mathematics

Article

Summation Formulas for Certain Combinatorial Sequences

Yulei Chen [1] and Dongwei Guo [2,*]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; ylchen.math@whu.edu.cn
[2] School of Economics and Management, Nanjing University of Science and Technology, Nanjing 210094, China
* Correspondence: guo.dongwei@njust.edu.cn

Abstract: In this work, we establish some characteristics for a sequence, $\mathcal{A}_\alpha(n,k)$, including recurrence relations, generating function and inversion formula, etc. Based on the sequence, we derive, by means of the generating function approach, some transformation formulas concerning certain combinatorial numbers named after Lah, Stirling, harmonic, Cauchy and Catalan, as well as several closed finite sums. In addition, the relationship between $\mathcal{A}_\alpha(n,k)$ and r-Whitney–Lah numbers is established, and some formulas for the r-Whitney–Lah numbers are obtained.

Keywords: Lah numbers; Stirling numbers of the first kind; multiple harmonic numbers; Cauchy numbers; Catalan numbers

MSC: 11B75; 11B73

check for updates

Citation: Chen, Y.; Guo, D. Summation Formulas for Certain Combinatorial Sequences. *Mathematics* **2024**, *12*, 1210. https://doi.org/10.3390/math12081210

Academic Editor: Mario Gionfriddo

Received: 21 February 2024
Revised: 16 April 2024
Accepted: 16 April 2024
Published: 17 April 2024

1. Introduction and Motivation

Recently, Choi et al. ([1], Definition 1) defined a new number sequence, which we restate as follows. For two indeterminates, $\alpha, x \in \mathbb{C}$ and $n \in \mathbb{N}$, the sequences $\{\mathcal{A}_\alpha(n,k)\}_{k=0}^{n}$ are defined by

$$(\alpha + x)_n = \sum_{k=0}^{n} \mathcal{A}_\alpha(n,k)\langle x\rangle_k \tag{1}$$

and $\mathcal{A}_\alpha(n,k) = 1$ when $n = k = 0$.

Here and throughout the paper, we shall make use of the following notations for shifted factorials. For an indeterminate x and a non-negative integer n, they are defined by $(x)_0 = \langle x\rangle_0 = 1$ and

$$(x)_n = x(x+1)\cdots(x+n-1),\ \langle x\rangle_n = x(x-1)\cdots(x-n+1).$$

The r-Whitney–Lah numbers $L(n,k;r,m)$ are defined as [2–4]:

$$(x+2r|m)_n = \sum_{k=0}^{n} L(n,k;r,m)\langle x|m\rangle_k,\ n \geq 0, \tag{2}$$

where $(x|m)_n = x(x+m)\cdots(x+(n-1)m)$ and $\langle x|m\rangle_n = x(x-m)\cdots(x-(n-1)m)$ are the generalized rising and falling factorials with $(x|m)_0 = \langle x|m\rangle_0 = 1$. Comparing (1) and (2), when $m = 1$, we obtain $\mathcal{A}_{2r}(n,k) = L(n,k;r,1)$.

Choi et al. ([1], Lemma 1) derived the explicit expression of $\mathcal{A}_\alpha(n,k)$:

$$\mathcal{A}_\alpha(n,k) = \binom{n}{k}\frac{(\alpha)_n}{(\alpha)_k} = \binom{n}{k}(\alpha+k)_{n-k}$$

and a summation formula, which is equivalent to

$$\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)s(k,m) = \sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{k-m},$$

where $s(n,k)$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the signed and unsigned Stirling numbers of the first kind defined by the generating functions [5–7]:

$$\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!} = \frac{\ln^k(1+x)}{k!} \quad \text{and} \quad \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{\{-\ln(1-x)\}^k}{k!}.$$

Obviously, when $\alpha = 0$, we have the relation

$$\mathcal{A}_0(n,k) = (-1)^n L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!},$$

where $L(n,k)$ is the Lah numbers defined by the generating function [5,6]:

$$\sum_{n=k}^{\infty} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}\Big(-\frac{x}{1+x}\Big)^k.$$

For function $f(x)$, denote its difference by

$$\Delta f(x) = f(x+1) - f(x)$$

and n-order difference

$$\Delta^n f(x) = \Delta^{n-1} f(x+1) - \Delta^{n-1} f(x), \quad n \in \mathbb{N}_0,$$

where the symbol Δ is called the difference operator. By means of induction, it is not difficult to obtain

$$\Delta^k \langle x \rangle_n = \begin{cases} 0, & n < k; \\ \langle n \rangle_k \langle x \rangle_{n-k}, & n \geq k, \end{cases} \tag{3}$$

and

$$\Delta^k (x)_n = \begin{cases} 0, & n < k; \\ \langle n \rangle_k (x+k)_{n-k}, & n \geq k. \end{cases} \tag{4}$$

Let $[\tau^n]f(\tau)$ denote the coefficient of τ^n in the formal power series $f(\tau)$. Then, we have the effective lemma [6,7] below, which will be frequently used in the next sections to establish summation formulas.

Lemma 1. *For the double indexed sequence $\{\Omega(n,k)\}_{n\geq k}$, subject to $\Omega(n,k) = 0$ when $n < k$, and sequence $\mathcal{D}_k$ that have generating functions*

$$\sum_{n=k}^{\infty} \Omega(n,k)\tau^n = h(\tau)g^k(\tau) \quad \text{and} \quad \sum_{k=\lambda}^{\infty} \mathcal{D}_k \tau^k = f(\tau),$$

where $\lambda \in \mathbb{N}$, then we have

$$\sum_{k=\lambda}^{n} \mathcal{D}_k \Omega(n,k) = [\tau^n]h(\tau)f(g(\tau)).$$

In this paper, we shall continue to explore the properties and satisfied identities of the numbers $\mathcal{A}_\alpha(n,k)$ that are not listed by Choi et al. in [1]. In addition, the r-Whitney–Lah numbers will be used as special cases to show some of our results. This study is an extension of the study of generalized Lah numbers and is instructive for the further study of other combinatorial sequences such as Stirling numbers, harmonic numbers, Cauchy numbers, Catalan numbers, etc.

