mathematics

Article

A Generalized Residual-Based Test for Fractional Cointegration in Panel Data with Fixed Effects

Saidat Fehintola Olaniran [1,†], Oyebayo Ridwan Olaniran [2,*,†], Jeza Allohibi [3], Abdulmajeed Atiah Alharbi [3] and Mohd Tahir Ismail [4]

[1] Department of Statistics and Mathematical Sciences, Faculty of Pure and Applied Sciences, Kwara State University, Malete 1530, Nigeria; saidat.olaniran@kwasu.edu.ng
[2] Department of Statistics, Faculty of Physical Sciences, University of Ilorin, Ilorin 1515, Nigeria
[3] Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawara 42353, Saudi Arabia; jlohibi@taibahu.edu.sa (J.A.); aahharbi@taibahu.edu.sa (A.A.A.)
[4] School of Mathematical Sciences, Universiti Sains Malaysia, Pulau Pinang 11800, Malaysia; m.tahir@usm.my
[*] Correspondence: olaniran.or@unilorin.edu.ng
[†] These authors contributed equally to this work.

Abstract: Asymptotic theories for fractional cointegrations have been extensively studied in the context of time series data, with numerous empirical studies and tests having been developed. However, most previously developed testing procedures for fractional cointegration are primarily designed for time series data. This paper proposes a generalized residual-based test for fractionally cointegrated panels with fixed effects. The test's development is based on a bivariate panel series with the regressor assumed to be fixed across cross-sectional units. The proposed test procedure accommodates any integration order between $[0, 1]$, and it is asymptotically normal under the null hypothesis. Monte Carlo experiments demonstrate that the test exhibits better size and power compared to a similar residual-based test across varying sample sizes.

Keywords: fractional cointegration; residual-based test; panel data model; fixed effects; asymptotic theory

MSC: 62F15; 62G20; 62G08

check for updates

Citation: Olaniran, S.F.; Olaniran, O.R.; Allohibi, J.; Alharbi, A.A.; Ismail, M.T. A Generalized Residual-Based Test for Fractional Cointegration in Panel Data with Fixed Effects. *Mathematics* **2024**, *12*, 1172. https://doi.org/10.3390/math12081172

Academic Editor: Heng Lian

Received: 15 March 2024
Revised: 4 April 2024
Accepted: 12 April 2024
Published: 13 April 2024

1. Introduction

Numerous studies have explored panel data analysis and cointegration either independently or in tandem. Various methods, including residual-based and spectral-based approaches, have been devised to tackle issues like unit roots, cross-sectional dependence, and heterogeneity. Panel data analysis has garnered considerable attention owing to its wide-ranging applicability across fields such as epidemiology, demography, finance, and economics [1–3]. Recently, researchers in engineering and reliability analysis have expanded panel data analysis to encompass count data [4]. These data comprise repeated observations of discrete events over time for each individual or unit. Overall, discussions in panel data encompass topics like developing robust statistical models, devising efficient estimation techniques, and addressing challenges related to data heterogeneity, unit root problems, and correlation structures [5–7].

Ref. [8] highlighted the increasing attention given to unit root problems in panel data and the consequent identification of cointegration relationships among variables. The existing panel cointegration techniques were initially designed for balanced panel data with moderate time and cross-sectional units. However, in scenarios involving large time and cross-sectional units with the potential for long memory, conventional panel cointegration tests are inadequate [9–11]. The presence of long memory often implies fractional mean reversion, suggesting equilibrium occurs over fractional time periods. Therefore, there is a need to explore fractional cointegration or equilibrium mean reversion within the context of panel data.

There have been numerous fractional cointegration tests developed within the realm of time-series analysis [12]. These tests are typically grouped into two categories: spectral-density-based and residual-based. Ref. [13] developed a residual-based test utilizing the residuals of the multivariate fractional cointegration common-components model with varying memory parameters. Ref. [14] compared semiparametric tests of fractional cointegration, evaluating nine tests for both spectral-density- and residual-based approaches. They found that several methods yield significantly different results when correlated short-run components are present. Moreover, when applied to common-component models rather than triangular systems, these methods exhibit varied power. Notably, there is a significant difference in the power of the tests between the two models.

In empirical studies, Ref. [15] investigated the memory of exchange rates, while ref. [16] explored the dynamics of interest rate futures markets and stock market prices. Both studies identified evidence of fractional cointegration in stock market prices, achieving satisfactory results under the assumption that the observations are $I(1)$ processes. Subsequently, testing for fractional cointegration was extended to fractionally integrated processes. Ref. [17] proposed a test based on the joint local Whittle estimation of all parameters, which eliminates the possibility of the two underlying series having equal integration orders. In [18], the authors developed a Hausman-type test to detect fractional cointegration, with the additional assumption that the cointegration error is nonstationary. Ref. [19] proposed a Hausman-type test for time series with equal integration orders and no cointegration, which involves determining a bandwidth. In panel settings, Ref. [9] studied a large cross-sectional and time unit heterogeneous panel data model with fixed effects. Their approach allows for cross-sectional dependency, persistency, and fractionally integrated errors. In addition, the methodology provides a general treatment for stationary and nonstationary indicators. Monte Carlo simulation showed that it works effectively in practice. Ref. [20] proposed an extension of the Generalized Method of Moments (GMM) for a fixed-effect, fractionally integrated panel model. Both [9]'s and [20]'s studies assumed that the fixed-effect parameter fizzled out in the long run. The method of [9] is limited in that fractional cointegration is assumed in a panel system if the estimate of mean reversion for the series is greater than that of the residuals [9,21]. This approach is expected to inflate the type I error, as there is an increased possibility of many rejections of mean reversion when there is none [22,23].

