


Article

Additive Biderivations of Incidence Algebras

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Abstract

We characterize all additive biderivations on the incidence algebra $I(P, \mathcal{R})$ of a locally finite poset P over a commutative ring with unity \mathcal{R} . By decomposing P into its connected chains, we prove that any additive biderivation splits uniquely into a sum of inner biderivations and extremal ones determined by chain components. In particular, when every maximal chain of P is infinite, all additive biderivations are inner.

Keywords: additive biderivation; incidence algebra; local finite poset; maximal chain

MSC: 16W25; 17C50; 16S90; 16W10; 06A11

1. Introduction

The concept of incidence algebras of partially ordered sets was first introduced by Rota in the 1960s [1] as a tool for solving combinatorial problems. Incidence algebras are fascinating objects that have been the subject of extensive research since their inception. For instance, Pierre and John [2] describe the relationships between the algebraic properties of incidence algebras and the combinatorial features of the partially ordered sets. In [3], Spiegel and O'Donnell provide a detailed analysis of the maximal and prime ideals, derivations and isomorphisms, radicals, and additional ring-theoretic properties of incidence algebras. Further developments on the structure of incidence algebras can be found in [4–7].

The study of derivations in the context of algebras is a valuable and significant endeavor. Yang provides a detailed account of the structure of nonlinear derivations on incidence algebras in [8], specifically decomposing the nonlinear derivations into three more specific forms. Regarding the decomposition of derivations, two significant findings pertaining to their structure have been established. Notably, Baclawski demonstrated that every derivation of the incidence algebra $I(P, \mathcal{R})$, when \mathcal{R} is a field and P is a finite locally poset, can be expressed as the sum of an inner derivation and an additive induced derivation [9]. This result was extended by Spiegel and O'Donnell [3] to cases where \mathcal{R} is a commutative ring. Additional insights into the structure of other special derivations on incidence algebra can be found in [10–13].

Building upon the aforementioned studies of derivations on incidence algebras, it is natural to investigate biderivations in this context. Kaygorodov and Khrypchenko delineate the structure of antisymmetric biderivations of finitary incidence algebras $FI(P, \mathcal{R})$, where P is an arbitrary poset and \mathcal{R} is a commutative ring with unity, in [14]. In [15], Benkovič proves that every biderivation of a triangular algebra is the sum of an inner biderivation and an external biderivation. Later, Ghosseiri demonstrates that every biderivation of upper triangular matrix rings is the sum of an inner biderivation, an external biderivation, and a distinct category of biderivations [16]. Ghosseiri also presents particular instances where every biderivation is inner.



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In the literature, many biderivations on different algebras have been shown to decompose into the sum of inner and extremal ones. Motivated by this, we ask whether general biderivations on incidence algebras admit a similar decomposition. Compared with [14], which characterizes antisymmetric biderivations on finitary incidence algebras $FI(P, \mathcal{R})$, our work extends the type of biderivations under consideration from antisymmetric to general ones, while the algebraic framework is more restrictive. Moreover, although most existing studies assume linearity, we find this condition unnecessarily strong. Therefore, in this paper, we focus on additive biderivations of incidence algebras and determine their precise structure. Specifically, let \mathcal{R} be a commutative ring with unity, and P a locally finite poset such that any maximal chain in P contains at least three elements. The additive biderivation b is the exact sum of several inner biderivations and extremal biderivations. Furthermore, we give that when the number of elements in any maximal chain in P is infinite, b is an inner biderivation.

2. Preliminaries

2.1. Incidence Algebra

Throughout this paper, \mathcal{R} denotes a commutative ring with unity. Recall that a relation \leq is said to be a *partial order* on a set S if it satisfies the following conditions:

1. Reflexivity: $a \leq a$ for all $a \in S$;
2. Antisymmetry: if $a \leq b$ and $b \leq a$, then $a = b$;
3. Transitivity: if $a \leq b$ and $b \leq c$, then $a \leq c$.

A *partially ordered set* (or *poset*) is a set equipped with a partial order. A poset (S, \leq) is *locally finite* if, for any $x \leq y$ in S , there are finitely many elements $u \in S$ satisfying $x \leq u \leq y$. Let (P, \leq) be a locally finite poset. We use the notation $x < y$ to mean that $x \leq y$ and $x \neq y$. Now, let us recall the definition of incidence algebra.

