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Generalized Weak Contractions Involving a Pair of Auxiliary Functions via Locally Transitive Binary Relations and Applications to Boundary Value Problems

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Abstract: The intent of this paper was to investigate the fixed-point results under relation-theoretic generalized weak contractivity condition employing a pair of auxiliary functions ϕ and ψ verifying appropriate properties. In proving our outcomes, we observed that the partial-ordered relation (even, transitive relation) adopted by earlier authors can be weakened to the extent of a locally $\mathcal{F}$-transitive binary relation. The findings proved herewith generalize, extend, improve, and unify a number of existing outcomes. To validate of our findings, we offer a number of illustrative examples. Our outcomes assist us to figure out the existence and uniqueness of solutions to a boundary value problem.

Keywords: fixed points; binary relations; generalized weak contractions; boundary value problems

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1. Introduction

There are numerous approaches in nonlinear functional analysis to handling various problems that emerge in real-world situations. Owing to their potential applications, various disciplines within nonlinear functional analysis have drawn significant attention in recent years. In this context, we mainly refer to the most recent outcomes, especially those contained in [1–5]. The oldest and a widely recognised outcome of nonlinear functional analysis is the classical BCP. Indeed, the BCP continues to inspire researchers of fixed point theory. A variety of extensions of BCP have been developed, incorporating relatively general contraction conditions. A self-map $\mathcal{F}$ composed on a MS $(\mathbf{W}, \varrho)$ is termed as a contraction if $\exists\, \delta \in [0,1)$, verifying

$$\varrho(\mathcal{F}w, \mathcal{F}v) \leq \delta \cdot \varrho(w, v). \tag{1}$$

Browder [6], in 1968, investigated the idea of φ-contraction. In this class of generalized contraction, the Lipschitzian constant $\delta \in [0,1)$ involved in inequality (1) is altered with a control function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Various researchers have generalized the Browder fixed point theorem by improving the characterizations of the control function φ. However, we indicate two well-known generalizations of the Browder fixed point theorem established by Boyd and Wong [7] and Matkowski [8].

Theorem 1 (Boyd–Wong Theorem [7])**.** *If $\mathcal{F}$ is a φ-contraction in a CMC $(\mathbf{W}, \varrho)$, wherein $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ remains a right upper semicontinuous function verifying $\varphi(t) < t$ for every $t > 0$, then $\mathcal{F}$ owns a unique fixed point.*

Theorem 2 (Matkowski Theorem [8])**.** *If $\mathcal{F}$ is a φ-contraction in a CMC $(\mathbf{W}, \varrho)$, wherein $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ remains an increasing function verifying $\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t > 0$, then $\mathcal{F}$ enjoys a unique fixed point.*

Note that both Theorems 1 and 2 are independent of each other. φ-contraction yields to (1) subject to the limitation $\varphi(t) = \delta \cdot t,\ 0 < \delta < 1$, and so each of Theorems 1 and 2 reduces to the classical BCP.

Needless to say, the contraction condition (1) is identical to

$$\varrho(\mathcal{F}w, \mathcal{F}v) \leq \varrho(w, v) - \lambda \cdot \varrho(w, v) \quad (0 < \lambda \leq 1), \tag{2}$$

which can be deduced from (1) by setting $\delta = 1 - \lambda$. A nonlinear version of inequality (2) is called weak contraction. Nevertheless, a self-map $\mathcal{F}$ composed on an MS $(\mathbf{W}, \varrho)$ is recognised as weak ψ-contraction if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a map verifying

$$\varrho(\mathcal{F}w, \mathcal{F}v) \leq \varrho(w, v) - \psi(\varrho(w, v)).$$

For the restriction $\psi(t) = (1 - \delta)t$, weak ψ-contraction turns into (1). It is also pointed out that φ-contraction and weak ψ-contraction are equivalent in the virtue of the relation $\varphi = I - \psi$, whereas I remains the identity function on $\mathbb{R}_+$.

The prospect of weak ψ-contraction was introduced by Krasnosel'skiǐ et al. [9]. In 1997, Alber and Guerre-Delabriere [10] invented conscious fixed-point outcomes involving weak ψ-contraction in Hilbert spaces. Following Khan et al. [11], a continuous as well as increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\psi^{-1}(0) = \{0\}$ is termed as an altering distance function . Rhoades [12] extended the BCP to weak ψ-contraction, which can be outlined as follows:

Theorem 3 ([12])**.** *If $\mathcal{F}$ is a weak ψ-contraction in a CMS in $(\mathbf{W}, \varrho)$, wherein $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ remains an altering distance function, then $\mathcal{F}$ enjoys a unique fixed point.*

The contraction inequality (1) can be referred to as follows:

$$\varrho(\mathcal{F}w, \mathcal{F}v) \leq (1 + \beta)\varrho(w, v) - (1 - \beta)\varrho(w, v) \quad \forall\ w, v \in \mathbf{W}.$$

where $\delta = 2\beta$ with $\beta \in [0, 1/2)$. Motivated by this fact, Rhoades et al. [13] generalized the prospect of weak ψ-contraction by investigating the prospect of weak $(\phi - \psi)$-contraction (see Cho [14]), as follows:

$$\varrho(\mathcal{F}w, \mathcal{F}v) \leq \phi(\varrho(w, v) - \psi(\varrho(w, v)) \quad \forall\ w, v \in \mathbf{W},$$

whereas the auxiliary functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ verify the following properties:

(a) ϕ remains upper semicontinuous and increasing;
(b) ψ remains lower semicontinuous and decreasing;
(c) $\phi(0) = \psi(0)$;
(d) for all $t > 0$, $\phi(t), \psi(t) > 0$;
(e) for every $t > 0$, $\phi(t) - \psi(t) < t$.