Furthermore, Ref. [24] introduced a new panel cointegration test that is robust to nonlinearity, structural breaks, and cross-sectional dependency. The proposed method is a bootstrap panel cointegration test called the Fractional Frequency Flexible Fourier Form for Panel Cointegration Test, and it was empirically illustrated by testing the Feldstein–Horioka paradox for 15 Asian countries. It was discovered that Indonesia, Philippines, Bangladesh, Japan, Thailand, and China are not among the countries that generate cointegration in the cross-section. The study is limited, however, as it only considered country-specific cointegration rather than the panel of interest.

In summary, the findings of the finite sample properties of various fractional cointegration tests for time-series data reviewed by [14] revealed the exemplary applicability of the residual-based methods of [13,25] for stationary systems under the common component model assumption. Also, the study showed that the class of tests with low power under the alternative hypothesis of fractional cointegration are [19,26]. However, the methods of Ref. [17,27,28] are resistant to short-term correlation and are commonly applied due to their simplistic framework. Of all the works conducted on fractional integration and cointegration, none have developed a test particularly for panel data.

Thus, in this paper, a generalized fractional cointegration testing procedure is developed, where both fractionally and non-fractionally cointegrated models are considered such that the observed series with cross-sections and cointegrating error are both fractional and non-fractional processes. In addition, a residual-based testing procedure for generalized fractional cointegration is proposed, and its performance is compared to an existing fractionally cointegrated test when $0 \leq d \leq 1$.

The proposed test involves modifying the residual-based test proposed by [25], which involves two integrated time series y_t and x_t, where the observed series are $I(d)$ processes, the regression residual $\epsilon_t = y_t - \beta x_t$ is an $I(\gamma)$, γ, and d can be real-valued. The test includes traditional cointegration as a special case. Ref. [25] constructed a test statistic which has an asymptotic standard normal distribution under the null hypothesis of no cointegration using a consistent estimate of d and γ obtained from x_t and the residual ϵ_t.

2. Wang et al. (2015) [25] Fractional Cointegration Test

Let x_t, y_t, where $t = 1, 2, 3, \ldots, T$, represent two processes, both characterized by $I(d)$. Ref. [29] demonstrated that for a given scalar $\beta \neq 0$, the linear combination $\boldsymbol{\epsilon}_t = y_t - \beta x_t$ also exhibits $I(d)$ behavior, potentially with ϵ_t being $I(d-b)$, where $b > 0$. Consequently, given real numbers d and b, the elements of a vector c_t are considered to be cointegrated of order d, b, denoted as $c_t \sim CI(d, b)$, if the following hold:

i. All elements of c_t follow $I(d)$.
ii. There exists a non-zero vector α such that $s_t = \alpha' c_t \sim I(\gamma) = I(d-b)$, where $b > 0$.

Here, α and s_t are termed the cointegration vector and error, respectively [30]. Also, it is assumed that $d \geq b$ such that $(d-b) \geq 0$. Therefore, a simple bivariate system of fractionally cointegrated x_t and y_t processes is defined as follows:

$$\begin{aligned} y_t &= \beta x_t + (1-L)^{-\gamma} \epsilon_{1t} \\ x_t &= (1-L)^{-d} \epsilon_{2t} \end{aligned} \tag{1}$$

for positive t. The vector $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \epsilon_{2t})'$ now represents a bivariate zero-mean covariance stationary $I(0)$ process, where $\beta \neq 0$ and $\gamma < d$. In Equation (1), both x_t and y_t are $I(d)$, and $\epsilon_{1t} = y_t - \beta x_t$ is $I(\gamma)$. The lag operator L is defined as $Ly_t = y_{t-1}$ and the difference operator Δ^{-d} is obtained using $(1-L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j$, where $\binom{d}{j} = \frac{d!}{j!(d-j)!}$. Unlike the standard $CI(1,1)$ cointegration, the memory parameter d remains unknown in fractionally cointegrated systems and requires estimation. The corresponding hypotheses to assess whether the two processes exhibit fractional cointegration are as follows:

Hypothesis 0 (H0): *x_t and y_t are not fractionally cointegrated $(d = \gamma)$,*

Hypothesis 1 (H1): *x_t and y_t are fractionally cointegrated $(d > \gamma)$.*

The fractional cointegration test proposed by [25] is based on the second component x_t of the process $\boldsymbol{X}_t = (y_t, x_t)$ and the associated residuals $\epsilon_t = y_t - \beta x_t$. Ref. [25] devised a simple t-like test statistic that employs the spectral density of the component x_t, denoted as $\hat{f}_{22} = \frac{1}{2\pi T} \sum_{t=1}^{T} (\Delta^{\hat{d}} x_t)^2$, and the fractional cointegration parameter γ of residual ϵ_{2t}. Thus, the statistic is expressed as follows:

$$F_w = \frac{\sum_{t=1}^{T} \Delta^{\hat{\gamma}} x_t}{\sqrt{2\pi T \hat{f}_{22}}} \xrightarrow{H_0} N(0,1). \tag{2}$$

The method requires $d > 0.5$ so that a consistent cointegrating vector β can be estimated using Ordinary Least Squares (OLS). The first step in the construction of the test is to estimate the cointegration parameter β using

$$\hat{\beta}_{ols} = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2} \tag{3}$$

and then obtain the residuals $\hat{\epsilon}_{1t} = y_t - \hat{\beta}_{ols}x_t$. The values $\hat{d}$ and $\hat{\gamma}$ are later estimated from the series x_t and $\hat{\epsilon}_{1t}$, respectively, using the method of [31]. Correspondingly, the differenced series $\Delta^{\hat{d}}x_t$ and $\Delta^{\hat{\gamma}}x_t$ are then calculated.