Definition 1. The incidence algebra $I(P, \mathcal{R})$ is defined as the set of functions

$$I(P, \mathcal{R}) = \{f : P \times P \rightarrow \mathcal{R} \mid f(x, y) = 0 \text{ if } x \not\leq y\}, \quad (1)$$

with multiplication given by

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y), \quad \text{for all } x, y \in P. \quad (2)$$

For each pair $x \leq y$ in P , let e_{xy} denote the element in $I(P, \mathcal{R})$ defined by

$$e_{xy}(u, v) = \begin{cases} 1, & \text{if } (x, y) = (u, v); \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

For brevity, we will use the notation α_{xy} to denote $\alpha(x, y)$, where $\alpha \in I(P, \mathcal{R})$ and $x, y \in P$. Consequently, any element $\alpha \in I(P, \mathcal{R})$ can be expressed as $\alpha = \sum_{x \leq y} \alpha_{xy} e_{xy}$. It is evident that the multiplication in $I(P, \mathcal{R})$ satisfies the following properties:

1. If $y = u$, then $e_{xy}e_{uv} = e_{xv}$;
2. If $y \neq u$, then $e_{xy}e_{uv} = 0$.

This relation allows us to derive the following formula, which will be used extensively in this article. For any $f \in I(P, \mathcal{R})$ and $x \leq y, u \leq v$ in P , we have

$$e_{xy}fe_{uv} = f_{yv}e_{xv}. \quad (4)$$

2.2. Additive Biderivations

A function $d : I(P, \mathcal{R}) \rightarrow I(P, \mathcal{R})$ is a *derivation* of $I(P, \mathcal{R})$ if, for any $\alpha, \beta \in I(P, \mathcal{R})$, it satisfies

$$d(\alpha\beta) = \alpha d(\beta) + d(\alpha)\beta.$$

A function $b : I(P, \mathcal{R}) \times I(P, \mathcal{R}) \rightarrow I(P, \mathcal{R})$ is said to be a *biderivation* of $I(P, \mathcal{R})$ if it is a derivation when fixing any one of its arguments. This means that for every $\alpha, \beta, \gamma \in I(P, \mathcal{R})$, b satisfies

$$\begin{aligned} b(\alpha\beta, \gamma) &= \alpha b(\beta, \gamma) + b(\alpha, \gamma)\beta, \\ b(\alpha, \beta\gamma) &= \beta b(\alpha, \gamma) + b(\alpha, \beta)\gamma. \end{aligned}$$

Moreover, b is an *additive biderivation* if b also satisfies

$$\begin{aligned} b(\alpha + \beta, \gamma) &= b(\alpha, \gamma) + b(\beta, \gamma), \\ b(\alpha, \beta + \gamma) &= b(\alpha, \beta) + b(\alpha, \gamma). \end{aligned}$$

The so-called *inner biderivation* is defined by

$$b(\alpha, \beta) = \lambda[\alpha, \beta], \quad (5)$$

where $\lambda \in \mathcal{R}$ is a fixed element, and $[\alpha, \beta]$ denotes the Lie bracket of α and β . It is known that in prime and semiprime rings, all biderivations are inner (Brešar [17]). However, in certain algebras, such as triangular algebras and upper triangular matrix algebras, non-inner biderivations do occur. To describe the structure of these non-inner parts, Benkovič [15] introduced the notion of extremal biderivations and showed that every biderivation on triangular algebras can be expressed as the sum of an inner biderivation and an extremal one.

More precisely, a biderivation b is called an *extremal biderivation* if there exists $\gamma \in I(P, \mathcal{R})$ such that

$$b(\alpha, \beta) = [\alpha, [\beta, \gamma]]. \quad (6)$$

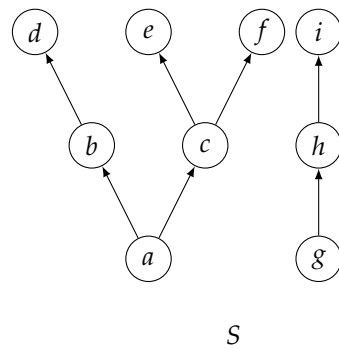
2.3. Decomposition of Posets

Our research reveals a close relationship between the structure of additive biderivations in $I(P, \mathcal{R})$ and that of the poset P . Accordingly, we proceed to decompose P in this section, which will subsequently be employed in constructing the structure of the additive biderivations. In detail, we will consider and decompose the cases where P is connected or not.

First, let us introduce some notation. Consider a relation \sim on P where $x \sim y$ indicates that x and y are comparable, i.e., either $x \leq y$ or $y \leq x$. The relation \nsim indicates that x and y are not comparable. A *totally ordered set* is a poset in which every pair of elements is comparable.

Definition 2. A chain in a poset P is a subset that is totally ordered with respect to \leq .

Example 1. Consider the poset $S = \{a, b, c, d, e, f, g, h, i\}$ represented by the following Hasse diagram, where $x \rightarrow y$ denotes $x \leq y$ for all $x, y \in S$. In this case, the pair of elements b and e is not comparable, and the subset $\{g, h, i\}$ of S constitutes a chain.



1. The first decomposition, when P is not connected.

First, let us introduce a relation between any two elements in \mathcal{R} .

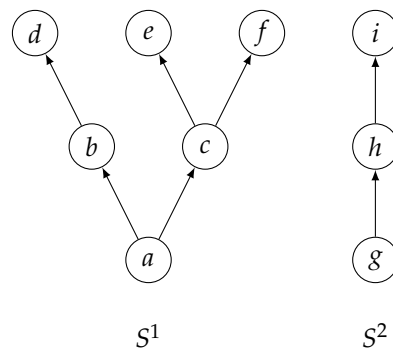
Definition 3. Two elements $x, y \in P$ are defined as being connected if there exists a sequence $u_0, u_1, \dots, u_n \in P$ such that $x \sim u_0$, $u_0 \sim u_1, \dots, u_{n-1} \sim u_n$, and $u_n \sim y$. A poset P is connected if any pair of elements in it are connected.