In particular, $\phi = I$, the identity function, weak $(\phi - \psi)$-contraction reduces to weak ψ-contraction.

In the last few years, a number of scholars have broadened and enhanced the BCP in the context of an MS that contains a partial order, c.f. [15–18]. In this direction, Rhoades et al. [13] established certain fixed-point findings under weak $(\phi - \psi)$-contraction in the framework of ordered MS. Alam and Imdad [19] investigated a natural extension of the BCP in an MS comprising an arbitrary BR. For further discussions on relation-theoretic metric fixed point theory, we refer to [20–29]. Such outcomes involve relatively weaker contraction conditions which hold for comparative elements only. Owing to such limitations, these findings are utilised to solve special kinds of matrix equations, integral equations, and BVP.

The purpose of this paper is three-fold:

- We noticed that the monotonicity requirement of ϕ and ψ and assumption (d) are unnecessary. Thus, we improve the class of weak $(\phi - \psi)$-contractions by removing these assumptions.
- Utilizing this enlarged class of weak $(\phi - \psi)$-contractions, we investigate the outcomes on the fixed points in the setup of an MS comprising locally $\digamma$-transitive BR. Demonstrating our findings, a variety of illustrative examples are proposed. In process, we derive a few existing outcomes from our findings.
- To put into use our findings in practice, we turn to a unique solution of a first-order BVP, verifying certain additional hypotheses in the presence of a lower solution.

Likewise, for relation-theoretic contraction principles [19], in order to prove their relation-theoretic formulations, a few generalized contractions require an arbitrary binary relation for the existence of fixed points of such a map. Apart from this, in the context of nonlinear contractions, the transitivity of the underlying relation is additionally required. But the transitivity requirement is very restrictive. In order to employ an optimal condition of transitivity, we utilized the concept of locally $\digamma$-transitive BR.

2. Preliminaries

Given a set $\mathbf{W}$, any subset $\mathcal{L} \subseteq \mathbf{W}^2$ is regarded as a BR on $\mathbf{W}$. In the upcoming definitions, $\mathbf{W}$ refers to a set, $\mathcal{L}$ refers to a BR on $\mathbf{W}$, ϱ refers to a metric on $\mathbf{W}$ and $\digamma : \mathbf{W} \to \mathbf{W}$ refers to a map.

Definition 1 ([19])**.** *Two points $w, v \in \mathbf{W}$ are named $\mathcal{L}$-comparative if $(w, v) \in \mathcal{L}$ or $(v, w) \in \mathcal{L}$. We denote these by $[w, v] \in \mathcal{L}$.*

Definition 2 ([30])**.** *$\mathcal{L}^{-1} := \{(w, v) \in \mathbf{W}^2 : (v, w) \in \mathcal{L}\}$ is named the inverse of $\mathcal{L}$. Moreover, $\mathcal{L}^s := \mathcal{L} \cup \mathcal{L}^{-1}$ is named the symmetric closure of $\mathcal{L}$.*

Remark 1 ([19])**.** $(w, v) \in \mathcal{L}^s \Longleftrightarrow [w, v] \in \mathcal{L}$.

Definition 3 ([31])**.** *The restriction to $\mathcal{L}$ on a given subset $\mathbf{U} \subseteq \mathbf{W}$ is described as the BR $\mathcal{L}|_{\mathbf{U}} := \mathcal{L} \cap \mathbf{U}^2$.*

Definition 4 ([21])**.** *$\mathcal{L}$ is regarded as locally $\digamma$-transitive if for every countably infinite set $\mathbf{U} \subset \digamma(\mathbf{W})$, $\mathcal{L}|_{\mathbf{U}}$ is transitive.*

Definition 5 ([32])**.** *A subset $\mathbf{U} \subseteq \mathbf{W}$ is regarded as $\mathcal{L}$-directed if for any $w, v \in \mathbf{U}$, $\exists\, u \in \mathbf{W}$ satisfies $(w, u) \in \mathcal{L}$ and $(v, u) \in \mathcal{L}$.*

Definition 6 ([19])**.** *$\mathcal{L}$ is named $\digamma$-closed if*

$$(w, v) \in \mathcal{L} \Longrightarrow (\digamma w, \digamma v) \in \mathcal{L}.$$

Definition 7 ([19]). *A sequence $\{w_n\} \subset \mathbf{W}$ is named $\mathcal{L}$-preserving, if for every $n \in \mathbb{N}$, it verifies $(w_n, w_{n+1}) \in \mathcal{L}$.*

Definition 8 ([19]). *$\mathcal{L}$ is named ϱ-self-closed if for any $\mathcal{L}$-preserving sequence $\{w_n\} \subset \mathbf{W}$ enjoying $w_n \xrightarrow{\varrho} w^*$, there exists $\exists$ a subsequence $\{w_{n_\iota}\}$ with $[w_{n_\iota}, w^*] \in \mathcal{L}$.*

Definition 9 ([20]). *An MS in which those Cauchy sequences are $\mathcal{L}$-preserving converge is named $\mathcal{L}$-complete.*

Definition 10 ([20]). *$\digamma$ is named $\mathcal{L}$-continuous if for every $w \in \mathbf{W}$ and for any $\mathcal{L}$-preserving sequence $\{w_n\} \subset \mathbf{W}$,*

$$w_n \xrightarrow{\varrho} w^* \Longrightarrow \digamma(w_n) \xrightarrow{\varrho} \digamma(w^*).$$

Let Γ denote the class of the pair (ϕ, ψ) of self-functions on $\mathbb{R}_+$, verifying the following properties:

Γ_1: ϕ remains upper semicontinuous;
Γ_2: ψ remains lower semicontinuous;
Γ_3: $\phi(0) = \psi(0)$;
Γ_4: for every $t > 0$, $\phi(t) - \psi(t) < t$.