3. Generalized Residual-Based Fractional Cointegration Test for Fixed-Effect Panel Model

Suppose we have a balanced fixed-effect panel model with n being the number of cross-sectional units and T being the time period such that $N = n \times T$ is the total sample size. The model is given as follows:

$$\begin{aligned} y_{it} &= \mu_i + \beta x_{it} + (1-L)^{-\gamma}\epsilon_{1it} \\ x_{it} &= (1-L)^{-d}\epsilon_{2it}. \end{aligned} \tag{4}$$

where β is the cointegration parameter and is assumed to be constant over i cross-sectional units, μ_i is the fixed-effect coefficient for the ith cross-sectional units, x_t and y_t represent simple bivariate processes denoting independent and dependent variables in the panel model, which are fractionally cointegrated if we can establish the following **Assumptions**:

A$_1$. x_t and y_t are both $I(d)$ with $0 \leq d \leq 1$ and $\epsilon_{1it} = y_t - \beta x_t$ is $I(\gamma)$;

A$_2$. The vector $\boldsymbol{\epsilon}_{it} = (\epsilon_{1it}, \epsilon_{2it})'$ is a bivariate zero mean covariance stationary $I(0)$ process which is independent across i, $\beta \neq 0$, and $\gamma < d$;

A$_3$. The vector μ_i fizzles out in the long-run such that $\mu_i = 0$ as $N \to \infty$.

The corresponding hypotheses to assess whether the two processes exhibit fractional cointegration are as follows:

Hypothesis 0 (H0): *x_{it} and y_{it} are not fractionally cointegrated $(d = \gamma)$.*

Hypothesis 1 (H1): *x_{it} and y_{it} are fractionally cointegrated $(d > \gamma)$.*

Notice that we can rewrite (4) as follows:

$$\begin{aligned} y_{it} - \hat{\mu}_i &= \beta x_{it} + (1-L)^{-\gamma}\epsilon_{1it} \\ x_{it} &= (1-L)^{-d}\epsilon_{2it}. \end{aligned} \tag{5}$$

where $\hat{\mu}_i = \bar{y}_i = T^{-1}\sum_{t=1}^{T} y_{it}$ and denotes $z_{it} = y_{it} - \hat{\mu}_i$, such that we have

$$\begin{aligned} z_{it} &= \beta x_{it} + (1-L)^{-\gamma}\epsilon_{1it} \\ x_{it} &= (1-L)^{-d}\epsilon_{2it}. \end{aligned} \tag{6}$$

It is clear that (6) is the demean transformed version of (1), which is reduced to the original time series model in (1) with cross-sectional parameter μ_i factored out. According to [13], when $d \leq 0.5$, a consistent β in (6) can be estimated using the Tapered Narrow Band Least Square (TNBLS). The TNBLS [13] procedure involves the estimation of a complex-valued taper q_t defined as

$$q_t = \frac{1}{2}\left(1 - \exp^{-i2\pi(t-1/2)T^{-1}}\right), t = 1, 2, \ldots, T. \tag{7}$$

The next step involves obtaining the discrete tapered Fourier transform of the series η_t and the cross-periodogram using

$$\omega'_{\eta,j} = \left(2\pi\sum_{t=1}^{T}|q_t^{p-1}|^2\right)^{-0.5}\sum_{t=1}^{T} q_t^{p-1}\eta_t \exp^{-i\lambda_j t}, \tag{8}$$

$$I'_{\eta\bar{\eta},j} = \omega'_{\eta,j}\bar{\omega}'_{\eta,j} \tag{9}$$

respectively. The averaged tapered periodogram obtained using m bandwidth is given by

$$\hat{F}'_{\eta\bar{\eta},j}(m) = 2\pi T^{-1} \sum_{j=1}^{m} \Re I'_{\eta\bar{\eta},j}, 1 \leq m \leq \frac{T}{2}. \tag{10}$$

Therefore, the estimate of a consistent long-memory parameter β in (4) when $d \leq 0.5$ is

$$\hat{\beta}_m = \frac{\hat{F}'_{xz}(m)}{\hat{F}'_{xx}(m)} \tag{11}$$

where $m \geq 1$ is fixed. If instead of keeping m fixed, we substitute $m = T/2$ and avoid differencing and tapering, we obtain the ordinary least squares (OLS) estimator [13].

Ref. [25] established that if $d > 0.5$, the OLS estimator is consistent, and otherwise, it is inconsistent. In order to develop a generalized test statistic that is usable for all d's in the range of $[0, 1]$, we developed a piecewise estimator for β for the two possible situations, that is, for $d \leq 0.5$ and $d > 0.5$. Thus, our proposed estimator for the long-memory parameter β when $0 < d < 1$ is denoted by $\hat{\beta}_{mix}$, and it can be estimated using

$$\hat{\beta}_{mix} = \begin{cases} \frac{\hat{F}'_{xz}(m)}{\hat{F}'_{xx}(m)} & 0 < d \leq 0.5 \\ \\ \frac{\sum_{i,t=1}^{N} x_{it} z_{it}}{\sum_{i,t=1}^{N} x_{it}^2} & 0.5 < d \leq 1 \end{cases} \tag{12}$$

Theorem 1. *For the fixed-effect fractional cointegrated panel model defined in (4) satisfying A1 and A2, the long-memory parameter can be estimated with (12). Thus, the modified test statistic* $M_w = \frac{\sum_{i,t=1}^{N} \Delta^{\hat{\gamma}} x_{it}}{\sqrt{2\pi N \hat{K}_{22}}}$, *where* $\hat{K}_{22} = \frac{1}{2\pi N} \sum_{i,t=1}^{N} (\Delta^{\hat{d}} x_{it})^2$, *converges;* $M_w \xrightarrow{d} N(0, 1)$ *under* H_0 *and diverges under* H_1.