It is evident that the relation of being connected is an equivalence relation on P . Thus, we can decompose P into the union of its connected components:

$$P = \bigcup_{i \in I} P^i, \quad (7)$$

where I is an index set and each P^i is a connected poset.

Example 2. Consider the poset S defined in Example 1. It can be observed that S can be expressed as the union of two disjoint sets: $S^1 = \{a, b, c, d, e, f\}$ and $S^2 = \{g, h, i\}$.



It is evident that $P^i \cap P^j = \emptyset$ for $i \neq j$. Therefore, for any $\alpha \in I(P, \mathcal{R})$, we have

$$\alpha = \sum_{i \in I} \alpha^i, \quad \alpha^i = \sum_{x \leq y \in P^i} \alpha_{xy} e_{xy} \in I(P^i, \mathcal{R}). \quad (8)$$

2. The second decomposition, when P is connected.

Next, we present a second decomposition of P when P is connected. We begin by defining a key term.

Definition 4. A chain in P is called maximal if adding any element from P to it would result in it no longer being a chain.

In this context, the notation $\{x, y, \dots\}^+$ will be used to denote a maximal chain in P that contains $\{x, y, \dots\}$.

We proceed to define a set of subsets of P :

$$L = \{l \subset P \mid l \text{ is a maximal chain}\}. \quad (9)$$

Definition 5. For any pair of elements $l', l'' \in L$, we define the relation $l' \approx l''$ if there exist elements $x < y$ in P such that $x, y \in l' \cap l''$. We say l' and l'' are connected if there exist chains $l_0, l_1, \dots, l_n \in L$ such that $l' \approx l_0, l_0 \approx l_1, \dots, l_{n-1} \approx l_n, l_n \approx l''$.

With respect to the connectedness relation on L , we can decompose L into the union of its equivalence classes:

$$L = \bigcup_{j \in \mathcal{J}} L_j, \quad (10)$$

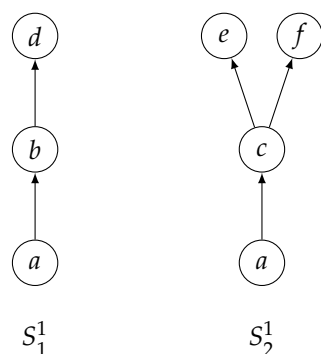
where \mathcal{J} is an index set corresponding to the decomposition of L .

For each $j \in \mathcal{J}$, define a subset of P :

$$P_j = \{x \in P \mid \text{there exists } l \in L_j \text{ such that } x \in l\}. \quad (11)$$

It is evident that $P = \bigcup_{j \in \mathcal{J}} P_j$. However, it is not necessarily the case that $P_i \cap P_j = \emptyset$ for all $i \neq j$ in \mathcal{J} .

Example 3. To illustrate this decomposition, consider $S^1 = \{a, b, c, d, e, f\}$ from Example 2. The set S^1 can be expressed as the union of two sets: $S_1^1 = \{a, b, d\}$ and $S_2^1 = \{a, c, e, f\}$.



An element $x \in P$ is said to be *maximal* if there is no element $y \in P$ such that $x < y$, and *minimal* if there is no element $y \in P$ such that $y < x$.

Lemma 1. Let $i \neq j$ in \mathcal{J} be defined in (10); then there does not exist a pair of elements $x < y$ in P such that $x, y \in P_i \cap P_j$. Additionally, any element in $P_i \cap P_j$ is either a minimal or a maximal element of P .

Proof. We begin by asserting that any element in $P_i \cap P_j$ is isolated, meaning that no other element in $P_i \cap P_j$ can be compared with it. Suppose, for contradiction, that there exists a pair of elements $x < y$ contained in $P_i \cap P_j$. Then, there exist chains $l_1 \in L_i$ and $l_2 \in L_j$ such that $x, y \in l_1$ and $x, y \in l_2$. It follows that $l_1 \approx l_2$, implying $P_i = P_j$ and $i = j$, which contradicts the assumption that $i \neq j$. Therefore, any element in $P_i \cap P_j$ is isolated.