The rest of this paper is organized as follows. In the next section, we shall use a difference operator to regain the explicit expression of the number $\mathcal{A}_\alpha(n,k)$ and some of its other characteristics, such as its recurrence relations, generating function and inversion formula, as well as some summation formulas involving Lah numbers and Stirling numbers of the first kind. In Section 3, some formulas, concerning classical and generalized harmonic numbers, will be established via the sequence $\mathcal{A}_\alpha(n,k)$ and Lemma 1. In Section 4, we shall derive several summation formulas concerning Cauchy numbers. Then, the paper will end in Section 5 with comments and some summation formulas involving Catalan numbers.

2. Some Results Involving $\mathcal{A}_\alpha(n,k)$, Lah and Stirling Numbers

In this section, firstly, we shall derive a trivial recursive relation for $\mathcal{A}_\alpha(n,k)$ by using (1) and its explicit expression by a difference operator, as well as a nontrivial recursive relation by a linear relation. Then, we shall establish the generating function of $\mathcal{A}_\alpha(n,k)$ and some transformation formulas.

Proposition 1 (**Trivial recursive relation**)**.** *For $n,k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, the following recursive relation holds*

$$\mathcal{A}_\alpha(n+1,k) = \mathcal{A}_\alpha(n,k-1) + (k+n+\alpha)\mathcal{A}_\alpha(n,k).$$

Proof. By using the (1), we have

$$(\alpha+x)_{n+1} = \sum_{k=0}^{n+1} \mathcal{A}_\alpha(n+1,k)\langle x\rangle_k.$$

The left hand side can be rewritten as

$$\begin{aligned}(\alpha+x)_{n+1} &= (\alpha+x+n)\sum_{k=0}^{n} \mathcal{A}_\alpha(n,k)\langle x\rangle_k \\ &= \sum_{k=0}^{n} \mathcal{A}_\alpha(n,k)(x-k)\langle x\rangle_k + \sum_{k=0}^{n}(\alpha+n+k)\mathcal{A}_\alpha(n,k)\langle x\rangle_k \\ &= \sum_{k=0}^{n+1} \mathcal{A}_\alpha(n,k-1)\langle x\rangle_k + \sum_{k=0}^{n+1}(\alpha+n+k)\mathcal{A}_\alpha(n,k)\langle x\rangle_k.\end{aligned}$$

Then, the proof follows by comparing the coefficients of $\langle x\rangle_k$. $\square$

Proposition 2 (**Explicit expression**)**.** *For $n,k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, we have the explicit expression*

$$\mathcal{A}_\alpha(n,k) = \binom{n}{k}(\alpha+k)_{n-k}.$$

Proof. By utilizing (3) and (4), we can evaluate the k-order difference

$$\begin{aligned}\Delta^k(\alpha+x)_n &= \sum_{r=k}^{n} \mathcal{A}_\alpha(n,r)\Delta^k\langle x\rangle_r \\ &= \sum_{r=k}^{n} \mathcal{A}_\alpha(n,r)\langle r\rangle_k\langle x\rangle_{r-k}.\end{aligned}$$

Letting $x=0$ and noting that $\langle x\rangle_{r-k}|_{x=0} = 0$ for $r>k$, we obtain

$$\mathcal{A}_\alpha(n,k)k! = \Delta^k(\alpha)_n = \langle n\rangle_k(\alpha+k)_{n-k},$$

which leads us to the explicit expression of $\mathcal{A}_\alpha(n,k)$. $\square$

Proposition 3 (**Nontrivial recursive relation**)**.** *For $n, k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, the following recursive relation holds*

$$k\mathcal{A}_\alpha(n,k) = kn\mathcal{A}_\alpha(n-1,k) + n\mathcal{A}_\alpha(n-1,k-1).$$

Proof. By means of the linear relation

$$k = \frac{k(n-k)}{\alpha+n-1} + \frac{k(\alpha+k-1)}{\alpha+n-1},$$

we have the following equation:

$$\begin{aligned}
k\mathcal{A}_\alpha(n,k) &= \frac{k(n-k)}{\alpha+n-1}\binom{n}{k}(\alpha+k)_{n-k} + \frac{k(\alpha+k-1)}{\alpha+n-1}\binom{n}{k}(\alpha+k)_{n-k} \\
&= kn\binom{n-1}{k}(\alpha+k)_{n-k-1} + n\binom{n-1}{k-1}(\alpha+k-1)_{n-k} \\
&= kn\mathcal{A}_\alpha(n-1,k) + n\mathcal{A}_\alpha(n-1,k-1).
\end{aligned}$$

□

By specifying the parameter $\alpha = 0$ in Proposition 3, we obtain the following recursive relation on Lah numbers:

Corollary 1. *For $n, k \in \mathbb{N}$, the following recursive relation holds:*

$$kL(n,k) = -knL(n-1,k) - nL(n-1,k-1).$$

Proposition 4 (**Exponential generating function**)**.**

$$\sum_{n=k}^{\infty} \mathcal{A}_\alpha(n,k)\frac{\tau^n}{n!} = \frac{1}{k!}\frac{\tau^k}{(1-\tau)^{\alpha+k}}.$$

Proof. According to the explicit expression of $\mathcal{A}_\alpha(n,k)$ (Proposition 2), we have

$$\begin{aligned}
\sum_{n=k}^{\infty} \mathcal{A}_\alpha(n,k)\frac{\tau^n}{n!} &= \sum_{n=k}^{\infty}\binom{n}{k}(\alpha+k)_{n-k}\frac{\tau^n}{n!} = \frac{1}{k!}\sum_{n=k}^{\infty}\binom{\alpha-1+n}{n-k}\tau^n \\
&= \frac{\tau^k}{k!}\sum_{n=0}^{\infty}\binom{\alpha-1+n+k}{n}\tau^n.
\end{aligned}$$

Then the proof follows by using the fact

$$\sum_{n=0}^{\infty}\binom{\alpha-1+n+k}{n}\tau^n = \frac{1}{(1-\tau)^{\alpha+k}}.$$