Proof. It is required to show that

$$M_w = \frac{\sum_{i,t=1}^{N} \Delta^{\hat{\gamma}} x_{it}}{\sqrt{2\pi N \hat{K}_{22}}} \xrightarrow{H_0} N(0, 1). \tag{13}$$

Equation (13) can be rewritten as

$$M_w = \frac{S_N}{\hat{k}_{22}\sqrt{N}}. \tag{14}$$

where $S_N = \sum_{i,t=1}^{N} \Delta^{\hat{\gamma}} x_{it}$, $\hat{k}_{22} = \sqrt{2\pi \hat{K}_{22}}$, since S_N is the sum of N independently and identically distributed random variables. Recall that the moment-generating function $Q(u) = E(e^{ux})$ of S_N and correspondingly M_w can be defined as

$$Q_{S_N}(u) = \left(Q(u)\right)^N;$$

$$Q_{M_w}(u) = \left[Q\left(\frac{u}{\hat{k}_{22}\sqrt{N}}\right)\right]^N.$$

Now, computing the Taylor's series expansion of $Q(u)$ around 0 leads to

$$Q(u) = Q(0) + Q^{'}(0)u + \frac{1}{2}Q^{''}(0)u^2 + rem = 1 + \frac{1}{2}K_{22}u^2 + \mathcal{O}(u^3),$$

since $Q(0) = E(e^0) = 1$, $Q^{'}(0) = \frac{d}{du}E(e^{ux}) = E(x) = 0$ (x is assumed to be the differenced x_{it} whose mean is zero under H_0), $Q^{''}(0) = \frac{d^2}{du^2}E(e^{ux}) = Var(x) = Var(\Delta^{\hat{\gamma}} x_{it} | H_0) = K_{22}$. Thus,

$$Q\left(\frac{u}{\hat{k}_{22}\sqrt{N}}\middle|H_0\right) = 1 + \frac{1}{2}K_{22}\left(\frac{u}{k_{22}\sqrt{N}}\right)^2 + \mathcal{O}\left[\left(\frac{u}{k_{22}\sqrt{N}}\right)^3\right]$$
$$= 1 + \frac{u^2}{2N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right)$$

$$Q_{M_w}(u|H_0) = \left[1 + \frac{u^2}{2N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right)\right]^N \xrightarrow{N\to\infty} e^{u/2}.$$

The moment-generating function of a Gaussian random variable $\varsigma \sim N(0,1)$ with mean 0 and variance 1 is defined as $Q_\varsigma(u) = E(e^{u\varsigma}) = e^{u/2}$. Thus, $M_w \xrightarrow{d} N(0,1)$ under H_0. On the other hand, under H_1, $E(e^{ux}) = Var(x) = Var(\Delta^{\hat{\gamma}}x_{it}|H_1) \neq K_{22}$. Let $Var(\Delta^{\hat{\gamma}}x_{it}|H_1) = G_{22}$; then, we have

$$Q\left(\frac{u}{\hat{k}_{22}\sqrt{N}}\middle|H_1\right) = 1 + \frac{1}{2}G_{22}\left(\frac{u}{k_{22}\sqrt{N}}\right)^2 + \mathcal{O}\left[\left(\frac{u}{k_{22}\sqrt{N}}\right)^3\right]$$
$$= 1 + \frac{u^2}{2N}\left(\frac{G_{22}}{\hat{K}_{22}}\right) + \mathcal{O}\left[\frac{1}{N^{3/2}}\left(\frac{G_{33}}{\hat{k}_{22}}\right)\right].$$

$$Q_{M_w}(u|H_1) = \left\{1 + \frac{u^2}{2N}\left(\frac{G_{22}}{\hat{K}_{22}}\right) + \mathcal{O}\left[\frac{1}{N^{3/2}}\left(\frac{G_{33}}{\hat{k}_{22}}\right)\right]\right\}^N \xrightarrow{N\to\infty} e^{u/2\left(\frac{G_{22}}{\hat{K}_{22}}\right)}.$$

□

4. Simulation Study

We consider the following balanced panel model with fixed effects, where n denotes the count of cross-sectional units and T represents the total time units, resulting in a total sample size of $N = n \times T$. The model is as follows:

$$\begin{aligned} y_{it} &= \mu_i + \beta x_{it} + (1-L)^{-\gamma}\epsilon_{1it} \\ x_{it} &= (1-L)^{-d}\epsilon_{2it}. \end{aligned} \tag{15}$$

where $\mu_i = (5, 10, 15, 20, 25)$ are the panel intercepts across the units $i = 1, 2, \dots, 5$. Monte Carlo experiments are conducted to examine the finite sample performance of the tests. Let $(y_{it}, x_{it})'$ be generated from model (1) with $\beta = 1$, $\epsilon_{it} = (\epsilon_{1it}, \epsilon_{2it})'$ being a Gaussian white noise with $E(\epsilon_{it}) = 0$, $Var(\epsilon_{1it}) = Var(\epsilon_{2it}) = 1$ and $Cov(\epsilon_{1it}, \epsilon_{2it}) = \rho$. We consider cases with $\rho = 0.0, 0.5$ and sample sizes $N = 500, 1250, 2500$ corresponding to $T = 100, 250, 1000$. Similar approaches were used in [14,22,25,32–39].