Now, suppose there exists an element $x \in P_i \cap P_j$ that is neither maximal nor minimal. Then there exist elements $u < x < v$ in P . According to the definition of P_i and P_j and the assumption that any maximal chain of P has at least three elements, there exist elements $y_i \in P_i$ and $y_j \in P_j$ that can be compared with x , and there exist maximal chains $l_i \in L_i$ and

$l_j \in L_j$ such that $x, y_i \in l_i$ and $x, y_j \in l_j$. Without loss of generality, suppose $x < y_i$. It is evident that $\{u, x, v\}^+ \approx \{u, x, y_i\}^+ \approx l_i$, so $u, x, v \in P_i$. Similarly, $u, x, v \in P_j$. This implies that both $u < x$ and $x < v$ are elements of $P_i \cap P_j$, which contradicts the previous conclusion. Thus, any element in $P_i \cap P_j$ must be either minimal or maximal. \square

By the lemma above, for any $\alpha \in I(P, \mathcal{R})$, where P is connected, we can decompose it as

$$\alpha = \sum_{j \in \mathcal{J}} \alpha'_j + \alpha^D, \quad \text{where } \alpha'_j = \sum_{x < y \in P_j} \alpha_{xy} e_{xy}, \text{ and } \alpha^D = \sum_{z \in P} \alpha_{zz} e_{zz}. \quad (12)$$

If $i \neq j$ and the product $\alpha'_i \beta'_j = \sum_{x < y \in P_i} \sum_{u < v \in P_j} \alpha_{xy} \beta_{uv} e_{xy} e_{uv}$ is non-zero, then there exist $x' < y'$ in P_i and $u' < v'$ in P_j such that $e_{xy} e_{uv} \neq 0$. Therefore, we conclude that $y = u$, implying that $y \in P_i \cap P_j$ and $x < y < v$, which contradicts Lemma 1. This leads to the following corollary.

Corollary 1. For any $i \neq j$ in \mathcal{J} , let $\alpha'_i \in I(P_i, \mathcal{R})$ and $\beta'_j \in I(P_j, \mathcal{R})$, then $\alpha'_i \beta'_j = 0$.

3. Additive Derivations of $I(P, \mathcal{R})$

In the study of additive biderivations, it is necessary to discuss some properties of additive derivations of $I(P, \mathcal{R})$, since an additive biderivation becomes an additive derivation when one of its arguments is fixed. Let P be a locally finite poset, let \mathcal{R} be a commutative ring with unity, and let $I(P, \mathcal{R})$ be the incidence algebra of P over \mathcal{R} . Suppose d is an additive derivation of $I(P, \mathcal{R})$.

We will denote $d_{xy}(\alpha)$ to represent $d(\alpha)(x, y)$ for any $\alpha \in I(P, \mathcal{R})$ and $x, y \in P$. We begin by demonstrating a lemma that shows the structure of the value of $d(re_{xy})$.

Lemma 2. Let $x \leq y \in P$ and $r \in \mathcal{R}$, then

$$d(re_{xy}) = \sum_{p \leq x} d_{py}(re_{xy}) e_{py} + \sum_{y < q} d_{xq}(re_{xy}) e_{xq}. \quad (13)$$

Proof. Consider a fixed pair $x \leq y \in P$ and an arbitrary element $r \in \mathcal{R}$. Suppose $u \leq v \in P$ with $u \neq x$ and $v \neq y$. We examine the expression $e_{uu} d(re_{xy}) e_{vv}$ by multiplying $d(re_{xy})$ on the left by e_{uu} and on the right by e_{vv} :

$$\begin{aligned} e_{uu} d(re_{xy}) e_{vv} &= e_{uu} d(re_{xx} e_{xy}) e_{vv} \\ &= e_{uu} e_{xx} d(re_{xy}) e_{vv} + e_{uu} d(e_{xx}) re_{xy} e_{vv} \\ &= 0. \end{aligned} \quad (14)$$

Since $e_{uu} d(re_{xy}) e_{vv} = d_{uv}(re_{xy}) e_{uv}$, it follows that $d_{uv}(re_{xy}) = 0$ unless $u = x$ or $v = y$. This establishes the desired equality (13). \square

The proof of the aforementioned lemma allows us to derive the following corollary.

Corollary 2. Let $x \leq y, u \leq v \in P$, then $d_{uv}(re_{xy}) = 0$ unless the elements x, y, u, v satisfy one of the following cases: (1) $u = x, v \leq y$; (2) $u \leq x, v = y$.

In describing the structure of biderivations, we will try to prove that some of them are equivalent. To do this, we introduce a number of instrumental lemmas as described below.

Lemma 3. Let $x \leq y \in P$ and $r \in \mathcal{R}$, then

- (1) For any $p < x$, $d_{py}(re_{xy}) = rd_{py}(e_{xy}) = rd_{px}(e_{xx})$;
- (2) For any $q > y$, $d_{xq}(re_{xy}) = rd_{xq}(e_{xy}) = rd_{yq}(e_{yy})$.

Proof. We will demonstrate part (1), and the proof of part (2) follows analogously. Consider fixed elements $x \leq y \in P$ and $r \in \mathcal{R}$. For any $p < x$, using Equation (4), we obtain

$$\begin{aligned} d_{py}(re_{xy})e_{py} &= e_{pp}d(re_{xy})e_{yy} \\ &= e_{pp}(e_{xx}d(re_{xy}) + d(e_{xx})re_{xy})e_{yy} \\ &= rd_{px}(e_{xx})e_{py}. \end{aligned} \quad (15)$$

Therefore, we deduce that $d_{py}(re_{xy}) = rd_{px}(e_{xx})$. Setting r to be the unity element of \mathcal{R} yields $d_{py}(e_{xy}) = d_{px}(e_{xx})$, thereby establishing part (1). \square

Lemma 4. Let $x < y \in P$, then $d_{xy}(e_{xx}) + d_{xy}(e_{yy}) = 0$.

