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Hyers–Ulam–Rassias Stability of Nonlinear Implicit Higher-Order Volterra Integrodifferential Equations from above on Unbounded Time Scales

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Abstract: In this paper, we present sufficient conditions for Hyers–Ulam-Rassias stability of nonlinear implicit higher-order Volterra-type integrodifferential equations from above on unbounded time scales. These new sufficient conditions result by reducing Volterra-type integrodifferential equations to Volterra-type integral equations, using the Banach fixed point theorem, and by applying an appropriate Bielecki type norm, the Lipschitz type functions, where Lipschitz coefficient is replaced by unbounded rd-continuous function.

Keywords: Volterra integrodifferential equations; time scales; Hyers–Ulam–Rassias stability; existence; uniqueness

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1. Introduction

In 1940, Stanislaw Ulam [1] formulated a generally applicable definition of stability. He wrote that "*For every equations one can ask the following question. When is it true that the solution of an equation differing slightly from the given one must of necessity be close to the solution of the given equation?*". Among those was the question concerning the stability of group homomorphisms. Hyers [2] solved the problem for the case of approximately additive mappings between Banach spaces. So, the stability concept proposed by Ulam and Hyers was named *Hyers–Ulam stability*. Afterwards, Rassias [3] introduced new ideas of Hyers–Ulam stability using unbounded right-hand side in the involved inequalities, depending on certain functions, therefore introducing the so-called *Hyers–Ulam–Rassias stability*.

In 2007, Jung [4] proved, using a fixed point approach, that the Volterra nonlinear integral equation is Hyers–Ulam–Rassias stable on a compact interval under certain conditions. Then, several authors [5–7] generalized the previous result on the Volterra integral equations on infinite interval in the case when the integrand is Lipschitz with a fixed Lipschitz constant. In the near past, many research papers have been published about Hyers–Ulam stability of Voltera integral equations of different types, including nonlinear Volterra integrodifferential equations, mixed integral dynamic systems with impulses, etc. [8–12].

The theory of time scales analysis has been rising fast, and has gained a lot of interest. The pioneer of this theory was Hilger [13]. He introduced this theory in 1988, with the inspiration to unify continuous and discrete calculus. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales, we recommend the books by Bohner et al. [14,15] and Georgiev [16]. In addition, the basic concepts and definitions of the time scale calculus are used in the article, which are described in Section 2. Furthermore, we used the traditional symbols and mathematical expressions adopted in the theory of time scales.

To the best of our knowledge, the first ones who pay attention to Hyers–Ulam stability for Volterra integral equations on time scales are Andras et al. [17] and Hua et al. [18]. However, they restricted their research to the case when an integrand satisfies the Lipschitz conditions with some fixed Lipschitz constant.

In 1956, Bielecki published a remark [19] in which he gave a new method for proving the global existence and unity of solutions of differential equations. His method has been applied to a wide range of classes of integral, integrodifferential and many other functional equations. For a review of the results obtained by the mentioned method, and many applications in various mathematical problems, see [16,20,21] and references therein for details.

Tisdell et al. [22,23] gave the basic qualitative and quantitative results to nonlinear Volterra integral equations on time scales in the case when the integrand is estimated by the Lipschitz type function with a fixed Lipschitz constant

$$x(t) = f(t) + \int_{t_0}^{t} k(t,s,x(s))\,\Delta s, \quad t_0, t \in I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}.$$

Reinfelds et al. [24–27] generalized previous results by analysing the case where the integrand can be evaluated by the Lipschitz type function and the corresponding Lipschitz coefficient can be unbounded rd-continuous function. Using the exponential function defined at the time scale calculus, it was possible to introduce the appropriate Bielecki norm to evaluate the corresponding expressions in the proofs.

Several authors [28–32] consider first order explicit and implicit Volterra integrodifferential equations on intervals, and also on time scales in which the integrand is Lipschitz with a fixed Lipschitz constant

$$x^{\Delta}(t) = f\left(t, x(t), x^{\Delta}(t), \int_{t_0}^{t} k(t,s,x(s),x^{\Delta}(s))\,\Delta s\right),\ t_0, t \in I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}, \quad x(t_0) = x_0.$$

Sikorska-Nowak [33] uses Henstok–Kurzweil–Pettis delta integral on compact time scale $I_{\mathbb{T}} = [0, t_0] \cap \mathbf{T}$, $t_0 \geq 0$.