The test statistic simulation procedure follows the same three-step approach used in [25], which is as follows:

Step 1: Estimate $\hat{d}$ using x_{it} by the method of [31].
Step 2: Compute $\hat{K}_{22} = \frac{1}{2\pi N}\sum_{i,t=1}^{N}(\Delta^{\hat{d}}x_{it})^2$.
Step 3: Compute the estimate of the long-memory parameter using $\hat{\beta}_{mix}$ and use it to estimate $\hat{\epsilon}_{1it} = y_{it} - \hat{\beta}_{mix}x_{it}$. Again, estimate $\hat{\gamma}$ using ϵ_{1it} by the method used in step 1. Thus, the test statistic M_w is computed. Each statistic is replicated 5000 times so as to estimate the empirical type 1 error rates at 1%, 5%, and 10%.

The empirical type 1 error rates and power are reported in Table 1. In Table 1, it was observed that the original [25] F_w test undersized the nominal size when $d < 0.5$, as expected. However, when $d > 0.5$, its empirical type 1 error rates compete with the modified test. On the other hand, the modified test's empirical type 1 error rates are slightly oversized and converge to the nominal size as $N \to \infty$, irrespective of the d values and the correlation values ρ. Overall, the empirical type 1 error rates returned by the modified test

M_w are relatively closer to the nominal size than the original [25] F_w test. This establishes the validity and applicability of the proposed M_w test for fractionally cointegrated panels.

Table 1. Empirical type I error rate for original [25] test (F_w) and proposed modified [25] test (M_w) at varying levels of $d = \gamma$, ρ, and sample sizes N.

			$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.10$		
	$d = \gamma$	**Method/N**	**500**	**1250**	**2500**	**500**	**1250**	**2500**	**500**	**1250**	**2500**
	0.3	F_w	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000
		M_w	0.015	0.014	0.013	0.066	0.055	0.052	0.118	0.115	0.106
$\rho = 0.0$	0.6	F_w	0.028	0.018	0.015	0.069	0.063	0.048	0.107	0.094	0.091
		M_w	0.026	0.016	0.011	0.079	0.069	0.056	0.145	0.120	0.115
	0.8	F_w	0.036	0.027	0.018	0.083	0.073	0.065	0.146	0.125	0.116
		M_w	0.015	0.013	0.010	0.068	0.064	0.059	0.128	0.112	0.101
	1	F_w	0.026	0.023	0.016	0.076	0.070	0.065	0.122	0.117	0.112
		M_w	0.008	0.011	0.013	0.059	0.051	0.051	0.114	0.104	0.101
	0.3	F_w	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
		M_w	0.012	0.014	0.011	0.059	0.054	0.052	0.123	0.111	0.103
$\rho = 0.5$	0.6	F_w	0.023	0.017	0.016	0.060	0.054	0.050	0.107	0.091	0.097
		M_w	0.020	0.012	0.013	0.084	0.067	0.062	0.148	0.120	0.116
	0.8	F_w	0.038	0.022	0.017	0.089	0.070	0.067	0.140	0.126	0.109
		M_w	0.014	0.011	0.010	0.060	0.059	0.057	0.127	0.119	0.109
	1	F_w	0.030	0.020	0.019	0.076	0.068	0.060	0.123	0.113	0.113
		M_w	0.008	0.010	0.010	0.046	0.044	0.043	0.110	0.105	0.102

For the power results in Table 2, we considered $\gamma < d$. The powers of the two tests approach 1 when N increases and when the effect size $\gamma - d$ is large. Again, the powers of the M_w are in most cases closer to 1 than F_w. The better performance of M_w observed in Tables 1 and 2 can be attributed to model adequacy. While M_w was developed using a panel data model assumption, F_w was developed using a time series model. Since the model simulated is a panel one, M_w is expected to be better than F_w.

Table 2. Empirical power for original [25] test (F_w) and proposed modified [25] test (M_w) at varying levels of d, γ, $\rho = 0.0$, and sample sizes N.

			$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.10$		
	γ	**Method/N**	**500**	**1250**	**2500**	**500**	**1250**	**2500**	**500**	**1250**	**2500**
$d = 1.0$	0.9	F_w	0.146	0.169	0.175	0.256	0.258	0.307	0.332	0.362	0.369
		M_w	0.169	0.186	0.197	0.286	0.310	0.352	0.390	0.410	0.434
	0.6	F_w	0.765	0.815	0.877	0.851	0.890	0.919	0.856	0.900	0.910
		M_w	0.837	0.860	0.899	0.889	0.907	0.937	0.906	0.919	0.929
	0.3	F_w	0.913	0.959	0.967	0.944	0.952	0.980	0.949	0.966	0.981
		M_w	0.969	0.983	0.984	0.974	0.994	0.994	0.982	0.992	0.990
	0.0	F_w	0.935	0.961	0.968	0.957	0.969	0.976	0.950	0.971	0.979
		M_w	0.993	0.997	1.000	0.995	0.998	1.000	0.999	0.997	1.000
$d = 0.9$	0.6	F_w	0.638	0.694	0.759	0.721	0.747	0.826	0.721	0.747	0.826
		M_w	0.737	0.743	0.803	0.795	0.796	0.850	0.795	0.796	0.850
	0.3	F_w	0.856	0.887	0.939	0.887	0.924	0.952	0.887	0.924	0.952
		M_w	0.950	0.971	0.980	0.961	0.977	0.986	0.961	0.977	0.986
$d = 0.6$	0.3	F_w	0.264	0.335	0.421	0.392	0.457	0.530	0.392	0.457	0.530
		M_w	0.728	0.739	0.813	0.788	0.793	0.847	0.788	0.793	0.847
	0.0	F_w	0.389	0.446	0.503	0.518	0.569	0.625	0.518	0.569	0.625
		M_w	0.966	0.973	0.977	0.978	0.978	0.979	0.978	0.978	0.979
$d = 0.4$	0.1	F_w	0.005	0.006	0.012	0.011	0.021	0.033	0.055	0.070	0.082
		M_w	0.712	0.735	0.803	0.815	0.819	0.868	0.817	0.820	0.870