□

Proposition 5. *The ordinary generating function of $\mathcal{A}_\alpha(n,k)$*

$$f_k(x) = \sum_{n=k}^{\infty} \mathcal{A}_\alpha(n,k)x^n$$

satisfies the difference-differential equation

$$\{1-(k+\alpha)x\}f_k(x) = x^2\frac{d}{dx}f_k(x) + xf_{k-1}(x).$$

Proof. By means of the recursive relation Proposition 1, we have

$$f_k(x) = \sum_{n=k}^{\infty} \mathcal{A}_\alpha(n,k)x^n = \sum_{n=k}^{\infty} \{\mathcal{A}_\alpha(n-1,k-1) + (\alpha+k+n-1)\mathcal{A}_\alpha(n-1,k)\}x^n$$
$$= x\sum_{n=k}^{\infty} \mathcal{A}_\alpha(n-1,k-1)x^{n-1} + (\alpha+k)x\sum_{n=k}^{\infty} \mathcal{A}_\alpha(n-1,k)x^{n-1}$$
$$+ x\sum_{n=k}^{\infty}(n-1)\mathcal{A}_\alpha(n-1,k)x^{n-1}.$$

Under the replacement $n \to n+1$, we get the equation

$$f_k(x) = x\sum_{n=k-1}^{\infty} \mathcal{A}_\alpha(n,k-1)x^n + x(\alpha+k)\sum_{n=k}^{\infty} \mathcal{A}_\alpha(n,k)x^n + x\sum_{n=k}^{\infty} n\mathcal{A}_\alpha(n,k)x^n$$
$$= xf_{k-1}(x) + x(\alpha+k)f_k(x) + x^2\frac{d}{dx}f_k(x),$$

which confirms the result stated in the proposition. □

Proposition 6. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)\mathcal{A}_\alpha(k,m) = 2^{n-m}\mathcal{A}_\alpha(n,m).$$

Proof. Using the explicit expression of $\mathcal{A}_\alpha(n,k)$ (Proposition 2), we have

$$\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)\mathcal{A}_\alpha(k,m) = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}(\alpha+k)_{n-k}(\alpha+m)_{k-m}$$
$$= (\alpha+m)_{n-m}\binom{n}{m}\sum_{k=m}^{n}\binom{n-m}{k-m}$$
$$= 2^{n-m}\mathcal{A}_\alpha(n,m).$$

□

For the particular case of $\alpha = 0$, the above Proposition 6 reduces to the corollary below.

Corollary 2. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} (-1)^k L(n,k)L(k,m) = 2^{n-m}L(n,m).$$

Similar to the proof of Proposition 6, we can get the following alternating summation formula:

Proposition 7 (Orthogonality relation). *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=m}^{n} (-1)^k \mathcal{A}_\alpha(n,k)\mathcal{A}_\alpha(k,m) = (-1)^m\delta_{m,n},$$

where $\delta_{m,n}$ be the Kronecker symbol $\delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$

By letting $\alpha = 0$, we can obtain, from the above proposition, the following known formula on Lah numbers.

Corollary 3. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} L(n,k)L(k,m) = \delta_{m,n}.$$

Proposition 8 (Reversion formula). *For two sequences $\{F(n)\}_{n\geq 0}$ and $\{G(n)\}_{n\geq 0}$, the following equivalent relations hold:*

$$F(n) = \sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)G(k) \Longleftrightarrow G(n) = \sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)F(k).$$

Proof. If the two sequences $\{F(n)\}_{n\geq 0}$ and $\{G(n)\}_{n\geq 0}$ satisfy

$$F(n) = \sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)G(k),$$

then we can evaluate the sum

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)F(k) &= \sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)\sum_{i=0}^{k}(-1)^i \mathcal{A}_\alpha(k,i)G(i)\\ &= \sum_{i=0}^{n}(-1)^i G(i)\sum_{k=i}^{n}(-1)^k \mathcal{A}_\alpha(n,k)\mathcal{A}_\alpha(k,i).\end{aligned}$$

By means of Proposition 7, the inner sum in the last line can be rewritten as

$$\sum_{k=i}^{n}(-1)^k \mathcal{A}_\alpha(n,k)\mathcal{A}_\alpha(k,i) = (-1)^i \delta_{n,i}.$$

We therefore obtain the following formula:

$$\sum_{k=0}^{n}(-1)^k \mathcal{A}_\alpha(n,k)F(k) = \sum_{i=0}^{n} G(i)\delta_{n,i} = G(n).$$

And vice versa. □

Proposition 9. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)s(k,m) = \sum_{k=m}^{n}\binom{n}{k}\begin{bmatrix}k\\m\end{bmatrix}(\alpha)_{n-k}.$$

Proof. By means of Lemma 1 and the exponential generating functions of $\mathcal{A}_\alpha(n,k)$ and $s(n,k)$, we can manipulate the sum

$$\begin{aligned}\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)s(k,m) &= \frac{n!}{(1-x)^\alpha}\sum_{k=m}^{n} s(k,m)[x^n]\frac{1}{k!}\left(\frac{x}{1-x}\right)^k\\ &= [x^n]\frac{n!}{(1-x)^\alpha}\sum_{k=m}^{\infty} s(k,m)\frac{1}{k!}\left(\frac{x}{1-x}\right)^k\\ &= n![x^n]\frac{\{-\ln(1-x)\}^m}{m!}(1-x)^{-\alpha}\\ &= n!\sum_{k=0}^{n}[x^k]\frac{\{-\ln(1-x)\}^m}{m!}[x^{n-k}](1-x)^{-\alpha}.\end{aligned}$$

Then, the proof follows by evaluating the coefficients

$$[x^k]\frac{\{-\ln(1-x)\}^m}{m!} = \frac{1}{k!}\begin{bmatrix}k\\m\end{bmatrix} \quad \text{and} \quad [x^{n-k}](1-x)^{-\alpha} = \frac{(\alpha)_{n-k}}{(n-k)!}.$$

□

Letting $\alpha = 0$ in Proposition 9, we have the following summation formula involving Lah numbers and Stirling numbers of the first kind:

Corollary 4. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} L(n,k)s(k,m) = (-1)^n \begin{bmatrix} n \\ m \end{bmatrix}.$$

Letting $\alpha = 2r$ in Proposition 9, we have the following summation formula involving r-Whitney–Lah numbers and Stirling numbers of the first kind: For $m, n \in \mathbb{N}_0$, the following formula holds:

$$\sum_{k=m}^{n} L(n,k;r,1)s(k,m) = \sum_{k=m}^{n} \binom{n}{k} \begin{bmatrix} k \\ m \end{bmatrix} (2r)_{n-k}.$$

Alternatively, by letting $\alpha = 1$ in Proposition 9, we have another formula below, involving signed and unsigned Stirling numbers of the first kind.