Let us note that many integrodifferential equations can be reduced to Volterra-type integral equations. Motivated by the above results, Reinfelds et al. [34] consider implicit Volterra integrodifferential equations on an arbitrary time scale $\mathbb{T}$

$$x^{\Delta}(t) = f(t) + \int_{t_0}^{t} k(t,\tau,x(\tau),x^{\Delta}(\tau))\,\Delta\tau, \quad x(t_0) = x_0.$$

Hyers–Ulam–Rassias stability of higher-order Volterra integrodifferential equations have been studied in the cases when $t \in (t_0, +\infty)$ [35,36]. Article [35] uses the Laplace transform method, while [36] uses the equation in explicit form.

In this paper, we consider nonlinear implicit k-th order Volterra-type integrodifferential equations from above on unbounded time scale $I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}$

$$x^{\Delta^k}(t) = f\left(t, x(t), x^{\Delta}(t), \dots, \Delta^k(t), \int_{t_0}^{t} k(t,s,x(s),x^{\Delta}(s),\dots,x^{\Delta^k}(s))\,\Delta s\right) \tag{1}$$

with initial conditions

$$x^{\Delta^i}(t_0) = x_i, \quad i = 0,1,2,\dots,k-1, \quad t_0, t \in I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T} \tag{2}$$

where $x \in \mathbb{R}^n$ is n-dimensional linear real space with the Euclidean norm $|\cdot|$. Let us note that

$$x^{\Delta^i}(t) = x_i + \int_{t_0}^{t} x^{\Delta^{i+1}}(s)\,\Delta s, \quad i = 0,1,2,\dots,k-1.$$

We define a new map $z\colon I_{\mathbb{T}} \to \mathbb{R}^{n(k+1)}$, where

$$\begin{aligned} z(t) &= (z_0(t), z_1(t), \ldots, z_k(t)) = \left(x(t), x^{\Delta}(t), \ldots, x^{\Delta^k}(t)\right) \\ &= \left(x_0 + \int_{t_0}^{t} x^{\Delta}(s)\,\Delta s, x_1 + \int_{t_0}^{t} x^{\Delta^2}(s)\,\Delta s, \ldots, x_{k-1}\right. \\ &+ \left.\int_{t_0}^{t} x^{\Delta^k}(s)\,\Delta s, f\left(t, x(t), x^{\Delta}(t), \ldots, x^{\Delta^k}(t), \int_{t_0}^{t} k(t,s,x(s),x^{\Delta}(s),\ldots,x^{\Delta^k}(s))\,\Delta s\right)\right). \end{aligned}$$

So, we have general implicit Volterra-type integral equation,

$$z(t) = F\left(t, z(t), \int_{t_0}^{t} K(t,s,z(s))\,\Delta s\right), \quad t_0, t \in I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}, \tag{3}$$

with integrand $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^{n(k+1)} \to \mathbb{R}^{n(k+1)}$

$$K(t,s,z(s)) = \left(x^{\Delta}(s) \ldots, x^{\Delta^k}(s), k(t,s,x(s),x^{\Delta}(s),\ldots,x^{\Delta^k}(s))\right),$$

where $z\colon I_{\mathbb{T}} \to \mathbb{R}^{n(k+1)}$ is the unknown map and $F(\cdot, z, w)\colon I_{\mathbb{T}} \to \mathbb{R}^{n(k+1)}$ is the rd-continuous map.

The main aim and innovation of the paper is the reduction of higher-order nonlinear implicit Volterra-type integrodifferential equations from above on unbounded time scales to Volterra-type integral equations without using repeated integration, which allows to find universal and, at the same time, conditionally simpler proofs for many basic properties of the Volterrra-type equations, including to prove the Hyers–Ulam–Rassias stability. In addition, repeated integration is quite inconvenient, and can be applied to explicit equations [32,36]. It can be applied by further deriving the implicit equation and imposing additional conditions on the smoothness of the right-hand side of equations [37].