Proof. Let $x < y \in P$ be fixed. Observe that

$$\begin{aligned} 0 &= e_{xx}d(e_{xx}e_{yy})e_{yy} \\ &= e_{xx}d(e_{xx})e_{yy} + e_{xx}d(e_{yy})e_{yy} \\ &= (d_{xy}(e_{xx}) + d_{xy}(e_{yy}))e_{xy}. \end{aligned} \quad (16)$$

Since the above expression equals zero and e_{xy} is a non-zero element of the incidence algebra, it follows that $d_{xy}(e_{xx}) + d_{xy}(e_{yy}) = 0$. \square

Lemma 5. Let $x < y < z \in P$, then $d_{xz}(e_{xz}) = d_{xy}(e_{xy}) + d_{yz}(e_{yz})$.

Proof. Let $x < y < z \in P$ be fixed. Then,

$$\begin{aligned} d_{xz}(e_{xz})e_{xz} &= e_{xx}d(e_{xy}e_{yz})e_{zz} \\ &= e_{xx}(d(e_{xy})e_{yz} + e_{xy}d(e_{yz}))e_{zz} \\ &= (d_{xy}(e_{xy}) + d_{yz}(e_{yz}))e_{xz}. \end{aligned} \quad (17)$$

Since e_{xz} is a non-zero element of the incidence algebra, we deduce that $d_{xz}(e_{xz}) = d_{xy}(e_{xy}) + d_{yz}(e_{yz})$. \square

4. Additive Biderivations of $I(P, \mathcal{R})$

In this section, we will employ the properties of additive derivations derived in Section 3 to prove our main theorem (Theorem 4), which elucidates the structure of additive biderivations on the incidence algebra $I(P, \mathcal{R})$. Let P be a locally finite poset, and let \mathcal{R} be a commutative ring with unity. In this section, we will use the notation $b_{xy}(\alpha, \beta)$ to denote the value $b(\alpha, \beta)(x, y)$ for any $\alpha, \beta \in I(P, \mathcal{R})$ and $x, y \in P$.

We begin by considering a corollary that can be readily extended from Lemma 2.

Corollary 3. Let $x \leq y, u \leq v \in P$, and $r_1, r_2 \in \mathcal{R}$, then

$$b(r_1e_{xy}, r_2e_{uv}) = \begin{cases} b_{xv}(r_1e_{xy}, r_2e_{uv})e_{xv} + b_{uy}(r_1e_{xy}, r_2e_{uv})e_{uy}, & \text{if } x \neq u, y \neq v; \\ \sum_{y \leq q, v \leq q} b_{xq}(r_1e_{xy}, r_2e_{xv})e_{xq}, & \text{if } x = u, y \neq v; \\ \sum_{p \leq x, p \leq u} b_{py}(r_1e_{xy}, r_2e_{uy})e_{py}, & \text{if } x \neq u, y = v; \\ \sum_{p \leq x} b_{py}(r_1e_{xy}, r_2e_{xy})e_{py} + \sum_{y < q} b_{xq}(r_1e_{xy}, r_2e_{xy})e_{xq}, & \text{if } x = u, y = v. \end{cases} \quad (18)$$

The preliminary stage will entail a reduction in the complexity of the structure of b . This will be achieved by establishing a sufficient condition that characterizes the circumstances under which b is equal to zero.

Lemma 6. Let $x \leq y, u \leq v \in P$; if at least one pair among $\{x, y, u, v\}$ is not comparable, then $b(r_1e_{xy}, r_2e_{uv}) = 0$ for any $r_1, r_2 \in \mathcal{R}$.

Proof. Consider $x \leq y, u \leq v \in P$. Suppose at least one pair among $\{x, y, u, v\}$ is not comparable. This situation can be divided into four cases: (1) $x \not\sim u$; (2) $x \not\sim v$; (3) $y \not\sim u$; (4) $y \not\sim v$.

Case 1: Suppose $x \not\sim u$. We consider two subcases: $y \neq v$ and $y = v$.

If $y \neq v$, by Corollary 3, we have

$$b(r_1e_{xy}, r_2e_{uv}) = b_{xv}(r_1e_{xy}, r_2e_{uv})e_{xv} + b_{uy}(r_1e_{xy}, r_2e_{uv})e_{uy}. \quad (19)$$

According to Corollary 2, both $b_{xv}(r_1e_{xy}, r_2e_{uv})$ and $b_{uy}(r_1e_{xy}, r_2e_{uv})$ are zero because $x \not\sim u$.