The following fact can be argued, implementing the symmetry of the metric ϱ.

Proposition 1. *Given $(\phi, \psi) \in \Gamma$, the following assumptions are identical:*

(I) $\varrho(\digamma w, \digamma v) \leq \phi(\varrho(w, v)) - \psi(\varrho(w, v)), \quad \forall\, (w, v) \in \mathcal{L},$
(II) $\varrho(\digamma w, \digamma v) \leq \phi(\varrho(w, v)) - \psi(\varrho(w, v)), \quad \forall\, [w, v] \in \mathcal{L}.$

Eventually, we address the following ancillary outcomes.

Proposition 2 ([21]). *For every $n \in \mathbb{N}_0$, $\mathcal{L}$ is $\digamma^n$-closed provided that $\mathcal{L}$ remains $\digamma$-closed.*

Lemma 1 ([17]). *If $\{w_n\}$ is a non-Cauchy sequence in an MS $(\mathbf{W}, \varrho)$, then we can determine $\epsilon > 0$ and two subsequences $\{w_{n_\iota}\}$ and $\{w_{m_\iota}\}$ of $\{w_n\}$, which verify*

(i) $\iota \leq m_\iota < n_\iota, \quad \forall\, \iota \in \mathbb{N},$
(ii) $\varrho(w_{m_\iota}, w_{n_\iota}) > \epsilon, \quad \forall\, \iota \in \mathbb{N},$
(iii) $\varrho(w_{m_\iota}, w_{n_{\iota-1}}) \leq \epsilon, \quad \forall\, \iota \in \mathbb{N}.$

Meanwhile, if $\lim\limits_{n\to\infty} \varrho(w_n, w_{n+1}) = 0$, then

(iv) $\lim\limits_{\iota\to\infty} \varrho(w_{m_\iota}, w_{n_\iota}) = \epsilon,$
(v) $\lim\limits_{\iota\to\infty} \varrho(w_{m_\iota}, w_{n_\iota+1}) = \epsilon,$
(vi) $\lim\limits_{\iota\to\infty} \varrho(w_{m_\iota+1}, w_{n_\iota}) = \epsilon,$
(vii) $\lim\limits_{\iota\to\infty} \varrho(w_{m_\iota+1}, w_{n_\iota+1}) = \epsilon.$

3. Main Results

The following outcome ensures the availability of a fixed point to a weak $(\phi - \psi)$-contraction mapping assigned to the relational MS.

Theorem 4. *Assuming that $(\mathbf{W}, \varrho)$ stays an MS, $\digamma : \mathbf{W} \to \mathbf{W}$ constitutes a map, and $\mathcal{L}$ is a BR on $\mathbf{W}$. Also,*

(i) *$(\mathbf{W}, \varrho)$ is $\mathcal{L}$-complete;*
(ii) *$\exists\, w_0 \in \mathbf{W}$ contains $(w_0, \digamma w_0) \in \mathcal{L}$;*

(iii) *$\mathcal{L}$ is $\digamma$-closed and locally $\digamma$-transitive;*
(iv) **W** *remains $\mathcal{L}$-continuous, or $\mathcal{L}$ constitutes ϱ-self-closed;*
(v) $\exists\,(\phi,\psi)\in\Gamma$ *has*

$$\varrho(\digamma w,\digamma v)\le\phi(\varrho(w,v))-\psi(\varrho(w,v)),\qquad\forall\,(w,v)\in\mathcal{L}.$$

Then, $\digamma$ admits a fixed point.

Proof. Considering $w_0\in\mathbf{W}$ as an initial point, we construct the sequence $\{w_n\}\subset\mathbf{W}$ verifying

$$w_n:=\digamma^n(w_0)=\digamma(w_{n-1}),\quad\forall\,n\in\mathbb{N}.\tag{3}$$

By (ii), $\digamma$-closedness of $\mathcal{L}$ and Proposition 2, we conclude

$$(\digamma^n w_0,\digamma^{n+1}w_0)\in\mathcal{L},$$

so that

$$(w_n,w_{n+1})\in\mathcal{L},\quad\forall\,n\in\mathbb{N}_0.\tag{4}$$

Hence, $\{w_n\}$ is a $\mathcal{L}$-preserving sequence.

Denote $\varrho_n:=\varrho(w_n,w_{n+1})$. If $\exists\,n_0\in\mathbb{N}_0$ with $\varrho_{n_0}=\varrho(w_{n_0},w_{n_0+1})=0$, then owing to (3), we find $\digamma(w_{n_0})=w_{n_0}$, and hence the result is concluded.

Another option is $\varrho_n>0$, $\forall\,n\in\mathbb{N}_0$. Utilizing (v), (3) and (4), we attain

$$\begin{aligned}\varrho_n&=\varrho(w_n,w_{n+1})=\varrho(\digamma w_{n-1},\digamma w_n)\\&\le\phi(\varrho(w_{n-1},w_n))-\psi(\varrho(w_{n-1},w_n)),\end{aligned}$$

thereby implying

$$\varrho_n\le\phi(\varrho_{n-1})-\psi(\varrho_{n-1}),\quad\forall\,n\in\mathbb{N}_0.\tag{5}$$