2. Elements of the Time Scale Calculus

A *time scale* is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the topology induced by the standard topology on the real numbers $\mathbb{R}$. We denote a time scale by the symbol $\mathbb{T}$. Since time scales may or may not be connected, we need the concept of jump operators. For $t \in \mathbb{T}$ the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by the equality

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$$

while the *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by the equality

$$\rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}.$$

In this definition, we put $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$. The jump operators allow the classification of points in a time scale $\mathbb{T}$. If $\sigma(t) > t$, then the point $t \in \mathbb{T}$ is called *right-scattered*, while if $\rho(t) < t$, then the point $t \in \mathbb{T}$ is called *left-scattered*. If $\sigma(t) = t$, then $t \in \mathbb{T}$ is called *right-dense*, while if $\rho(t) = t$ then $t \in \mathbb{T}$ is called *left dense*. A function $g\colon \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$ and its left sided limits exist (finite) at left-dense points in $\mathbb{T}$. We define the *graininess function* $\mu\colon \mathbb{T} \to [0, +\infty)$ by the relation

$$\mu(t) = \sigma(t) - t.$$

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The function $g\colon \mathbb{T} \to \mathbb{R}$ is *regressive* if

$$1 + \mu(t)g(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

Assume $g\colon \mathbb{T} \to \mathbb{R}$ is a function and fix $t \in \mathbb{T}^\kappa$. The *delta derivative* (also called the Hilger derivative) $g^\Delta(t)$ exists if, for every $\varepsilon > 0$, there exists a neighbourhood $U = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$, such that

$$\left|(g(\sigma(t)) - g(s)) - g^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

Moreover, we say that g is *delta differentiable* on $\mathbb{T}^\kappa$, provided that $g^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The higher-order derivatives are denoted by g^{Δ^i}, where $i = 2, 3, \ldots$ and $g^{\Delta^0} = g$, $g^{\Delta^1} = g^\Delta$.

If g is rd-continuous, than there is function G such that $G^\Delta(t) = g(t)$. In this case, we define the (Cauchy) *delta integral* by

$$\int_r^s g(t)\,\Delta t = G(s) - G(r), \text{ for all } r, s \in \mathbb{T}.$$

Whether map $g\colon \mathbb{T} \to \mathbb{R}^n$ is rd-continuous or regressive is defined analogically. The same can be said about delta derivatives and delta integrals.

Let $\beta\colon \mathbb{T} \to \mathbb{R}$ be a nonnegative (and therefore regressive) and rd-continuous scalar function. The Cauchy initial value problem for scalar linear equation

$$x^\Delta = \beta(t)x, \quad x(t_0) = 1, \quad t_0 \in \mathbb{T}$$

has the unique solution $e_\beta(\cdot, t_0)\colon \mathbb{T} \to \mathbb{R}$ [14,15]. More explicitly, using the cylinder transformation, the *exponential function* $e_\beta(\cdot, t_0)$ is given by

$$e_\beta(t, t_0) = \exp\left(\int_{t_0}^t \xi_{\mu(s)}(\beta(s))\,\Delta s\right),$$

where

$$\xi_h(z) = \begin{cases} z, & h = 0; \\ \frac{1}{h}\log(1 + hz), & h > 0. \end{cases}$$

Observe that we also have Bernoulli's type estimate [38]

$$1 + \int_{t_0}^t \beta(s)\,\Delta s \le e_\beta(t, t_0) \le \exp\left(\int_{t_0}^t \beta(s)\,\Delta s\right)$$

for all $t \in I_\mathbb{T} = [t_0, +\infty) \cap \mathbb{T}$.

3. Volterra-Type Integral Equations

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. If $z \in \mathbb{R}^{n(k+1)}$, then $|z| = \max_{0 \le i \le k} |z_i|$. We will consider the linear space of k times delta differentiable functions $C^k(I_\mathbb{T}; \mathbb{R}^n)$, such that

$$\sup_{t \in I_\mathbb{T}} \frac{\max_{0 \le i \le k}\{|x^{\Delta^i}(t)|\}}{e_\beta(t, t_0)} < \infty.$$

and denote this special space by $C^k_\beta(I_\mathbb{T}; \mathbb{R}^{n(k+1)})$. The space $C^k_\beta(I_\mathbb{T}; \mathbb{R}^{n(k+1)})$ endowed with Bielecki type norm

$$\|z\|^k_\beta = \sup_{t \in I_\mathbb{T}} \frac{|z(t)|}{e_\beta(t, t_0)} = \sup_{t \in I_\mathbb{T}} \frac{\max_{0 \le i \le k}\{|z_i(t)|\}}{e_\beta(t, t_0)} = \sup_{t \in I_\mathbb{T}} \frac{\max_{0 \le i \le k}\{|x^{\Delta^i}(t)|\}}{e_\beta(t, t_0)}$$

is a Banach space [19,22,23].