If $y = v$, we obtain that

$$\begin{aligned} & b(r_1e_{xy}, r_2e_{uy}) \\ &= \sum_{p \leq x, p \leq u} b_{py}(r_1e_{xy}, r_2e_{uy})e_{py} \\ &= \sum_{p < x, p < u} b_{py}(r_1e_{xy}, r_2e_{uy})e_{py} + b_{xy}(r_1e_{xy}, r_2e_{uy})e_{xy} + b_{uy}(r_1e_{xy}, r_2e_{uy})e_{uy}. \end{aligned} \quad (20)$$

In (20), both $b_{xy}(r_1e_{xy}, r_2e_{uy})e_{xy}$ and $b_{uy}(r_1e_{xy}, r_2e_{uy})e_{uy}$ are zero by Corollary 2. For the remaining terms in (20), using Lemma 3 on the first argument of each term, we have

$$\sum_{p < x, p < u} b_{py}(r_1e_{xy}, r_2e_{uy})e_{py} = \sum_{p < x, p < u} b_{px}(r_1e_{xx}, r_2e_{uy})e_{py} = 0. \quad (21)$$

Noting that $p < u$ and $x < y$ (if $x = y$, we obtain $u < x = v$, which contradicts $x \not\sim u$), $b_{px}(r_1e_{xx}, r_2e_{uy})e_{py}$ in (21) is zero by Corollary 2. Therefore, the lemma is proved when $x \not\sim u$.

Case 2: If $x \not\sim v$, then $x \neq u$ and $y \neq v$. From Corollary 3, we have

$$b(r_1e_{xy}, r_2e_{uv}) = b_{uy}(r_1e_{xy}, r_2e_{uv})e_{uy}. \quad (22)$$

We assert that either $u < x$ or $u \not\sim x$, because if $x \leq u$, then $x \leq u \leq v$, which contradicts $x \not\sim v$. For $b_{uy}(r_1e_{xy}, r_2e_{uv})$, it is zero by Lemma 2 if $u \not\sim x$. If $u < x$, by Lemma 2, we get

$$b_{uy}(r_1e_{xy}, r_2e_{uv}) = b_{ux}(r_1e_{xx}, r_2e_{uv}) = 0. \quad (23)$$

Therefore, the lemma is proved when $x \not\sim v$.

Cases 3 and 4: If $y \not\sim u$ or $y \not\sim v$, the proof is similar to Cases 1 and 2.

Thus, the lemma is proved. \square

Remark 1. Intuitively, the lemma asserts that an additive biderivation must vanish on any pair of basis elements that are incomparable. The multiplication rules of incidence basis elements (where $e_{xy}e_{uv} = 0$ for many incomparable configurations), together with the derivation property in each

argument, force all such coefficients to be zero. Hence, nontrivial behavior can only occur along comparable pairs (chains), which is the crucial reduction used in the sequel.

According to the decomposition (8), we can suppose that for any $\alpha, \beta \in I(P, \mathcal{R})$,

$$\alpha = \sum_{i \in \mathcal{I}} \alpha^i, \quad \beta = \sum_{i \in \mathcal{I}} \beta^i,$$

where $\alpha^i, \beta^i \in I(P^i, \mathcal{R})$. By applying the decomposition of P , as defined in (7), and the aforementioned lemma, it is straightforward to conclude that $b(\alpha^{i_1}, \beta^{i_2}) = 0$ if $i_1 \neq i_2 \in \mathcal{I}$, thereby yielding the following result:

$$b(\alpha, \beta) = \sum_{i \in \mathcal{I}} b(\alpha^i, \beta^i). \quad (24)$$

Accordingly, this decomposition permits us to limit our analysis to the case where P is connected, as will be discussed subsequently. We will subsequently present the theorem, in the case where P is connected, which describes the conditions under which certain terms in Equation (18) are equal to zero. Prior to this, we will present an instrumental lemma.

Lemma 7. Let $\alpha, \beta, \gamma, \delta \in I(P, \mathcal{R})$, then

$$b(\alpha, \beta)[\gamma, \delta] = [\alpha, \beta]b(\gamma, \delta). \quad (25)$$

Proof. Let $\alpha, \beta, \gamma, \delta \in I(P, \mathcal{R})$ be arbitrary. Since b is a biderivation, it satisfies the derivation property in each argument separately. We begin by applying the derivation property to the first argument of $b(\alpha\gamma, \beta\delta)$, followed by applying it to the second argument. This yields

$$\begin{aligned} b(\alpha\gamma, \beta\delta) &= ab(\gamma, \beta\delta) + b(\alpha, \beta\delta)\gamma \\ &= \alpha\beta b(\gamma, \delta) + ab(\gamma, \beta)\delta + \beta b(\alpha, \delta)\gamma + b(\alpha, \beta)\delta\gamma. \end{aligned} \quad (26)$$