5. A Fractional Cointegration Panel Model for Realized Industry and Market Volatilities in the U.S. Economy

The data used here were drawn from Yahoo Finance and Kenneth French's Data Library. Five industry portfolios (Cnsmr: Consumer Durables, Nondurables, Wholesale, Retail, and Some Services; Manuf: Manufacturing, Energy, and Utilities; HiTec: Business Equipment, Telephone and Television Transmission; Hlth: Healthcare, Medical Equipment, and Drugs; and Other: Mines, Construction, Building Materials, Transport, Hotels, Bus Services, Entertainment, Finance) in the U.S. economy spanning the time period from 2000 to 2019 ($N_t = 240$ months) were extracted from Kenneth French's Data Library. This dataset was employed to calculate the realized volatility within the industry. The market volatility data were sourced from Yahoo Finance for three composite portfolios (NYSE, NASDAQ, and AMEX). These market portfolios were consolidated to serve as constant input for the realized industry portfolios. The computation of returns followed the method outlined in [9].

We denote IV_{it} for $i = 1, 2, 3, 4, 5$ and $t = 1, \ldots, 240$ as industry volatility, and MV_{it} as market volatility. The corresponding fractional cointegrated panel model is formulated as follows:

$$\begin{aligned} IV_{it} &= \mu_i + \beta MV_{it} + \Delta^{-\gamma}\epsilon_{1it} \\ MV_{it} &= \Delta^{-d}\epsilon_{2it}. \end{aligned} \tag{16}$$

We conducted estimation for the fractional cointegration parameters d and γ using a bandwidth of $\eta_m = 0.75$, corresponding to $m = 240^{0.75} = 61$ for each industry as well as the pooled industries (panel). It is crucial to examine the equality of d across portfolios to ensure the validity of the pooling. We applied the tests proposed by [19] as cited in [40], yielding estimated results of $(T_{stat} = 0.38, p = 0.353)$. These findings indicate that the null hypothesis of the equality of d across different portfolios remains valid.

Table 3 presents the estimates of $\hat{d}$, $\hat{\gamma}$, $\hat{\beta}_{mix}$, and F_w, M_w tests of no fractional cointegration for the five industry portfolios and market average. The $\hat{\beta}_{mix} = \hat{\beta}_{ols}$ since $d_{market} = 0.55 > 0.5$, thus the TNBLS approach was not employed here. All the estimates of d's for both market and industry portfolios are all less than 1 indicating the validity of fractional integration for the U.S. volatilities. Furthermore, the F_w test showed that of the five industry portfolio volatilities, only HiTec is not fractionally cointegrated with market-realized volatility. Also, the F_w fractional panel cointegration test obtained by pooling all industries showed that there is no fractional panel cointegration $(p > 0.05)$ for the combined industries against the market. On the other hand, the M_w test showed that all five industry portfolio volatilities are not fractionally cointegrated with market-realized volatility. In addition, the fractional panel cointegration test obtained by adjusting for the fixed effect showed that there is also no fractional panel cointegration for the combined industries against the market. The results of M_w are more reliable compared to F_w, as all the individual fractional cointegration tests agree with the overall results obtained for the panel of industries.

Table 3. Estimates of $\hat{d}$, $\hat{\gamma}$, $\hat{\beta}_{mix}$, and F_w and M_w tests of no fractional cointegration for the five industry portfolios and market average.

	Market	Cnsmr	Manuf	HiTec	Hlth	Other	Panel
$\hat{d}$	0.55	0.55	0.52	0.61	0.46	0.74	0.54
$\hat{\gamma}$		0.20	0.42	0.87	0.32	0.34	0.52
$\hat{\beta}_{mix}$		0.75	0.98	1.11	0.68	1.26	0.96
$SE(\hat{\beta}_{mix})$		0.016	0.022	0.043	0.028	0.029	0.014
F_w		7.54	2.16	0.79	2.80	12.69	1.28
$p(>\lvert F_w\rvert)$		0.000	0.031	0.432	0.005	0.000	0.201
M_w		0.38	0.25	0.23	0.21	0.49	0.12
$p(>\lvert M_w\rvert)$		0.705	0.799	0.821	0.832	0.626	0.903

6. Conclusions

This paper introduces a generalized residual-based test for a fractionally cointegrated panel model with fixed effects. The test's development is centered on bivariate panel series y_{it} and x_{it}, where x_{it} is assumed to remain fixed across cross-sectional units. Similar to other fractional cointegration tests, both y_{it} and x_{it} are $I(d)$ and the residual $\epsilon_{it} = y_{it} - \beta x_{it}$ is $I(\gamma)$. The proposed test procedure accommodates any values of d and γ between $[0,1]$. The modified test, denoted as M_w, converges to an asymptotically normal distribution under the null hypothesis and diverges under the alternative. Compared to the test by [25], M_w demonstrates superior size and power across varying sample sizes and simulation conditions. Furthermore, its real-life application to realized industry and market volatilities for the U.S. economy showcases the practicality of the test. However, it is worth noting that the proposed method has limitations; it has not been tested for imbalanced panel data, and it assumes normal distribution of the model's error.