By Γ_4 in (5), we find

$$\varrho_n<\varrho_{n-1},\quad\forall\,n\in\mathbb{N}.$$

Therefore, $\{\varrho_n\}\subset\mathbb{R}_+$ is a decreasing sequence, so exists $\exists\,l\ge0$ with

$$\lim_{n\to\infty}\varrho_n=l.\tag{6}$$

If possible, assuming that $l>0$, utilizing the upper limit in (5) and by Γ_1,Γ_2 and (6), we conclude the following:

$$\begin{aligned}\limsup_{n\to\infty}\varrho_n&\le\limsup_{n\to\infty}\phi(\varrho_{n-1})+\limsup_{n\to\infty}[-\psi(\varrho_{n-1})]\\&=\limsup_{n\to\infty}\phi(\varrho_{n-1})-\liminf_{n\to\infty}\psi(\varrho_{n-1})\\&\le\phi(l)-\psi(l)\end{aligned}$$

so that $l\le\phi(l)-\psi(l)$, which contradicts Γ_4. Hence, $l=0$. Consequently, we have

$$\lim_{n\to\infty}\varrho_n=0.\tag{7}$$

Now, we aim to verify the Cauchyness of $\{w_n\}$. On the contrary, we assume that $\{w_n\}$ is not Cauchy, due to Lemma 1, $\exists\,\epsilon>0$, and two subsequences, $\{w_{n_\iota}\}$ and $\{w_{m_\iota}\}$, of $\{w_n\}$, which verify

$$\iota\le m_\iota<n_\iota,\ \varrho(w_{m_\iota},w_{n_\iota})>\epsilon\ge\varrho(w_{m_\iota},w_{n_{\iota-1}}),\quad\forall\,\iota\in\mathbb{N}.$$

Denote $\delta_\iota := \varrho(w_{m_\iota}, w_{n_\iota})$. Since $\{w_n\}$ is $\mathcal{R}$-preserving (owing to (4)) and $\{w_n\} \subset F(\mathbf{W})$ (owing to (3)), via the local F-transitivity of $\mathcal{L}$, we have $(w_{m_\iota}, w_{n_\iota}) \in \mathcal{L}$. Using (v), we obtain

$$\begin{aligned} \varrho(w_{m_\iota+1}, w_{n_\iota+1}) &= \varrho(F w_{m_\iota}, F w_{n_\iota}) \\ &\leq \phi(\varrho(w_{m_\iota}, w_{n_\iota})) - \psi(\varrho(w_{m_\iota}, w_{n_\iota})) \end{aligned}$$

so that

$$\varrho(w_{m_\iota+1}, w_{n_\iota+1}) \leq \phi(\delta_\iota) - \psi(\delta_\iota). \tag{8}$$

Utilizing the upper limit in (8) and by Γ_1, Γ_2 and Lemma 1, we find

$$\begin{aligned} \epsilon &= \limsup_{\iota\to\infty} \varrho(w_{m_\iota+1}, w_{n_\iota+1}) \\ &\leq \limsup_{\iota\to\infty} \phi(\delta_\iota) - \liminf_{\iota\to\infty} \phi(\delta_\iota) \\ &\leq \phi(\epsilon) - \psi(\epsilon) \\ &< \epsilon, \end{aligned}$$

which yields a contradiction. Hence, $\{w_n\}$ is Cauchy, and also remains $\mathcal{L}$-preserving. Consequently, due to the $\mathcal{L}$-completeness of $\mathbf{W}$, $\exists\, w \in \mathbf{W}$ exists with $w_n \xrightarrow{\varrho} w$.

Finally, we will apply condition (iv) to prove that $w \in Fix(F)$. Firstly, assuming the $\mathcal{L}$-continuity of F, we find $w_{n+1} = F(w_n) \xrightarrow{\varrho} F(w)$, which yields $F(w) = w$. In the case that $\mathcal{L}$ is ϱ-self-closed, we can pinpoint a subsequence $\{w_{n_\iota}\}$ of $\{w_n\}$ satisfying $[w_{n_\iota}, w] \in \mathcal{L}$, $\forall\, \iota \in \mathbb{N}$. Utilizing (v) and Proposition 1, we find

$$\begin{aligned} \varrho(w_{n_\iota+1}, F w) &= \varrho(F w_{n_\iota}, F w) \\ &\leq \phi(\varrho(w_{n_\iota}, w)) - \psi(\varrho(w_{n_\iota}, w)). \end{aligned}$$

We assert that

$$\varrho(w_{n_\iota+1}, F w) \leq \varrho(w_{n_\iota}, w), \quad \forall\, \iota \in \mathbb{N}. \tag{9}$$

Consider the subsets $\mathbb{N}^0$ and $\mathbb{N}^+$ of $\mathbb{N}$ such that $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$; these verify that

(a) $\varrho(w_{n_\iota}, w) = 0, \quad \forall\, \iota \in \mathbb{N}^0,$

(b) $\varrho(w_{n_\iota}, w) > 0, \quad \forall\, \iota \in \mathbb{N}^+.$

In case (a), utilizing Γ_3, we have $\varrho(F w_{n_\iota}, F w) = 0$, $\forall\, \iota \in \mathbb{N}^0$, implying that $\varrho(w_{n_\iota+1}, F w) = 0$, $\forall\, \iota \in \mathbb{N}^0$, so (9) holds for every $\iota \in \mathbb{N}^0$. In case (b), using Γ_4, we have

$$\varrho(w_{n_\iota+1}, F w) \leq \phi(\varrho(w_{n_\iota}, w)) - \psi(\varrho(w_{n_\iota}, w)) < \varrho(w_{n_\iota}, w),\ \forall\, \iota \in \mathbb{N}^+;$$

so (9) holds for every $\iota \in \mathbb{N}^+$. Hence, (9) holds for every $\iota \in \mathbb{N}$. Utilizing the limit of (9) as well as $w_{n_\iota} \xrightarrow{\varrho} w$, we find $w_{n_\iota+1} \xrightarrow{\varrho} F(w)$, which yields $F(w) = w$. Thus, w remains a fixed point of F. □

Afterwards, we deliver the uniqueness outcome.