Corollary 5. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} \frac{1}{k!} \binom{n}{k} s(k,m) = \frac{1}{n!} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}.$$

Proof. By setting $\alpha = 1$, the formula stated in Proposition 9 can be rewritten as

$$\sum_{k=m}^{n} \frac{n!}{k!} \binom{n}{k} s(k,m) = \sum_{k=m}^{n} \frac{n!}{k!} \begin{bmatrix} k \\ m \end{bmatrix}.$$

Then, the identity desired follows by employing the known Formula ([8], Eq. 6.16)

$$\sum_{k=m}^{n} \frac{n!}{k!} \begin{bmatrix} k \\ m \end{bmatrix} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}.$$

□

Proposition 10. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=m}^{n} \binom{n}{k} \begin{bmatrix} k \\ m \end{bmatrix} (\alpha)_{n-k} = \sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{k-m}.$$

Proof. In ([1], Lemma 1), Choi et al. obtained the following transformation formula:

$$\sum_{k=m}^{n} \mathcal{A}_\alpha(n,k)s(k,m) = \sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{k-m}.$$

Combining it with Proposition 9, we can obtain the desired symmetric transformation formula. □

For the particular case of $\alpha = 1$, the Proposition 10 reduces the following identity.

Corollary 6. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} \frac{n!}{k!} \begin{bmatrix} k \\ m \end{bmatrix} = \sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix}.$$

In fact, we have the closed expression

$$\sum_{k=m}^{n} \frac{n!}{k!} \begin{bmatrix} k \\ m \end{bmatrix} = \sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix},$$

which are recorded in ([8], Eqs. 6.16 and 6.21) by Graham et al.

3. Summation Formulas Concerning Harmonic Numbers

It is well known that the classical harmonic numbers are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N},$$

and its generating function is given by

$$\sum_{n=1}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x}.$$

Harmonic numbers have wide applications in number theory, combinatorics, and computer science. Their properties and identities have been explored extensively. In addition, many authors also have studied other harmonic-like numbers defined in various ways. For instance, Cheon and El-Mikkawy [9,10] (also see [7,11]) studied the following multiple harmonic-like numbers, which reduce, when $\ell = 1$, to the ordinary harmonic numbers

$$H_n(\ell) = \sum_{1 \le k_1 + k_2 + \cdots + k_\ell \le n} \frac{1}{k_1 k_2 \cdots k_\ell},$$

and obtained its generating function:

$$\sum_{n=1}^{\infty} H_n(\ell) x^n = \frac{\{-\ln(1-x)\}^\ell}{1-x}.$$

In [7], Guo and Chu also studied the alternating harmonic numbers

$$\mathcal{H}_0 = 0 \quad \text{and} \quad \mathcal{H}_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \text{for} \quad n \in \mathbb{N}$$

as well as the multiple alternating harmonic numbers

$$\mathcal{H}_n(\ell) = \sum_{1 \le k_1 + k_2 + \cdots + k_\ell \le n} \frac{(-1)^{k_1 + k_2 + \cdots + k_\ell}}{k_1 k_2 \cdots k_\ell},$$

and obtained their generating functions:

$$\sum_{n=1}^{\infty} \mathcal{H}_n x^n = \frac{-\ln(1+x)}{1-x} \quad \text{and} \quad \sum_{n=1}^{\infty} \mathcal{H}_n(\ell) x^n = \frac{\{-\ln(1+x)\}^\ell}{1-x}.$$

Now, we further explore the summation formulae concerning (generalized) harmonic numbers and the sequence $\mathcal{A}_\alpha(n,k)$, whose particular cases reduces to several interesting identities on Stirling numbers.

Proposition 11. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=1}^{n} (-1)^k k! H_k \mathcal{A}_\alpha(n,k) = -n! \sum_{k=1}^{n} \frac{(\alpha-1)_{n-k}}{k(n-k)!}.$$

Proof. With the aid of Lemma 1 and the generating functions of $\mathcal{A}_\alpha(n,k)$ and H_n, the sum on the left hand side can be evaluated as

$$\sum_{k=1}^{n}(-1)^k k! H_k \mathcal{A}_\alpha(n,k) = n! \sum_{k=1}^{n}(-1)^k k! H_k [x^n] \frac{1}{k!} \frac{x^k}{(1-x)^{\alpha+k}}$$
$$= n![x^n](1-x)^{-\alpha} \sum_{k=1}^{\infty} H_k \Big(-\frac{x}{1-x}\Big)^k$$
$$= n![x^n](1-x)^{1-\alpha} \ln(1-x).$$

Extracting the coefficients of $[x^n]$ from the last line

$$[x^n](1-x)^{1-\alpha}\ln(1-x) = \sum_{k=1}^{n}[x^k]\ln(1-x)[x^{n-k}](1-x)^{1-\alpha}$$
$$= -\sum_{k=1}^{n}\frac{(\alpha-1)_{n-k}}{k(n-k)!},$$

we therefore confirm the desired formula. □

For the particular cases $\alpha = 1$ and $\alpha = 2$, Proposition 11 reduces the following well-known formulas

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k = -\frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}H_k = -\frac{H_n}{n+1}.$$

Analogously, we have the following formula on alternating harmonic numbers $\mathcal{H}_k$.

Proposition 12. *For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=1}^{n}(-1)^k k! \mathcal{H}_k \mathcal{A}_\alpha(n,k) = n! \sum_{k=1}^{n}(2^k-1)\frac{(\alpha-1)_{n-k}}{k(n-k)!}.$$

Particularly, when $\alpha = 1$ and $\alpha = 2$, we have, respectively, the formula below.

