position $X \in \mathcal{X}$ we obviously have $X \in \{Y \in \mathcal{X} \mid \pi(Y) \leqslant \pi(X)\}$, and this together with the definition of $G(\pi, t)$ imply

$$\inf_{\pi \in \mathcal{X}'} G(\pi, \pi(X)) \geqslant a\iota(X).$$

It remains to show the reverse inequality and eventually that the infimum can be taken over normalized positive linear functionals $\pi \in \mathcal{B} \subseteq \mathcal{X}'_+$, where

$$\mathcal{X}'_+ := \{\pi \in \mathcal{X}' \mid \exists\, X \in \mathcal{X} \text{ with } \pi(X) \geqslant 0\}$$

is the positive cone of linear functionals. Beforehand, take $\epsilon > 0$ and let

$$\mathcal{C}_\epsilon := \{Y \in \mathcal{X} \mid a\iota(Y) \geqslant a\iota(X) + \epsilon\},$$

which is a convex superlevel set since $a\iota$ is quasi-concave. Since $\mathcal{C}_\epsilon \cap \{X\} = \varnothing$, where $\{X\} \subseteq \mathcal{X}$ is also convex, we can invoke Edelheit's separation theorem and find some non-zero linear functional $\tilde{\pi} \in \mathcal{X}'$ such that

$$\tilde{\pi}(Y) \geqslant \tilde{\pi}(X), \quad \text{for all } Y \in \mathcal{C}_\epsilon.$$

Now, the complement of $\mathcal{C}_\epsilon$, i.e., $\mathcal{C}^c_\epsilon := \{Y \in \mathcal{X} \mid a\iota(Y) < a\iota(X) + \epsilon\}$, includes the set of those positions $Y \in \mathcal{X}$ for which $\tilde{\pi}(Y) < \tilde{\pi}(X)$, for a fixed $X \in \mathcal{X}$, due to the weak separation above. As byproduct:

$$\begin{aligned} a\iota(X) &\leqslant \inf_{\pi \in \mathcal{X}'} G(\pi, \pi(X)) \leqslant G(\tilde{\pi}, \tilde{\pi}(X)) \\ &\leqslant \sup_{\substack{Y \in \mathcal{X} \\ a\iota(Y) < a\iota(X) + \epsilon}} a\iota(Y) \leqslant a\iota(X) + \epsilon. \end{aligned}$$

The above inequalities show that $a\iota(X) = \inf_{\pi \in \mathcal{X}'} G(\pi, \pi(X))$. Now the infimum can be taken over $\mathcal{X}'_+$, because $\tilde{\pi} \in \mathcal{X}'_+$ as in the proof of [Th.40] in [12], mainly because $G(\pi, \pi(X)) = G(\lambda\pi, \pi(\lambda X))$ for every linear functional π and $\lambda \neq 0$. Eventually, since $\inf_{\pi \in \mathcal{B}} G(\pi, \pi(X)) \geqslant \inf_{\pi \in \mathcal{X}'_+} G(\pi, \pi(X))$ we are done. □

Note that Theorem 8 above and its proof are the algebraic version of [Th.40] in [12], where originally the set $\mathcal{X}$ of financial positions is assumed to be a locally convex topological linear space.

Remark 3. *The construction of a quasi-concave acceptability index of performance in Theorem 8 is mainly based on the insights of [4]. Relaxing the convexity property, the authors provide a dual representation of quasi-convex risk measures based on acceptability systems, generalizing those considered in [11]. Specifically, any acceptability set* $\mathcal{A}_x$ *posses additional features than convexity and gives rise to a quasi-convex* ρ *via the association* $\mathcal{A}_x := \{X \in \mathcal{X} \mid \rho(X) \leqslant x\}$ *if and only if* $\rho(X) := \inf\{x \in \mathbb{R} \mid X \in \mathcal{A}_x\}$. *A main difference with the acceptability families in [11] is that now* $\mathcal{A}_x$ *is increasing in* x. *Further, any such set is monotone for a fixed* x *in the sense of the membership relation and is right-continuous; see [Def 3] in [4]. Quasi-convex risk measures were motivated by the concept of cash-subadditivity in [5] and, in general, by the problem of finding a dual representation of quasi-convex and lower semicontinuous functions; see, for example, [29]. Thus, when* $\mathcal{X}$ *is a locally convex topological linear space with dual* $\mathcal{X}^*$ *the robust representation of a quasi-convex, monotone decreasing and a lower semicontinuous risk measure is given (uniquely) by* $\rho(X) = \sup_\pi R(\pi, \pi(-X))$, *where* $R(\pi, t) := \inf\{m \in \mathbb{R} \mid g_{min}(\pi, m) \geqslant t\}$ *and* $g_{min}(\pi, m) := \sup\{\pi(-X) \mid \rho(X) \leqslant m\}$. *The supremum is taken over all the normalized nonnegative functionals* π. *It is worth noting that the function* R *possesses some interesting properties; see [4].*

As said before, Theorem 7 can be given for convex risk measures ρ_x, which are also quasi-convex, in order to get the dual representation of quasi-concave acceptability indices like that in Equation (11). On the other hand, consider the function

$$\alpha(X) := \hat{\pi}(X) + \frac{1}{\lambda} \inf_{\pi \in \mathcal{B}} \pi(X), \quad \lambda > 0,$$

where $\hat{\pi} \in \mathcal{X}'$ and $\mathcal{B}$ is the set of dual functionals in Theorem 2 for coherent risk measures on partially ordered linear spaces. It is easily seen that $\alpha(X)$ is a monotone increasing and concave (thus quasi-concave) acceptability index, which is by no means scale invariant, and then it cannot be represented via Theorem 7. Observe that the equivalence $\alpha(X) \geqslant x$ if and only if $\rho_x(X) \leqslant 0$ is violated in this case since the condition

$$\hat{\pi}(X) + \frac{1}{\lambda} \inf_{\pi \in \mathcal{B}} \pi(X) - x \geqslant 0$$

cannot be reduced to $\rho_x(X) = \inf_{\pi \in \mathcal{B}_\S} \pi(X) \geqslant 0$, neither if we take $\mathcal{B}_x := \frac{1}{1+x}\{\hat{\pi}\} + \frac{x}{1+x}\mathcal{B}$, for every $x \in \mathbb{R}_+$, where the indexed sets $\mathcal{B}_x$ of linear functionals are understood as supporting the representation (10).

7. Conclusions

Convex duality theory is actively used in many financial problems. In this paper, we focus on the measurement of market risk associated with traded financial positions represented by a set of profit and loss random variables, considered only as a partially ordered linear space. By weak separation of convex sets properly characterized as acceptable financial positions, we deduce robust representations of coherent and convex risk measures without requiring neither any topological structure of the underlying space nor continuity of the functionals used in the dual representation. A further application of these results to acceptability indices of performance emphasizes the relevance of our algebraic approach to risk and reward measurement. A further result establishing the equivalence between the representation given in Theorem 8 and a representation like that in Theorem 7 is on the agenda.

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