Theorem 5. *Assume, in addition to the suppositions of Theorem 4, that $F(\mathbf{W})$ is $\mathcal{L}$-directed. Then, F enjoys a unique fixed point.*

Proof. With regard to Theorem 4, let us take $w, v \in Fix(\digamma)$, i.e.,

$$\digamma(w) = w \text{ and } \digamma(v) = v. \tag{10}$$

As $w, v \in \digamma(\mathbf{W})$, according to the given hypothesis, $\exists\, u \in \mathbf{W}$, with

$$(w, u) \in \mathcal{L} \quad \text{and} \quad (v, u) \in \mathcal{L}. \tag{11}$$

Denote $\varrho_n := \varrho(w, \digamma^n u)$. Using (10), (11) and (v), we find

$$\begin{aligned} \varrho_n = \varrho(w, \digamma^n u) &= \varrho(\digamma u, \digamma(\digamma^{n-1} u)) \\ &\leq \phi(\varrho(w, \digamma^{n-1} u)) - \psi(\varrho(w, \digamma^{n-1} u)) \\ &= \phi(\varrho_{n-1}) - \psi(\varrho_{n-1}) \end{aligned}$$

which implies

$$\varrho_n \leq \phi(\varrho_{n-1}) - \psi(\varrho_{n-1}). \tag{12}$$

If $\exists\, n' \in \mathbb{N}$, for which $\varrho_{n'} = 0$, then one can conclude that $\varrho_{n'} \leq \varrho_{n'-1}$. When $\varrho_n > 0$, $\forall\, n \in \mathbb{N}$, according to (12) and Γ_4, we obtain $\varrho_n < \varrho_{n-1}$. Thus, in both cases, we obtain

$$\varrho_n \leq \varrho_{n-1}.$$

Applying justifications similar to Theorem 4, the aforementioned inequality leads to

$$\lim_{n \to \infty} \varrho_n = \lim_{n \to \infty} \varrho(w, \digamma^n u) = 0. \tag{13}$$

Similarly, we can find

$$\lim_{n \to \infty} \varrho(v, \digamma^n u) = 0. \tag{14}$$

Using (13) and (14), we obtain

$$\varrho(w, v) = \varrho(w, \digamma^n u) + \varrho(\digamma^n u, v) \to 0 \quad \text{as} \quad n \to \infty$$

i.e., $w = v$. Hence, $\digamma$ admits a unique fixed point. □

Remark 2. *Theorems 4 and 5 generalize several of the noted results from the existing literature as follows:*

- *For $\mathcal{L} := \preceq$ (partial ordering), we obtain the main results of Rhoades et al.* [13]*;*
- *For $\phi = I$ (the identity function) and $\mathcal{L} := \mathbf{W}^2$, we obtain the main results of Rhoades* [12] *(i.e., Theorem 3);*
- *For $\phi = I$ (the identity function), we obtain a consequence of the main result of Hossain et al.* [23]*;*
- *For $\phi(t) = t + \dfrac{\delta t}{2}$, $\psi(t) = t - \dfrac{\delta t}{2}$, where $\delta \in [0, 1)$, and removing transitivity requirement of $\mathcal{L}$, we obtain the main result of Alam and Imdad* [19]*;*
- *For $\phi = 0$ (the zero function), we obtain a consequence of the results of Alam and Imdad* [21] *and Alam et al.* [24]*;*
- *For $\phi = I$ (the identity function) and $\mathcal{L} := \preceq$ (partial ordering), we obtain the main results of Harjani and Sadarangani* [18].

4. Illustrative Examples

To elaborate on our findings, we deliver the following instances.

Example 1. *Take* $\mathbf{W} = [0,1]$ *with standard metric* ϱ. *On* $\mathbf{W}$, *let us take a BR* $\mathcal{L} = \{w, v \in \mathbf{W}$ *such that* $0 \le w < v \le \frac{1}{2}\}$ *and a self-map* $\digamma$ *defined by*

$$\digamma(w) = \begin{cases} \frac{1}{5} + \frac{w}{3}, & \text{if } w \in [0, \frac{1}{2}] \\ \frac{1}{4}, & \text{if } w \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $(\mathbf{W}, \varrho)$ *remains* $\mathcal{L}$*-complete,* $\mathcal{L}$ *remains* $\digamma$*-closed and locally* $\digamma$*-transitive, and* $\digamma$ *is* $\mathcal{L}$*-continuous. Also, we have*

$$\begin{aligned} \varrho(\digamma w, \digamma v) &= \left|\frac{1}{5} + \frac{w}{3} - \frac{1}{5} - \frac{v}{3}\right| \\ &= \left|\frac{w}{3} - \frac{v}{3}\right| \\ &= \frac{1}{3}|w - v| \\ &\le \frac{1}{6} \quad \text{as} \quad (w, v) \in \mathcal{L} \quad \text{then} \quad |w - v| \le \frac{1}{2} \\ &< |w - v|(1 - |w - v|) \\ &= |w - v| - |w - v|^2 \\ &= \phi(\varrho(w, v)) - \psi(\varrho(w, v)). \end{aligned}$$

This shows that $\digamma$ *verifies the suppositions that* (v) *for* $\phi(t) = t$ *and* $\psi(t) = t^2$. *Thus, each of the hypotheses of Theorem 4 holds; hence,* $\digamma$ *enjoys a fixed point. Moreover, in light of Theorem 5, the fixed point remains unique. Note that* $w = \frac{3}{10}$ *is the fixed point of* $\digamma$.