Corollary 7. *For $n \in \mathbb{N}_0$, the following two formulas hold:*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\mathcal{H}_k = \frac{2^n-1}{n};$$
$$\sum_{k=1}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}\mathcal{H}_k = \frac{1}{n+1}\Big(\sum_{k=1}^{n}\frac{2^k}{k} - H_n\Big).$$

When $\alpha = 2r$ in Propositions 11 and 12, we have, respectively,

$$\sum_{k=1}^{n}(-1)^k k! H_k L(n,k;r,1) = -n! \sum_{k=1}^{n}\frac{(2r-1)_{n-k}}{k(n-k)!},$$
$$\sum_{k=1}^{n}(-1)^k k! \mathcal{H}_k L(n,k;r,1) = n! \sum_{k=1}^{n}(2^k-1)\frac{(2r-1)_{n-k}}{k(n-k)!}.$$

Proposition 13 (Explicit expression of $H_n(m)$).

$$H_n(m) = \frac{m!}{n!}\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}.$$

Proof. According to the proof of Proposition 9 and the generating function of $H_n(m)$, when $\alpha = 1$, we have

$$\sum_{k=m}^{n}\frac{n!}{k!}\binom{n}{k}s(k,m) = \frac{n!}{m!}[x^n]\frac{\{-\ln(1-x)\}^m}{1-x} = n! H_n(m).$$

Combining this with (5), we obtain the explicit expression of $H_n(m)$. □

This explicit expression was also found by Cheon and El-Mikkawy [10], but there does not exist such an elegant expression for $\mathcal{H}_n(m)$ as that which Guo and Chu pointed out in [7].

Proposition 14. *For $n, \ell \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=1}^{n}(-1)^k k! H_k(\ell)\mathcal{A}_\alpha(n,k) = \ell! \sum_{k=1}^{n}(-1)^k \binom{n}{k} s(k,\ell)(\alpha-1)_{n-k}.$$

Proof. By employing Lemma 1 and the generating functions of $\mathcal{A}_\alpha(n,k)$ and $H_n(\ell)$, we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k k! H_k(\ell)\mathcal{A}_\alpha(n,k) &= \sum_{k=1}^{n}(-1)^k k! H_k(\ell)[x^n]\frac{n!}{k!}\frac{x^k}{(1-x)^{\alpha+k}}\\ &= n![x^n]\frac{1}{(1-x)^\alpha}\sum_{k=1}^{\infty} H_k(\ell)\Big(-\frac{x}{1-x}\Big)^k\\ &= n!\ell![x^n](1-x)^{1-\alpha}\frac{\ln^\ell(1-x)}{\ell!}\\ &= n!\ell!\sum_{k=1}^{n}[x^k]\frac{\ln^\ell(1-x)}{\ell!}[x^{n-k}](1-x)^{1-\alpha}.\end{aligned}$$

By extracting the coefficients

$$[x^k]\frac{\ln^\ell(1-x)}{\ell!} = \frac{(-1)^k}{k!}s(k,\ell) \quad \text{and} \quad [x^{n-k}](1-x)^{1-\alpha} = \frac{(\alpha-1)_{n-k}}{(n-k)!},$$

we therefore obtain the formula stated in the proposition. □

Setting $\alpha = 2r$ in Proposition 14, we obtain that

$$\sum_{k=1}^{n}(-1)^k k! H_k(\ell) L(n,k;r,1) = \ell! \sum_{k=1}^{n}(-1)^k \binom{n}{k} s(k,\ell)(2r-1)_{n-k}.$$

Letting $\alpha = 1$ and $\alpha = 2$ in Proposition 14, we obtain the following summation formulas.

Corollary 8. *For $n, \ell \in \mathbb{N}$, the following two formulas hold:*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} H_k(\ell) = (-1)^n \frac{\ell!}{n!} s(n,\ell);$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k+1}\binom{n}{k} H_k(\ell) = \frac{\ell!}{n+1}\sum_{k=1}^{n}\frac{(-1)^k}{k!}s(k,\ell).$$

Analogously, for the multiple alternating harmonic numbers $\mathcal{H}_n(\ell)$, we have the expression

$$\sum_{k=1}^{n} k! \mathcal{H}_k(\ell)\mathcal{A}_\alpha(n,k) = n![x^n](1-x)^{1-\alpha}\frac{\ln^\ell(1-x)}{1-2x}. \tag{5}$$

The special cases $\alpha = 1$ and $\alpha = 2$ of (5) lead to the following identities.

Corollary 9. *For $n, \ell \in \mathbb{N}$, the following two formulas hold:*

$$\sum_{k=1}^{n}\binom{n}{k}\mathcal{H}_k(\ell)=2^n\ell!\sum_{k=1}^{n}\Big(-\frac{1}{2}\Big)^k\frac{1}{k!}s(k,\ell);$$
$$\sum_{k=1}^{n}\frac{1}{k+1}\binom{n}{k}\mathcal{H}_k(\ell)=2^n\frac{(-1)^\ell}{n+1}\sum_{k=1}^{n}\frac{1}{2^k}H_k(\ell).$$

Proposition 15. *For $m,n\in\mathbb{N}_0$ and $\alpha\notin\{\cdots,-2,-1,0,1\}$, the following two formulas hold:*

$$\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\frac{\alpha-1+2k}{\alpha-1+k}\mathcal{H}_{n-k}(m)(\alpha)_k=(-1)^n\frac{m!}{n!}\sum_{k=m}^{n}\mathcal{A}_\alpha(n,k)s(k,m); \tag{6}$$
$$\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\frac{\alpha-1+2k}{\alpha-1+k}\mathcal{H}_{n-k}(m)(\alpha)_k=(-1)^n\frac{m!}{n!}\sum_{k=m}^{n}\binom{n}{k}\begin{bmatrix}k\\m\end{bmatrix}(\alpha)_{n-k}. \tag{7}$$

Proof. Letting $\tau=-x$ in Proposition 4, we obtain the generating function

$$\sum_{n=k}^{\infty}(-1)^n\mathcal{A}_\alpha(n,k)\frac{x^n}{n!}=\frac{1}{k!}\frac{(-x)^k}{(1+x)^{\alpha+k}}.$$