Conversely, we first apply the derivation property to the second argument and then to the first argument, obtaining

$$\begin{aligned} b(\alpha\gamma, \beta\delta) &= \beta b(\alpha\gamma, \delta) + b(\alpha\gamma, \beta)\delta \\ &= \beta ab(\gamma, \delta) + \beta b(\alpha, \delta)\gamma + ab(\gamma, \beta)\delta + b(\alpha, \beta)\gamma\delta. \end{aligned} \quad (27)$$

Subtracting Equation (26) from Equation (27) gives

$$0 = [\alpha, \beta]b(\gamma, \delta) - b(\alpha, \beta)[\gamma, \delta], \quad (28)$$

which rearranges to the desired identity:

$$b(\alpha, \beta)[\gamma, \delta] = [\alpha, \beta]b(\gamma, \delta). \quad (29)$$

The proof is completed. \square

In accordance with the aforementioned lemma, it is possible to select specific values for x, y, u, v in order to demonstrate that $b(e_{xy}, e_{uv}) = 0$. For $x, y, u, v \in P$ that are comparable with each other and have $[e_{xy}, e_{uv}] = 0$, the following two subcases can be identified: (1) $x = y = u = v$; (2) $x \neq v, y \neq u$. The following two theorems will address these subcases in greater detail.

Theorem 1. Let $x \in P$, then

$$b(e_{xx}, e_{xx}) = \begin{cases} \sum_{\substack{x < y, \\ y \text{ is max}}} b_{xy}(e_{xx}, e_{xx})e_{xx}, & x \text{ is a minimal element;} \\ \sum_{\substack{y < x, \\ y \text{ is min}}} b_{xy}(e_{xx}, e_{xx})e_{xx}, & x \text{ is a maximal element;} \\ 0, & \text{otherwise.} \end{cases} \quad (30)$$

Proof. We prove the lemma by considering three cases based on the position of x in P : minimal, maximal, and neither.

Case 1: x is a minimal element in P . By Corollary 3, we have

$$b(e_{xx}, e_{xx}) = \sum_{x \leq q} b_{xq}(e_{xx}, e_{xx})e_{xq}. \quad (31)$$

For any pair $x \leq y$ where y is not a maximal element, let $z > y$. By Lemma 7, we know that

$$b_{xy}(e_{xx}, e_{xx})e_{xz} = b(e_{xx}, e_{xx})[e_{yz}, e_{zz}] = [e_{xx}, e_{xx}]b(e_{yz}, e_{zz}) = 0. \quad (32)$$

Thus, when x is a minimal element, $b(e_{xx}, e_{xx})$ reduces to the given form

$$\sum_{x < y, y \text{ is max}} b_{xy}(e_{xx}, e_{xx})e_{xx}.$$

Case 2: x is a maximal element in P . The procedure is similar to Case 1.

Case 3: x is neither a minimal nor maximal element. In this case, there exist y, z such that $y < x < z$. For any $p \leq x$, we have

$$e_{pp}b(e_{xx}, e_{xx})e_{xz} = e_{pp}b(e_{xx}, e_{xx})[e_{xz}, e_{zz}] = e_{pp}[e_{xx}, e_{xx}]b(e_{xz}, e_{zz}) = 0. \quad (33)$$

This implies $b_{px}(e_{xx}, e_{xx}) = 0$. Similarly, for any $x \leq q$, we can deduce $b_{xq}(e_{xx}, e_{xx}) = 0$. Hence, when x is neither a minimal nor maximal element, we have $b(e_{xx}, e_{xx}) = 0$.

Combining the three cases, the lemma is proven. \square

Remark 2. The theorem implies that the values of a biderivation on diagonal basis elements e_{xx} can be non-zero only at boundary points of the poset, that is, at minimal or maximal elements. Intuitively, this follows from inserting $b(e_{xx}, e_{xx})$ between other basis elements and applying the commutation identities, which eliminate contributions from interior points: whenever a third comparable element exists, the corresponding terms vanish. Thus, the diagonal component of a biderivation is necessarily supported at the endpoints of the poset, providing the foundation for the extremal components constructed later.

Theorem 2. Let P be a connected poset such that any maximal chain has at least three elements. For any $r_1, r_2 \in \mathcal{R}$ and $x, y, u, v \in P$ that are comparable with each other and have $x \neq v, y \neq u$. The additive biderivation b satisfies $b(r_1e_{xy}, r_2e_{uv}) = 0$, except in the case where $x = y \neq u = v$ and one of x and u is the maximal element in P and the other is the minimal element.