Example 2. *Take* $\mathbf{W} = [0,2]$ *with standard metric* ϱ. *On* $\mathbf{W}$, *let us take a BR* $\mathcal{L} = \{(0,0), (0,1), (1,0), (1,1), (0,2)\}$ *and a self-map* $\digamma$ *defined by*

$$\digamma(w) = \begin{cases} 0, & \text{if } 0 \le w \le 1 \\ 1, & \text{if } 1 < w \le 2. \end{cases}$$

Then, $(\mathbf{W}, \varrho)$ *remains an* $\mathcal{L}$*-complete MS and* $\mathcal{L}$ *constitutes* $\digamma$*-closed and locally* $\digamma$*-transitive. Assuming that* $\{w_n\} \subset \mathbf{W}$ *is an* $\mathcal{L}$*-preserving sequence verifying* $w_n \xrightarrow{\varrho} w$. *Consequently, for each* $n \in \mathbb{N}$, *we have* $(w_n, w_{n+1}) \in \{(0,0), (0,1), (1,0), (1,1)\}$ *as* $(w_n, w_{n+1}) \notin \{(0,2)\}$. *This implies that* $\{w_n\} \subset \{0,1\}$, *which gives rise to* $[w_n, w] \in \mathcal{L}$. *Hence,* $\mathcal{L}$ *is* ϱ*-self-closed. Trivially, we can verify the supposition that* (v) *with* $\phi(t) = \frac{3t}{4}$ *and* $\psi(t) = \frac{t}{4}$. *Hence, according to Theorem 4,* $\digamma$ *enjoys a unique fixed point, say* $w = 0$.

5. Applications to BVP

This section aims to describe the validity of a unique solution for the BVP:

$$\begin{cases} \mathbf{w}'(\jmath) = \Delta(\jmath, \mathbf{w}(\jmath)), & \jmath \in [0, R] \\ \mathbf{w}(0) = \mathbf{w}(R) \end{cases} \tag{15}$$

where the function $\Delta : [0,R] \times \mathbb{R} \to \mathbb{R}$ is continuous. Following Nieto and Rodríguez-López [16], a function $\bar{\mathbf{w}} \in \mathcal{C}'[0,R]$ is termed as a lower solution of (15) if

$$\begin{cases} \bar{\mathbf{w}}'(\jmath) \leq \Delta(\jmath, \bar{\mathbf{w}}(\jmath)), & \jmath \in [0,R] \\ \bar{\mathbf{w}}(0) \leq \bar{\mathbf{w}}(R). \end{cases}$$

We are now equipped to establish an outcome that ensures a unique solution to Problem (15).

Theorem 6. *Together with Problem* (15)*, if* $\exists$ *has* $\varepsilon > 0$ *for every* $r, s \in \mathbb{R}$ *with* $r \leq s$*,*

$$0 \leq [\Delta(\jmath, s) + \varepsilon s] - [\Delta(\jmath, r) + \varepsilon r] \leq \varepsilon \ln(s - r + 1), \tag{16}$$

then the availability of a lower solution of Problem (15) *yields the availability of the unique solution of the problem.*

Proof. Problem (15) can be re-expressed as

$$\begin{cases} \mathbf{w}'(\jmath) + \varepsilon \mathbf{w}(\jmath) = \Delta(\jmath, \mathbf{w}(\jmath)) + \varepsilon \mathbf{w}(\jmath), & \forall \jmath \in [0,R] \\ \mathbf{w}(0) = \mathbf{w}(R). \end{cases} \tag{17}$$

Trivially, (17) transforms into the following integral equation:

$$\mathbf{w}(\jmath) = \int_0^R \Upsilon(\jmath, \tau)[\Delta(\tau, \mathbf{w}(\tau)) + \varepsilon \mathbf{w}(\tau)] d\tau \tag{18}$$

where $\Upsilon(\jmath, \tau)$ is a Green function represented by

$$\Upsilon(\jmath, \tau) = \begin{cases} \frac{e^{\varepsilon(R+\tau-\jmath)}}{e^{\varepsilon R}-1}, & 0 \leq \tau < \jmath \leq R \\ \frac{e^{\varepsilon(\tau-\jmath)}}{e^{\varepsilon R}-1}, & 0 \leq \jmath < \tau \leq R. \end{cases}$$

Denote $\mathbf{W} := \mathcal{C}[0,R]$. Define a map $\digamma : \mathbf{W} \to \mathbf{W}$ by

$$(\digamma \mathbf{w})(\jmath) = \int_0^R \Upsilon(\jmath, \tau)[\Delta(\tau, \mathbf{w}(\tau)) + \varepsilon \mathbf{w}(\tau)] d\tau, \quad \forall \jmath \in [0,R]. \tag{19}$$

On $\mathbf{W}$, obtain a metric ϱ by

$$\varrho(\mathbf{w}, \mathbf{v}) = \sup_{\jmath \in [0,R]} |\mathbf{w}(\jmath) - \mathbf{v}(\jmath)|, \quad \forall \mathbf{w}, \mathbf{v} \in \mathbf{W}. \tag{20}$$

On $\mathbf{W}$, consider a BR $\mathcal{L}$ by

$$\mathcal{L} = \{(\mathbf{w}, \mathbf{v}) \in \mathbf{W} \times \mathbf{W} : \mathbf{w}(\jmath) \leq \mathbf{v}(\jmath), \ \forall \jmath \in [0,R]\}. \tag{21}$$

Right now, we must validate all the criteria of Theorem 5.