Theorem 1. *Consider the integral Equation (3) with $I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}$. Let $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^{n(k+1)} \to \mathbb{R}^{n(k+1)}$ be rd-continuous in its first and second variable, $F(\cdot, z, w)\colon I_{\mathbb{T}} \to \mathbb{R}^{n(k+1)}$ and $L\colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$, $\beta(s) = L(s)\gamma$,*

$$|F(t,z,w) - F(t,\bar{z},\bar{w})| \le M(|z-\bar{z}| + |w - \bar{w}|), \quad z, \bar{z}, w, \bar{w} \in \mathbb{R}^{n(k+1)},$$

$$|K(t,s,z) - K(t,s,\bar{z})| \le L(s)|z - \bar{z}|, \quad s < t, \tag{4}$$

$$m = \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \left| F\left(t, 0, \int_{t_0}^{t} K(t,s,0)\,\Delta s\right) \right| < \infty. \tag{5}$$

If $M(1 + 1/\gamma) < 1$, then integral Equation (3) has a unique solution $z_* \in C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$.

Proof. Let operator $H\colon C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)}) \to C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$ be defined by

$$H(z(t)) = F\left(t, z(t), \int_{t_0}^{t} K(t,s,z(t))\,\Delta s\right). \tag{6}$$

The fixed point of operator H will be the solution of integral Equation (3). Thus, we want to prove that there exists a unique $z_* \in C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$, such that $Hz_* = z_*$. To do this, we show that the conditions of Banach's fixed point theorem are satisfied. Taking norms in (6), we obtain

$$\begin{aligned}
\|Hz\|^k_\beta &\le \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \left| F\left(t, 0, \int_{t_0}^{t} K(t,s,0)\,\Delta s\right) \right| \\
&+ \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \left| F\left(t, z(t), \int_{t_0}^{t} K(t,s,z(t))\,\Delta s\right) - F\left(t, 0, \int_{t_0}^{t} K(t,s,0)\,\Delta s\right) \right| \\
&\le m + \sup_{t \in I_{\mathbb{T}}} \frac{M}{e_\beta(t,t_0)} \left(|z(t)| + \int_{t_0}^{t} L(s)|z(s)|\,\Delta s \right) \\
&\le m + M\|z\|^k_\beta \left(1 + \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^{t} L(s) e_\beta(s,t_0)\,\Delta s \right) \\
&= m + M\|z\|^k_\beta \left(1 + \frac{1}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^{t} \beta(s) e_\beta(s,t_0)\,\Delta s \right) \\
&= m + M\|z\|^k_\beta \left(1 + \frac{1}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^{t} e^\Delta_\beta(s,t_0)\,\Delta s \right) \\
&= m + M\|z\|^k_\beta \left(1 + \frac{1}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} [e_\beta(s,t_0)]^t_{t_0} \right) \\
&= m + M\|z\|^k_\beta \left(1 + \frac{1}{\gamma} \sup_{t \in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,t_0)} \right) \right) \\
&\le m + M\|z\|^k_\beta \left(1 + \frac{1}{\gamma} \right) < \infty.
\end{aligned}$$

This proves that the operator H maps $C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$ into itself.

Next, we verify that H is a contraction map. For any $z, \bar{z} \in C^k_\beta(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$, we have the estimate

$$
\begin{aligned}
\|Hz - H\bar{z}\|_\beta^k &= \sup_{t\in I_{\mathbb{T}}} \frac{|[H(z(t)) - H(\bar{z}(t))|}{e_\beta(t,t_0)} \\
&= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \left| F\left(t, z(t), \int_{t_0}^t K(t,s,z(s))\,\Delta s\right) - F\left(t, \bar{z}(t), \int_{t_0}^t K(t,s,\bar{z}(s))\,\Delta s\right) \right| \\
&\le M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \left((|z(t) - \bar{z}(t)|) + \left| \int_{t_0}^t K(t,s,z(s))\,\Delta s - \int_{t_0}^t K(t,s,\bar{z}(s))\,\Delta s \right| \right) \\
&\le M\left(\|z-\bar{z}\|_\beta^k + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^t L(s)|z(s) - \bar{z}(s)|\,\Delta s \right) \\
&\le M\|z-\bar{z}\|_\beta^k \left(1 + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^t L(s) e_\beta(s,t_0)\,\Delta s \right) \\
&= M\|z-\bar{z}\|_\beta^k \left(1 + \frac{1}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^t \beta(s) e_\beta(s,t_0)\,\Delta s \right) \\
&= M\|z-\bar{z}\|_\beta^k \left(1 + \frac{1}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} \int_{t_0}^t e_\beta^\Delta(s,t_0)\,\Delta s \right) \\
&= M\|z-\bar{z}\|_\beta^k \left(1 + \frac{1}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,t_0)} [e_\beta(s,t_0)]_{t_0}^t \right) \\
&= M\|z-\bar{z}\|_\beta^k \left(1 + \frac{1}{\gamma} \sup_{t\in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,t_0)} \right) \right) \\
&\le M\left(1 + \frac{1}{\gamma}\right) \|z-\bar{z}\|_\beta^k.
\end{aligned}
$$