Author Contributions: Conceptualization, S.F.O. and M.T.I.; methodology, S.F.O.; software, S.F.O. and O.R.O.; validation, J.A. and A.A.A.; formal analysis, S.F.O. and O.R.O.; investigation, J.A. and A.A.A.; resources, S.F.O.; data curation, S.F.O.; writing—original draft preparation, S.F.O.; writing—review and editing, S.F.O., O.R.O., J.A. and A.A.A.; supervision, M.T.I.; funding acquisition, J.A. and A.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Taibah University (Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia), grant number 445-9-892.

Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

n	Total number of cross-sectional units.
i	A cross-sectional unit.
t	A unit of time.
T	Total time units in a time series.
N	Total sample size in panel data.
m	Time series bandwidth size.
x_t	Time series process x_t.
y_t	Time series process y_t.
η_t :	Time series process η_t.
d	Fractional integration order of the series.
γ :	Fractional integration order of the residuals.
$I(0)$	Integration at level.
$I(1)$	Integration at first difference.
$I(d)$	Integration at fractional order d.
$I(\gamma)$	Integration at fractional order γ.
$I(d,b)$	Cointegration order d,b.
β	Cointegration parameter.
ρ	Correlation between residuals.
$\hat{f}_{22}$	The spectral density of the component x_t.
L	Lag operator.
Δ	Differencing operator.
$\Gamma(.)$	Gamma function.
$\Re$	Set of real numbers.
q_t	Complex-valued taper.
$\omega'_{\eta,j}$	Discrete tapered Fourier transform of the series η_t and cross-periodogram.
$\hat{F}'_{\eta\bar{\eta},j}(m)$	The averaged tapered periodogram of the series η_t using m bandwidth.
F_w	Wang et al.'s (2015) [25] fractional cointegration test for time series data.
M_w	The proposed fractional cointegration test for panel data with fixed effects.