(i) Trivially, $(\mathbf{W}, \varrho)$ is an $\mathcal{L}$-complete MS.

(ii) If $\bar{\mathbf{w}} \in \mathcal{C}'[0,R]$ is a lower solution of (15), then we conclude

$$\bar{\mathbf{w}}'(\jmath) + \varepsilon \bar{\mathbf{w}}(\jmath) \leq \Delta(\jmath, \bar{\mathbf{w}}(\jmath)) + \varepsilon \bar{\mathbf{w}}(\jmath), \quad \forall \jmath \in [0,R].$$

Multiplying by $e^{\varepsilon s}$, one obtains

$$(\bar{\mathbf{w}}(\jmath) e^{\varepsilon \jmath})' \leq [\Delta(\jmath, \bar{\mathbf{w}}(\jmath)) + \varepsilon \bar{\mathbf{w}}(\jmath)] e^{\varepsilon s}, \quad \forall \jmath \in [0,R]$$

therefore implying

$$\bar{\mathbf{w}}(\jmath)e^{\varepsilon\jmath} \leq \bar{\mathbf{w}}(0) + \int_0^{\jmath} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)]e^{\varepsilon\tau} d\tau, \quad \forall\, \jmath \in [0, R]. \tag{22}$$

Due to $\bar{\mathbf{w}}(0) \leq \bar{\mathbf{w}}(R)$, we obtain

$$\bar{\mathbf{w}}(0)e^{\varepsilon R} \leq \bar{\mathbf{w}}(R)e^{\varepsilon R} \leq \bar{\mathbf{w}}(0) + \int_0^{R} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)]e^{\varepsilon\tau} d\tau$$

so that

$$\bar{\mathbf{w}}(0) \leq \int_0^{R} \frac{e^{\varepsilon\tau}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau. \tag{23}$$

By (22) and (23), we find

$$\begin{aligned}
\bar{\mathbf{w}}(\jmath)e^{\varepsilon\jmath} &\leq \int_0^{R} \frac{e^{\varepsilon\tau}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau + \int_0^{\jmath} e^{\varepsilon\tau} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau \\
&= \int_0^{\jmath} \frac{e^{\varepsilon(R+\tau)}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau + \int_{\jmath}^{R} \frac{e^{\varepsilon\tau}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau
\end{aligned}$$

so that

$$\begin{aligned}
\bar{\mathbf{w}}(\jmath) &\leq \int_0^{\jmath} \frac{e^{\varepsilon(R+\tau-\jmath)}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau + \int_{\jmath}^{R} \frac{e^{\varepsilon(\tau-\jmath)}}{e^{\varepsilon R} - 1} [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau \\
&= \int_0^{R} \Upsilon(\jmath, \tau) [\Delta(\tau, \bar{\mathbf{w}}(\tau)) + \varepsilon\bar{\mathbf{w}}(\tau)] d\tau \\
&= (\digamma\bar{\mathbf{w}})(\jmath), \quad \forall\, \jmath \in [0, R]
\end{aligned}$$

so that $(\bar{\mathbf{w}}, \digamma\bar{\mathbf{w}}) \in \mathcal{L}$.

(iii) Take $\mathbf{w}, \mathbf{v} \in \mathbf{W}$ with $(\mathbf{w}, \mathbf{v}) \in \mathcal{L}$. By (16), we find

$$\Delta(\jmath, \mathbf{w}(\jmath)) + \varepsilon\mathbf{w}(\jmath) \leq \Delta(\jmath, \mathbf{v}(\jmath)) + \varepsilon\mathbf{v}(\jmath), \quad \forall\, \jmath \in [0, R]. \tag{24}$$

Utilizing (19), (24) and $\Upsilon(\jmath, \tau) > 0, \forall\, \jmath, \tau \in [0, R]$, we obtain

$$\begin{aligned}
(\digamma\mathbf{w})(\jmath) &= \int_0^{R} \Upsilon(\jmath, \tau) [\Delta(\tau, \mathbf{w}(\tau)) + \varepsilon\mathbf{w}(\tau)] d\tau \\
&\leq \int_0^{R} \Upsilon(\jmath, \tau) [\Delta(\tau, \mathbf{v}(\tau)) + \varepsilon\mathbf{v}(\tau)] d\tau \\
&= (\digamma\mathbf{v})(\jmath), \quad \forall\, \jmath \in [0, R],
\end{aligned}$$

which, making use of (21), implies that $(\digamma\mathbf{w}, \digamma\mathbf{v}) \in \mathcal{L}$; thus, $\mathcal{L}$ is $\digamma$-closed.

(iv) Assume that $\{\mathbf{w}_n\} \subset \mathbf{W}$ is a $\mathcal{L}$-preserving sequence that converges to $\mathbf{w} \in \mathbf{W}$; thus, $\mathbf{w}_n(\jmath) \leq \mathbf{w}(\jmath), \forall\, n \in \mathbb{N}$ and $\forall\, \jmath \in [0, R]$. By (21), we find $(\mathbf{w}_n, \mathbf{w}) \in \mathcal{L}, \forall\, n \in \mathbb{N}$; thus, $\mathcal{L}$ constitutes ϱ-self-closed.