As $M\left(1+\frac{1}{\gamma}\right) < 1$, we obtain that H has a unique fixed point $z_* \in C_\beta^k(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$. The fixed point of H is, however, a solution of (3). The proof is complete. □

4. Hyers–Ulam–Rassias Stability

Definition 1. *We say that integral Equation (3) is Hyers–Ulam–Rassias stable if there exists a constant $C > 0$ such that, for each real number $\varepsilon > 0$, and for each solution $z \in C_\beta^k(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$ of the inequality*

$$\left\| z(t) - F\left(t, z(t), \int_{t_0}^t K(t,s,z(s))\,\Delta s\right) \right\|_\beta^k \le \varepsilon,$$

there exists the exact solution $z_* \in C_\beta^k(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$ of the integral Equation (3) with the property

$$\|z(t) - z_*(t)\|_\beta^k \le C\varepsilon.$$

Let us find sufficient conditions for the Hyers–Ulam–Rassias stability of nonlinear Volterra-type integral equation on arbitrary time scales.

Theorem 2. *Consider the integral Equation (3) satisfying conditions of Theorem 1. Suppose $z \in C_\beta^k(I_{\mathbb{T}}; \mathbb{R}^{n(k+1)})$ is such a map that satisfies the inequality*

$$\left\| z(t) - F\left(t, z(t), \int_{t_0}^t K(t,s,z(s))\,\Delta s\right) \right\|_\beta^k \le \varepsilon.$$

Then, integral Equation (3) is Hyers–Ulam–Rassias stable.

Proof. According to Theorem 1, there is unique solution $z_* \in C^k_\beta(I_\mathbb{T};\mathbb{R}^{n(k+1)})$ of the integral Equation (3). Let $z \in C^k_\beta(I_\mathbb{T};\mathbb{R}^{n(k+1)})$. From the proof of Theorem 1, we obtain the estimate

$$\begin{aligned} & \left|\int_{t_0}^t K(t,s,z(s))\,\Delta s - \int_{t_0}^t K(t,s,z_*(s))\,\Delta s\right| \\ \leq\ & \int_{t_0}^t L(s)|z(s)-z_*(s)|\,\Delta s \\ =\ & \frac{1}{\gamma}\int_{t_0}^t \beta(s)e_\beta(s,t_0)\frac{|z(s)-z_*(s)|}{e_\beta(s,t_0)}\,\Delta s \\ \leq\ & \frac{1}{\gamma}\int_{t_0}^t \beta(s)e_\beta(s,t_0)\|z(s)-z_*(s)\|^k_\beta\,\Delta s \\ =\ & \frac{\|z-z_*\|^k_\beta}{\gamma}\int_{t_0}^t e^\Delta_\beta(s,t_0)\,\Delta s \leq \frac{\|z-z_*\|^k_\beta}{\gamma}e_\beta(t,t_0). \end{aligned}$$

Therefore, from the triangle inequality, we obtain the upper bound

$$\begin{aligned} \|z(t)-z_*(t)\|^k_\beta \leq & \left\|z(t) - F\left(t, z(t), \int_{t_0}^t K(t,s,z(s))\,\Delta s\right)\right\|^k_\beta \\ & + \left\|F\left(t,z(t),\int_{t_0}^t K(t,s,z(s))\,\Delta s\right) - F\left(t,z_*(t),\int_{t_0}^t K(t,s,z_*(s))\,\Delta s\right)\right\|^k_\beta \\ & \leq \varepsilon + M\left(1+\frac{1}{\gamma}\right)\|z(t)-z_*(t)\|^k_\beta. \end{aligned}$$

Hence,

$$\|z(t)-z_*(t)\|^k_\beta \leq C\varepsilon, \tag{7}$$

where $C = \left(1 - M\left(1+\frac{1}{\gamma}\right)\right)^{-1}$. □

Corollary 1. *We will prove that nonlinear implicit k-th order Volterra-type integrodifferential Equations* (1) *and* (2) *are Hyers–Ulam–Rassias stable.*