References

1. Ghojagh, H.M.N.; Agheli, L.; Dizaji, S.F.; Kabir, M.J. Economic development and health status in Iran: A panel data analysis. *Environ. Sustain. Indic.* **2023**, *17*, 100221.
2. Zaman, K. A note on cross-panel data techniques. *Latest Dev. Econom.* **2023**, *1*, 1–7.
3. Song, M.; Zheng, H.; Shen, Z. Whether the carbon emissions trading system improves energy efficiency–Empirical testing based on China's provincial panel data. *Energy* **2023**, *275*, 127465. [CrossRef]
4. Wang, W.; Cui, Z.; Chen, R.; Wang, Y.; Zhao, X. Regression analysis of clustered panel count data with additive mean models. *Stat. Pap.* **2023**, 1–22. [CrossRef]
5. Sarsar, L.; Echaoui, A. Empirical analysis of the economic complexity boost on the impact of energy transition on economic growth: A panel data study of 124 countries. *Energy* **2024**, *294*, 130712. [CrossRef]
6. Abujassar, M. CEO Duality, Managerial Ownership, Large Shareholders, and Firm Performance: An Analysis Using Panel Data Approaches. *Kurd. Stud.* **2024**, *12*, 5907–5917.
7. Wang, Q.; Yang, T.; Li, R. Does income inequality reshape the environmental Kuznets curve (EKC) hypothesis? A nonlinear panel data analysis. *Environ. Res.* **2023**, *216*, 114575. [CrossRef] [PubMed]
8. Westerlund, J. Testing for error correction in panel data. *Oxf. Bull. Econ. Stat.* **2007**, *69*, 709–748. [CrossRef]
9. Ergemen, Y.E.; Velasco, C. Estimation of Fractionally Integrated Panels with Fixed Effects and Cross-section Dependence. *J. Econom.* **2017**, *196*, 248–258. [CrossRef]
10. Abdi, A.H.; Warsame, A.A.; Sheik-Ali, I.A. Modelling the impacts of climate change on cereal crop production in East Africa: evidence from heterogeneous panel cointegration analysis. *Environ. Sci. Pollut. Res.* **2023**, *30*, 35246–35257. [CrossRef]
11. Dritsaki, M.; Dritsaki, C. R&D Expenditures on Innovation: A Panel Cointegration Study of the EU Countries. *Sustainability* **2023**, *15*, 6637. [CrossRef]
12. Malmierca-Ordoqui, M.; Gil-Alana, L.A.; Monge, M. Fractional cointegration between energy imports to the EURO area and exchange rates to the US dollar. *Empir. Econ.* **2024**, *66*, 859–882. [CrossRef]
13. Chen, W.W.; Hurvich, C.M. Semiparametric Estimation of Fractional Cointegrating Subspaces. *Ann. Stat.* **2006**, *34*, 2939–2979. [CrossRef]
14. Leschinski, C.; Voges, M.; Sibbertsen, P. A Comparison of Semiparametric Tests for Fractional Cointegration. *Stat. Pap.* **2020**, *62*, 1997–2030. [CrossRef]
15. Hassler, U.; Marmol, F.; Velasco, C. Residual log-periodogram inference for long-run relationships. *J. Econom.* **2006**, *130*, 165–207. [CrossRef]
16. Dittmann, I. Residual-Based Tests For Fractional Cointegration: A Monte Carlo Study. *J. Time Ser. Anal.* **2000**, *21*, 615–647. [CrossRef]
17. Robinson, P.M. Diagnostic Testing for Cointegration. *J. Econom.* **2008**, *143*, 206–225. [CrossRef]
18. Marmol, F.; Velasco, C. Consistent Testing of Cointegrating Relationships. *Econometrica* **2004**, *72*, 1809–1844. [CrossRef]
19. Robinson, P.M.; Yajima, Y. Determination of Cointegrating Rank in Fractional Systems. *J. Econom.* **2002**, *106*, 217–241. [CrossRef]
20. Robinson, P.M.; Velasco, C. Efficient inference on fractionally integrated panel data models with fixed effects. *J. Econom.* **2015**, *185*, 435–452. [CrossRef]
21. Ergemen, Y.E. System estimation of panel data models under long-range dependence. *J. Bus. Econ. Stat.* **2019**, *37*, 13–26. [CrossRef]
22. Olaniran, S.F.; Ismail, M.T. A Comparative Analysis of Semiparametric Tests for Fractional Cointegration in Panel Data Models. *Austrian J. Stat.* 2021, *in press*. [CrossRef]
23. Olaniran, S.F.; Ismail, M.T. Purchasing power parity: The West African experience. *Proc. Aip Conf. Proc.* **2021**, *2423*, 070013.
24. Olayeni, R.O.; Tiwari, A.K.; Wohar, M.E. Fractional frequency flexible Fourier form (FFFFF) for panel cointegration test. *Appl. Econ. Lett.* **2021**, *28*, 482–486. [CrossRef]
25. Wang, B.; Wang, M.; Chan, N.H. Residual-Based Test for Fractional Cointegration. *Econ. Lett.* **2015**, *126*, 43–46. [CrossRef]
26. Hualde, J.; Velasco, C. Distribution-Free Tests of Fractional Cointegration. *Econom. Theory* **2008**, *24*, 216–255. [CrossRef]
27. Nielsen, M. Nonparametric Cointegration Analysis of Fractional Systems with Unknown Integration Orders. *J. Econom.* **2010**, *155*, 170–187. [CrossRef]
28. Zhang, R.; Robinson, P.; Yao, Q. Identifying cointegration by eigenanalysis. *J. Am. Stat. Assoc.* **2019**, *114*, 916–927. [CrossRef]
29. Engle, R.F.; Granger, C.W. Co-integration and Error correction: Representation, Estimation, and Testing. *Econom. J. Econom. Soc.* **1987**, *55*, 251–276. [CrossRef]
30. Caporale, G.M.; Gil-Alana, L.A. Fractional Integration and Cointegration in US Financial Time Series Data. *Empir. Econ.* **2014**, *47*, 1389–1410. [CrossRef]
31. Beran, J. Maximum likelihood estimation of the differencing parameter for invertible short and long memory autoregressive integrated moving average models. *J. R. Stat. Soc. Ser. (Methodol.)* **1995**, *57*, 659–672. [CrossRef]
32. Gutierrez, L. On the power of panel cointegration tests: A Monte Carlo comparison. *Econ. Lett.* **2003**, *80*, 105–111. [CrossRef]
33. Davidson, J. A model of fractional cointegration, and tests for cointegration using the bootstrap. *J. Econom.* **2002**, *110*, 187–212. [CrossRef]
34. Jamil, S.A.M.; Abdullah, M.A.A.; Kek, S.L.; Olaniran, O.R.; Amran, S.E. Simulation of Parametric Model towards the Fixed Covariate of Right Censored Lung Cancer Data. *J. Phys. Conf. Ser.* **2017**, *890*, 012172. [CrossRef]

35. Olaniran, O.R.; Abdullah, M.A.A. Subset Selection in High-Dimensional Genomic Data using Hybrid Variational Bayes and Bootstrap priors. *J. Phys. Conf. Ser.* **2020**, *1489*, 012030. [CrossRef]
36. Banerjee, A.; Carrion-i Silvestre, J.L. Panel data cointegration testing with structural instabilities. *J. Bus. Econ. Stat.* **2024**, 1–25.
37. Olaniran, O.R.; Abdullah, M.A.A. Bayesian Analysis of Extended Cox model with Time-Varying Covariates using Bootstrap Prior. *J. Mod. Appl. Stat. Methods* **2019**, *18*, 7. [CrossRef]
38. Caporale, G.M.; Claudio-Quiroga, G.; Gil-Alana, L.A. Analysing the relationship between CO_2 emissions and GDP in China: A fractional integration and cointegration approach. *J. Innov. Entrep.* **2021**, *10*, 1–16. [CrossRef]
39. Popoola, J.; Yahya, W.B.; Popoola, O.; Olaniran, O.R. Generalized Self-Similar First Order Autoregressive Generator (GSFO-ARG) for Internet Traffic. *Stat. Optim. Inf. Comput.* **2020**, *8*, 810–821. [CrossRef]
40. Nielsen, M.Ø.; Shimotsu, K. Determining the Cointegrating Rank in Nonstationary Fractional Systems by the Exact Local Whittle Approach. *J. Econom.* **2007**, *141*, 574–596. [CrossRef]