(v) Let $(\mathbf{w}, \mathbf{v}) \in \mathcal{L}$. Using (16), (19) and (20), we find

$$\begin{aligned}
\varrho(\digamma\mathbf{w}, \digamma\mathbf{v}) &= \sup_{\jmath \in [0,R]} |(\digamma\mathbf{w})(\jmath) - (\digamma\mathbf{v})(\jmath)| = \sup_{\jmath \in [0,R]} \big((\digamma\mathbf{v})(\jmath) - (\digamma\mathbf{w})(\jmath)\big) \\
&\leq \sup_{\jmath \in [0,R]} \int_0^{R} \Upsilon(\jmath, \tau) [\Delta(\tau, \mathbf{v}(\tau)) + \varepsilon\mathbf{v}(\tau) - \Delta(\tau, \mathbf{w}(\tau)) - \varepsilon\mathbf{w}(\tau)] d\tau \\
&\leq \sup_{\jmath \in [0,R]} \int_0^{R} \Upsilon(\jmath, \tau) \varepsilon \ln(\mathbf{v}(\tau) - \mathbf{w}(\tau) + 1) d\tau.
\end{aligned} \tag{25}$$

Since $\theta(t) = \ln(t+1)$ is increasing and $0 \leq \mathbf{v}(\tau) - \mathbf{w}(\tau) \leq \varrho(\mathbf{w}, \mathbf{v})$, we attain

$$\ln(\mathbf{v}(\tau) - \mathbf{w}(\tau) + 1) \leq \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1).$$

Thus, (25) becomes

$$\begin{aligned}
\varrho(\digamma\mathbf{w}, \digamma\mathbf{v}) &\leq \varepsilon \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1) \sup_{\jmath \in [0,R]} \int_0^R \Upsilon(\jmath, \tau) d\tau \\
&= \varepsilon \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1) \sup_{\jmath \in [0,R]} \frac{1}{e^{\varepsilon R} - 1} \left[\frac{1}{\varepsilon} e^{\varepsilon(R+\tau-\jmath)} \Big|_0^{\jmath} + \frac{1}{\varepsilon} e^{\varepsilon(\tau-\jmath)} \Big|_{\jmath}^{R} \right] \\
&= \varepsilon \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1) \frac{1}{\varepsilon(e^{\varepsilon R} - 1)} (e^{\varepsilon R} - 1) = \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1) \\
&= \varrho(\mathbf{v}, \mathbf{w}) - [\varrho(\mathbf{v}, \mathbf{w}) - \ln(\varrho(\mathbf{v}, \mathbf{w}) + 1)].
\end{aligned}$$

Define $\phi(t) = t$ and $\psi(t) = t - \ln(t+1)$. Then $(\phi, \psi) \in \Gamma$. Thus, the last inequality becomes

$$\varrho(\digamma\mathbf{w}, \digamma\mathbf{v}) \leq \phi(\varrho(\mathbf{w}, \mathbf{v})) - \psi(\varrho(\mathbf{w}, \mathbf{v})), \quad \forall\, (\mathbf{w}, \mathbf{v}) \in \mathcal{L}.$$

Thus, all the hypotheses of Theorem 4 is satisfied. Now, take $\digamma(\mathbf{w}), \digamma(\mathbf{v}) \in \digamma(\mathbf{W})$, where $\mathbf{w}, \mathbf{v} \in \mathbf{W}$. Write $\mathbf{z} := \max\{\digamma\mathbf{w}, \digamma\mathbf{v}\}$. Then, we have $(\digamma\mathbf{w}, \mathbf{z}) \in \mathcal{L}$ and $(\digamma\mathbf{v}, \mathbf{z}) \in \mathcal{L}$. It follows that $\digamma(\mathbf{W})$ is $\mathcal{L}$-directed.

Hence, by Theorem 5, $\digamma$ enjoys a unique fixed point, which, equivalently, will form the solution (unique) of Problem (15). □

Intending to illustrate Theorem 6, we consider the following numerical example.

Example 3. *Let $\Delta(\jmath, w(\jmath)) = \cos\jmath$ for $0 \leq \jmath \leq \pi$. Then, Δ is a continuous function. Note that $\bar{w} = 0$ is a lower solution for $\dfrac{dw}{d\jmath} = \cos\jmath$. Therefore, Theorem 6 can be applied for the given problem and, hence, $w(\jmath) = \sin\jmath$ forms the unique solution.*

6. Conclusions

This manuscript concludes with some outcomes on fixed points enjoying the weak ψ-contraction condition in the setup of a an MS comprising a locally $\digamma$-transitive BR. The outcomes presented herewith reflect a weaker contraction inequality that is exclusively pertinent to the comparative elements only. To further demonstrate our outcomes, we created two examples. Our findings generalize the noted outcomes contained in Rhoades et al. [13], Rhoades [12], Hossain et al. [23], Alam and Imdad [19], Alam and Imdad [21], Alam et al. [24], and Harjani and Sadarangani [18]. By employing our findings, we looked into the validity of a unique solution for BVP whenever it admits a lower solution.

Due to the importance of the relation-theoretic fixed point theory, we consider the following possible future research works, which would be highly relevant and prominent areas on their own.

- Varying the properties on the involved auxiliary functions ϕ and ψ;
- Introducing a variety of metrical frameworks, such as semi-metric space, quasi metric space, dislocated space, partial metric space, fuzzy metric space, and cone metric space, equipped with locally $\digamma$-transitive relation;
- Proving the analogues of our findings to a couple of mappings;
- Proving an analogue of Theorem 6 for solving BVP (15) in the presence of an upper solution rather than the presence of a lower solution;
- Applying our results to special types of nonlinear integral equations and nonlinear matrix equations.

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of whole numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of non-negative real numbers
$\mathcal{C}[0,R]$	family of real continuous functions on the interval $[0,R]$
$\mathcal{C}'[0,R]$	the family of real continuously differentiable functions in $[0,R]$
BCP	Banach contraction principle
MS	metric space
CMS	complete metric space
BR	binary relation
BVP	boundary value problem(s)
$Fix(\mathcal{F})$	fixed-point set of a self-map $\mathcal{F}$

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