By means of Lemma 1 and the generating function of $s(n,k)$, we can evaluate

$$\begin{aligned}\sum_{k=m}^{n}(-1)^n\mathcal{A}_\alpha(n,k)s(k,m)&=\sum_{k=m}^{n}s(k,m)[x^n]\frac{n!}{k!}\frac{(-x)^k}{(1+x)^{\alpha+k}}\\&=n![x^n]\frac{1}{(1+x)^\alpha}\sum_{k=m}^{\infty}s(k,m)\frac{1}{k!}\Big(-\frac{x}{1+x}\Big)^k\\&=\frac{n!}{m!}[x^n]\frac{\big\{-\ln(1+x)\big\}^m}{(1+x)^\alpha}\\&=\frac{n!}{m!}[x^n]\frac{1-x}{(1+x)^\alpha}\frac{\big\{-\ln(1+x)\big\}^m}{1-x}\\&=\frac{n!}{m!}[x^n]\frac{\big\{-\ln(1+x)\big\}^m}{(1-x)(1+x)^\alpha}-\frac{n!}{m!}[x^{n-1}]\frac{\big\{-\ln(1+x)\big\}^m}{(1-x)(1+x)^\alpha}.\end{aligned} \tag{8}$$

By using the generating function of $\mathcal{H}_n(\ell)$ and the coefficient

$$[x^{n-k}]\frac{1}{(1+x)^\alpha}=(-1)^{n-k}\frac{(\alpha)_{n-k}}{(n-k)!},$$

we can evaluate the coefficients

$$\begin{aligned}&[x^n]\frac{\big\{-\ln(1+x)\big\}^m}{(1-x)(1+x)^\alpha}-[x^{n-1}]\frac{\big\{-\ln(1+x)\big\}^m}{(1-x)(1+x)^\alpha}\\&=\sum_{k=m}^{n}\frac{(-1)^{n-k}}{(n-k)!}\mathcal{H}_k(m)(\alpha)_{n-k}\Big\{1+\frac{n-k}{\alpha-1+n-k}\Big\}\\&=\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\frac{\alpha-1+2k}{\alpha-1+k}\mathcal{H}_{n-k}(m)(\alpha)_k,\end{aligned}$$

which confirms the identity (6). The second identity (7) follows by using Proposition 9. □

Proposition 16. *For $n,m\in\mathbb{N}$, the following relation holds:*

$$\mathcal{H}_n(m)-\mathcal{H}_{n-1}(m)=(-1)^n\frac{m!}{n!}\begin{bmatrix}n\\m\end{bmatrix}.$$

Proof. By setting $\alpha=0$, the equation (8) becomes

$$\begin{aligned}\sum_{k=m}^{n}(-1)^n\mathcal{A}_0(n,k)s(k,m) &= \frac{n!}{m!}[x^n]\frac{\{-\ln(1+x)\}^m}{1-x} - \frac{n!}{m!}[x^{n-1}]\frac{\{-\ln(1+x)\}^m}{1-x} \\ &= \frac{n!}{m!}\{\mathcal{H}_n(m) - \mathcal{H}_{n-1}(m)\}.\end{aligned}$$

Keeping in mind Corollary 4, we have the equation

$$\sum_{k=m}^{n}(-1)^n\mathcal{A}_0(n,k)s(k,m) = \sum_{k=m}^{n}L(n,k)s(k,m) = (-1)^n\begin{bmatrix}n\\m\end{bmatrix},$$

which completes the proof. □

By telescoping, we can obtain, from Proposition 16, the following summation formula.

Corollary 10. *For $n, m \in \mathbb{N}_0$, the following relation holds:*

$$\sum_{k=m}^{n}(-1)^k\frac{1}{k!}\begin{bmatrix}k\\m\end{bmatrix} = \frac{\mathcal{H}_n(m)}{m!}.$$

Proposition 17. *For $n, m \in \mathbb{N}_0$, the following relation holds:*

$$\sum_{k=m}^{n}(-1)^k\mathcal{H}_k(m) = \frac{m!}{2n!}\begin{bmatrix}n+1\\m+1\end{bmatrix} + \frac{(-1)^n}{2}\mathcal{H}_n(m).$$

Proof. By letting $\alpha = 1$ in (8), we obtain the identity

$$\begin{aligned}\sum_{k=m}^{n}(-1)^n\mathcal{A}_1(n,k)s(k,m) &= \frac{n!}{m!}[x^n]\frac{\{-\ln(1+x)\}^m}{1-x}\frac{1}{1+x} \\ &\quad - \frac{n!}{m!}[x^{n-1}]\frac{\{-\ln(1+x)\}^m}{1-x}\frac{1}{1+x} \\ &= \frac{n!}{m!}\left(2\sum_{k=0}^{n}(-1)^{n-k}\mathcal{H}_k(m) - \mathcal{H}_n(m)\right).\end{aligned}$$

According to Corollary 5, we have the summation formula

$$\sum_{k=m}^{n}(-1)^n\mathcal{A}_1(n,k)s(k,m) = (-1)^n n!\sum_{k=m}^{n}\frac{1}{k!}\binom{n}{k}s(k,m) = (-1)^n\begin{bmatrix}n+1\\m+1\end{bmatrix}.$$

Combining the above two equations, we can complete the proof. □

Proposition 18. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}\backslash\{\cdots,-2,-1,0\}$, the following formula holds:*

$$\sum_{k=m}^{n}\binom{n}{k}\begin{bmatrix}k\\m\end{bmatrix}(\alpha)_{n-k}H_{n-k}\langle\alpha-1\rangle = \sum_{k=m}^{n}\binom{k}{m}\begin{bmatrix}n\\k\end{bmatrix}(k-m)\alpha^{k-m-1},$$

where $H_n\langle\lambda\rangle$ denotes the generalized harmonic numbers defined by

$$H_n\langle\lambda\rangle = \sum_{k=1}^{n}\frac{1}{k+\lambda}, \quad \textit{with} \quad \lambda \in \mathbb{C}\backslash\mathbb{Z}^-.$$

Proof. It is easy to verify the relation

$$\frac{d}{d\alpha}(\alpha)_{n-k} = (\alpha)_{n-k}H_{n-k}\langle\alpha-1\rangle.$$

By differentiating, with respect to α, both sides of Proposition 9, we can obtain the desired identity. □

For the special case of $\alpha = 1$ in Proposition 18, we can find the formula below.

Corollary 11. *For $m, n \in \mathbb{N}_0$, the following formula holds:*

$$\sum_{k=m}^{n} \binom{k}{m} \begin{bmatrix} n \\ k \end{bmatrix} (k-m) = n! \sum_{k=m}^{n} \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix} H_{n-k}.$$

By denoting $h_n = H_{2n} - \frac{1}{2}H_n$, we have the following proposition.