Proof. We assume that the k-times delta differentiable map $x_* \colon I_\mathbb{T} \to \mathbb{R}^n$ with initial conditions (2) is a solution of Equation (1). In addition, we assume that k-times delta differentiable map $x\colon I_\mathbb{T} \to \mathbb{R}^n$ satisfies the initial conditions (2) and inequality

$$\left|x^{\Delta^k}(t) - f\left(t, x(t), x^\Delta(t), \dots, \Delta^k(t), \int_{t_0}^t k(t,s,x(s),x^\Delta(s),\dots,x^{\Delta^k}(s))\,\Delta s\right)\right| \leq \varepsilon\exp_\beta(t,t_0).$$

Then, according to Theorem 2, we obtain the estimate

$$|x(t)-x_*(t)| \leq C\varepsilon\exp_\beta(t,t_0)$$

for all $t \in I_\mathbb{T}$ or, in other words, Equation (1) together with the initial conditions (2) is Hyers–Ulam–Rassias stable.

If, in addition to the time scale, $I_\mathbb{T}$ is bounded, then

$$\sup_{t\in I_\mathbb{T}} |\exp_\beta(t,t_0)| \leq N \leq +\infty.$$

Then, for all $t \in I_\mathbb{T}$, we obtain a more accurate inequality

$$|x(t)-x_*(t)| \leq CN\varepsilon.$$

□

Example 1. *Consider the scalar nonlinear Volterra integrodifferential equation on arbitrary time scale* $I_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}$

$$x^{\Delta}(t) = \frac{1}{2}\left(t^2 + x(t) + \int_{t_0}^{t} (2 + s + \sigma(s))[x(s)^2 + x^{\Delta}(s)^2 + 1]^{\frac{1}{2}}\,\Delta s\right),$$

where $x(t_0) = x_0,\ t_0, t \in I_{\mathbb{T}}$ *and* $t_0 \geq 0$.

We will prove that this integrodifferential equation has a unique solution, and evaluate the Hyers–Ulam–Rassias constant of stability.

Proof. We will first apply Theorem 1 and check the fact that

$$K(t, s, q, r) = (2 + s + \sigma(s))(q^2 + r^2 + 1)^{\frac{1}{2}}$$

has the bounded partial derivatives with respect to q and r everywhere. So, we have

$$|K(t, s, q, r) - K(t, s, \bar{q}, \bar{r}])| \leq \sqrt{2}(2 + s + \sigma(s)) \max(|q - \bar{q}|, |r - \bar{r}|),$$

where we used Hadamard's Lemma. Therefore, (4) can be defined with $L(s) = \sqrt{2}(2 + s + \sigma(s))$. We choose $\gamma = \sqrt{2}$, then we have $\beta(s) = 2(2 + s + \sigma(s))$. Considering that

$$\int_{t_0}^{t} (2 + s + \sigma(s))\,\Delta s = 2t + t^2 - 2t_0 - t_0^2$$

and, according to Bernoulli's type estimate $e_{\beta}(t, t_0) \geq 1 + 2\big(2t + t^2 - 2t_0 - t_0^2\big)$, we verified that (5) holds. The existence and uniqueness of solution now follows from Theorem 1.

Additionally, in our example, $M = 1/2$. Therefore, the constant in the evaluation of Hyers–Ulam–Rassias stability is $C = \frac{2\sqrt{2}}{\sqrt{2}-1}$. □

5. Conclusions

In this article, we studied the Hyers–Ulam–Rassias stability of nonlinear implicit higher-order Volterra-type integrodifferential equations on time scales, using the Banach fixed point theorem and a generalization of the Bielecki-type norm. We presented sufficient conditions for Hyers–Ulam–Rassias stability of nonlinear implicit higher-order Volterra-type integrodifferential equations from above on unbounded time scales. We reduced the higher-order integrodifferential equation to the system of integral equations. It allows the theory of Volterra integral equations to be used. We used the Lipschitz type rd-continuous function $L\colon I_{\mathbb{T}} \to \mathbb{R}$ instead of the Lipschitz coefficient, which can be an unbounded function in our result. Such an approach allows for the use of the Banach contraction principle. The aim of this article was achieved by proving the Hyers–Ulam–Rassias stability in the more general case. An example was presented to illustrate the theoretical results.

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