Proposition 19. *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formula holds:*

$$\sum_{k=0}^{n} (-1)^k \frac{k!}{4^k} \binom{2k}{k} h_k \mathcal{A}_\alpha(n,k) = -\frac{n!}{2} \sum_{k=0}^{n-1} \frac{(\alpha - \frac{1}{2})_k}{(n-k)k!}.$$

Proof. In ([12], Eq. 20), Chen obtained the generating function

$$\sum_{n=0}^{\infty} \binom{2n}{n} h_n x^n = -\frac{1}{\sqrt{1-4x}} \ln \sqrt{1-4x}.$$

By employing Lemma 1, we can compute the sum

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \frac{k!}{4^k} \binom{2k}{k} h_k \mathcal{A}_\alpha(n,k) &= n![x^n] \frac{1}{(1-x)^\alpha} \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} \Big(-\frac{x}{1-x} \Big)^k \\ &= \frac{n!}{2} [x^n] (1-x)^{\frac{1}{2}-\alpha} \ln(1-x). \end{aligned}$$

Evaluating the coefficient

$$[x^n](1-x)^{\frac{1}{2}-\alpha} \ln(1-x) = -\sum_{k=0}^{n} \frac{(\alpha - \frac{1}{2})_k}{(n-k)k!},$$

we therefore obtain the formula stated in the proposition. □

Let $\alpha = 2r$ in Proposition 19, we have that

$$\sum_{k=0}^{n} (-1)^k \frac{k!}{4^k} \binom{2k}{k} h_k L(n,k;r,1) = -\frac{n!}{2} \sum_{k=0}^{n-1} \frac{(2r - \frac{1}{2})_k}{(n-k)k!}.$$

For the special case of $\alpha = 1$, Proposition 19 gives the formula below.

Corollary 12. *For $n \in \mathbb{N}$, the following formula holds*

$$\sum_{k=0}^{n} \Big(-\frac{1}{4} \Big)^k \binom{n}{k} \binom{2k}{k} h_k = -\frac{1}{2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{4^k (n-k)}.$$

4. Formulas Concerning $\mathcal{A}_\alpha(n,k)$ and Cauchy Numbers

The first- and second-kind Cauchy numbers a_n and b_n are defined, respectively, by the integrals (cf. [5,13])

$$a_n = \int_0^1 \langle x \rangle_n dx \quad \text{and} \quad b_n = \int_0^1 (x)_n dx.$$

Their generating functions are given by

$$\sum_{k=0}^{\infty} a_k \frac{x^k}{k!} = \frac{x}{\ln(1+x)} \quad \text{and} \quad \sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{-x}{(1-x)\ln(1-x)}.$$

They satisfy the following relations:

$$a_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{\langle n \rangle_k}{k+1} a_{n-k} \quad \text{and} \quad b_n = n! - \sum_{k=1}^{n} \frac{\langle n \rangle_k}{k+1} b_{n-k},$$

where the latter one corrects the error recorded in ([5], p. 294).

Similar to the last section, by means of Lemma 1, we can establish the following summation formulas.

Proposition 20. *For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the following formulas hold:*

$$\sum_{k=0}^{n} a_k \mathcal{A}_\alpha(n,k) = \sum_{k=0}^{n} \binom{n}{k} b_k (\alpha)_{n-k}; \tag{9}$$

$$\sum_{k=0}^{n} (-1)^k b_k \mathcal{A}_\alpha(n,k) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k (\alpha)_{n-k}. \tag{10}$$

Proof. For the first Formula (9), by applying Lemma 1 to the left hand side, we can evaluate

$$\begin{aligned}\sum_{k=0}^{n} a_k \mathcal{A}_\alpha(n,k) &= n![x^n] \frac{1}{(1-x)^\alpha} \sum_{k=0}^{\infty} a_k \frac{1}{k!} \left(\frac{x}{1-x}\right)^k \\ &= n![x^n] \frac{1}{(1-x)^\alpha} \frac{-x}{(1-x)\ln(1-x)}.\end{aligned}$$

The proof follows by extracting the coefficient

$$[x^n] \frac{1}{(1-x)^\alpha} \frac{-x}{(1-x)\ln(1-x)} = \sum_{k=0}^{n} \frac{b_k (\alpha)_{n-k}}{k!(n-k)!}.$$

Analogously, we can obtain the second one (10). □

By setting $\alpha = 2r$ in Proposition 20, we obtain

$$\sum_{k=0}^{n} a_k L(n,k;r,1) = \sum_{k=0}^{n} \binom{n}{k} b_k (2r)_{n-k};$$

$$\sum_{k=0}^{n} (-1)^k b_k L(n,k;r,1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k (2r)_{n-k}.$$

When setting $\alpha = 0$, Proposition 20 reduces to the following two formulas concerning Lah numbers, which can be found in ([6], Eqs. 3.1 and 3.2) (also see [13]).

$$\sum_{k=0}^{n} L(n,k) a_k = (-1)^n b_n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k L(n,k) b_k = a_n.$$

Alternatively, by letting $\alpha = 1$ in Proposition 20, we obtain another two summation formulas.

Corollary 13. *For $n \in \mathbb{N}_0$, the following formulas hold:*

$$\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} a_k = \sum_{k=0}^{n} \frac{1}{k!} b_k;$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} b_k = \sum_{k=0}^{n} \frac{(-1)^k}{k!} a_k.$$

Further, it is not difficult to confirm ([1], Eq. 35)

$$\frac{d}{d\alpha} \mathcal{A}_\alpha(n,k) = \mathcal{A}_\alpha(n,k) \{ H_n \langle \alpha - 1 \rangle - H_k \langle \alpha - 1 \rangle \}.$$

Then, by differentiating, with respect to α, both sides of (9) and (10), we obtain the formulas below.

Proposition 21. *For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}\backslash\{\cdots,-2,-1,0\}$, the following formulas hold:*

$$\sum_{k=0}^{n} a_k \mathcal{A}_\alpha(n,k)\{H_n\langle\alpha-1\rangle - H_k\langle\alpha-1\rangle\} = \sum_{k=0}^{n}\binom{n}{k} b_k(\alpha)_{n-k}H_{n-k}\langle\alpha-1\rangle;$$
$$\sum_{k=0}^{n}(-1)^k b_k \mathcal{A}_\alpha(n,k)\{H_n\langle\alpha-1\rangle - H_k\langle\alpha-1\rangle\} = \sum_{k=0}^{n}(-1)^k\binom{n}{k} a_k(\alpha)_{n-k}H_{n-k}\langle\alpha-1\rangle.$$

Particularly, when $\alpha = 1$, the above proposition reduce to the following identities.

Corollary 14. *For $n \in \mathbb{N}_0$, the following formulas hold:*

$$\sum_{k=0}^{n}\frac{1}{k!}\binom{n}{k}a_k H_k = \sum_{k=0}^{n}\frac{1}{k!}(H_n - H_{n-k})b_k;$$
$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}b_k H_k = \sum_{k=0}^{n}\frac{(-1)^k}{k!}(H_n - H_{n-k})a_k.$$

5. Concluding Comments

By means of the numbers $\mathcal{A}_\alpha(n,k)$ and Lemma 1, we may establish more summation formulas. For example, recalling the Catalan numbers C_n [14] defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

with the generating function

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x},$$

we can obtain the following summation formula:

$$\sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k k! C_k \mathcal{A}_\alpha(n,k) = \frac{2}{n+1}\Big\{\Big(\alpha-\frac{1}{2}\Big)_{n+1} - (\alpha-1)_{n+1}\Big\}. \tag{11}$$

By specifying $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, the above formula reduces to the identities below.

$$\boxed{\alpha=0}\quad \sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k k! C_k L(n,k) = \frac{2n-1}{n+1}\Big\langle\frac{1}{2}\Big\rangle_n, \quad (n>0);$$
$$\boxed{\alpha=1}\quad \sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k\binom{n}{k}C_k = \frac{2(n+2)}{4^{n+1}}C_{n+1};$$
$$\boxed{\alpha=2}\quad \sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k\frac{1}{k+1}\binom{n}{k}C_k = \frac{(n+1)(2n+3)}{2^{2n+1}}C_{n+1} - \frac{2}{n+1}.$$

By differentiating, with respect to α, both sides of (11), we obtain, for $\alpha \notin \{\cdots,-2,-1,0,1\}$, the following formula involving harmonic numbers.

$$\sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k k! C_k \mathcal{A}_\alpha(n,k)\{H_n\langle\alpha-1\rangle - H_k\langle\alpha-1\rangle\}$$
$$= \frac{2}{n+1}\Big\{\Big(\alpha-\frac{1}{2}\Big)_{n+1}H_{n+1}\Big\langle\alpha-\frac{3}{2}\Big\rangle - (\alpha-1)_{n+1}H_{n+1}\langle\alpha-2\rangle\Big\}.$$

For $\alpha = 2$, it reduces to the following summation formula:

$$\sum_{k=0}^{n}\Big(-\frac{1}{4}\Big)^k\frac{1}{k+1}\binom{n}{k}C_k H_{k+1} = \frac{(n+2)(2n+3)}{4^n(n+1)}C_{n+1}\{H_{n+1} - H_{2n+3} + 1\}.$$

In this paper, we mainly obtain some properties of the numbers $\mathcal{A}_\alpha(n,k)$ and some identities concerning them and other combinatorial numbers, which is different from the results obtained by Choi et al., which mainly obtain some transformation formulas regarding hypergeometric series through $\mathcal{A}_\alpha(n,k)$.

Author Contributions: Methodology, D.G.; Writing—original draft, Y.C.; Writing—review & editing, Y.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Zhoukou Normal University high-level talents start-up funds research project, China, (ZKNUC2022007).

Data Availability Statement: All data generated or analyzed during this study are included in this manuscript.

Acknowledgments: The authors are sincerely grateful to the anonymous referees for their careful reading and valuable suggestions, which have improved the manuscript during the revision.

Conflicts of Interest: The authors declare that they have no conflicts of interest regarding the publication of this paper.

References

1. Choi, J.; Qureshi, M.I.; Bhat, A.H.; Majid, J. Reduction formulas for generalized hypergeometric series associated with new sequences and applications. *Fractal Frac.* **2021**, *5*, 150. [CrossRef]
2. Cheon, G.S.; Jung, J.H. r-Whitney numbers of Dowling lattices. *Discrete Math.* **2012**, *312*, 2337–2348. [CrossRef]
3. Gyimesi, E.; Nyul, G. New combinatorial interpretations of r-Whitney and r-Whitney–Lah numbers. *Discrete Appl. Math.* **2019**, *255*, 222–233. [CrossRef]
4. Ramirez, J.L.; Shattuck, M. A (p,q)-analogue of the r-Whitney-Lah numbers. *J. Integer Seq.* **2016**, *19*, 16.5.6.
5. Comtet, L. *Advanced Combinatorics*; D. Reidel Pub. Co.: Dordrecht, The Netherlands; Boston, MA, USA, 1974.
6. Guo, D. Summation formulae involving Stirling and Lah numbers. *Forum Math.* **2020**, *32*, 1407–1414. [CrossRef]
7. Guo, D.; Chu, W. Summation formulae involving multiple harmonic numbers. *Appl. Anal. Discr. Math.* **2021**, *15*, 201–212. [CrossRef]
8. Graham, R.L.; Knuth, D.E.; Patashnik, O. *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed.; Addison–Wesley Publishing Company: Reading, MA, USA, 1994.
9. Cheon, G.S.; El-Mikkawy, M.E.A. Generalized harmonic number identities and a related matrix representation. *J. Korean Math. Soc.* **2007**, *44*, 487–498. [CrossRef]
10. Cheon, G.S.; El-Mikkawy, M.E.A. Generalized harmonic numbers with Riordan arrays. *J. Number Theory* **2008**, *128*, 413–425. [CrossRef]
11. Prodinger, H.; Tauraso, R. New multiple harmonic sum identities. *Electron. J. Comb.* **2014**, *21*, 1240–1248.
12. Chen, H. Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers. *J. Integer Seq.* **2016**, *19*, 16.1.5.
13. Zhao, F. Sums of products of Cauchy numbers. *Discrete Math.* **2009**, *309*, 3830–3842. [CrossRef]
14. Chu, W. Further identities on Catalan numbers. *Discrete Math.* **2018**, *343*, 3159–3164. [CrossRef]