Proof. Let $r_1, r_2 \in \mathcal{R}$ and $x, y, u, v \in P$ satisfy the conditions of the theorem. We further divide this case into four subcases:

Case 1: If $x \neq u$ and $y \neq v$, except in the case where $x = y \neq u = v$ and one of x and u is the maximal element in P and the other is the minimal element, then by Corollary 3, we have

$$b(r_1e_{xy}, r_2e_{uv}) = b_{xv}(r_1e_{xy}, r_2e_{uv})e_{xv} + b_{uy}(r_1e_{xy}, r_2e_{uv})e_{uy}. \quad (34)$$

We have $b_{xv}(r_1e_{xy}, r_2e_{uv}) = b_{yv}(r_1e_{yy}, r_2e_{uv})$ by using Lemma 3 in its first argument when $x \leq y < v$. If $v < y$, it is also correct because $b_{xv}(r_1e_{xy}, r_2e_{uv}) = b_{yv}(r_1e_{yy}, r_2e_{uv}) = 0$ by Corollary 3. By the same reasoning in its second argument, we obtain $b_{yv}(r_1e_{yy}, r_2e_{uv}) = b_{yu}(r_1e_{yy}, r_2e_{uu})$. Without loss of generality, we can assume that $y < u$. It is evident that y is not minimal or u is not maximal; otherwise, the case is excluded. Thus, by Lemma 4 and Theorem 1, we have

$$b_{yu}(r_1e_{yy}, r_2e_{uu}) = -r_1r_2b_{yu}(e_{yy}, e_{yy}) = 0 \quad (35)$$

Similarly, $b_{uy}(r_1e_{xy}, r_2e_{uv})$ is also equal to 0. Therefore, we have $b(r_1e_{xy}, r_2e_{uv}) = 0$ in this case.

Case 2: If $x = u$ and $y \neq v$. We consider two subcases: $v < y$ and $y < v$.

(a) If $v < y$, we have

$$b(r_1e_{xy}, r_2e_{xv}) = \sum_{v < y \leq q} b_{xq}(r_1e_{xy}, r_2e_{xv})e_{xq} \stackrel{\text{Lemma 3}}{=} \sum_{v < y \leq q} b_{vq}(r_1e_{xy}, r_2e_{vv})e_{xq}. \quad (36)$$

Since $x = u < v$ and $q \geq y$, $b_{vq}(r_1e_{xy}, r_2e_{vv}) = 0$ by Lemma 2 and Corollary 2. Therefore, $b(r_1e_{xy}, r_2e_{xv}) = 0$.

(b) If $y < v$, the proof is similar.

Case 3: If $x \neq u$ and $y = v$.

The proof is similar to subcase 2.

Case 4: If $x = u$ and $y = v$, implying $x = u < y = v$. By Corollary 3, we have

$$b(r_1e_{xy}, r_2e_{xy}) = \sum_{p < x} b_{py}(r_1e_{xy}, r_2e_{xy})e_{py} + \sum_{y < q} b_{xq}(r_1e_{xy}, r_2e_{xy})e_{xq} + b_{xy}(r_1e_{xy}, r_2e_{xy})e_{xy}. \quad (37)$$

For any pair $p < x$, the term $b_{py}(r_1e_{xy}, r_2e_{xy})e_{py}$ is equal to $b_{px}(r_1e_{xx}, r_2e_{xy})e_{py}$ by Lemma 3. Let us consider its second argument, whereby because $p \neq x$ and $x \neq y$, it is equal to 0 by Corollary 2. In the same way, for any $y < q$, we have $b_{xq}(r_1e_{xy}, r_2e_{xy})e_{xq} = 0$. Thus,

$$b(r_1e_{xy}, r_2e_{xy}) = b_{xy}(r_1e_{xy}, r_2e_{xy})e_{xy}. \quad (38)$$

By the assumption on P , there exists an element $z \neq x, y$ such that z is comparable with x and y . We consider three cases:

1. If $z < x < y$, applying Lemma 7, we obtain

$$b(e_{zx}, e_{xx})[r_1e_{xy}, r_2e_{xy}] = [e_{zx}, e_{xx}]b(r_1e_{xy}, r_2e_{xy}). \quad (39)$$

The left-hand side is zero, and the right-hand side equals $b_{xy}(r_1e_{xy}, r_2e_{xy})e_{zy}$, implying $b(r_1e_{xy}, r_2e_{xy}) = 0$.

2. If $x < y < z$, the proof is similar to (a).

3. If $x < z < y$, applying Lemma 5, we have

$$b_{xy}(r_1e_{xy}, r_2e_{xy}) = b_{xz}(r_1e_{xz}, r_2e_{xy}) + b_{zy}(r_1e_{zy}, r_2e_{xy}) \stackrel{\text{Lemma 2}}{=} 0, \quad (40)$$

Consider the second argument of $b_{xz}(r_1e_{xz}, r_2e_{xy})$ and $b_{zy}(r_1e_{zy}, r_2e_{xy})$, respectively. Because $x < z < y$, we have $b_{xz}(r_1e_{xz}, r_2e_{xy}) = b_{zy}(r_1e_{zy}, r_2e_{xy}) = 0$ by Corollary 2. This implies $b(r_1e_{xy}, r_2e_{xy}) = 0$.

Considering all the above cases, the theorem is proved. \square

Remark 3. The theorem shows—after a case-by-case analysis—that most configurations of comparable indices force the corresponding biderivation entries to vanish; only special “endpoint–endpoint” configurations, where one index is minimal and the other maximal, may yield nontrivial terms. The underlying mechanism is the repeated application of the commutation identity (Lemma 7) together with the vanishing on incomparable pairs: whenever an auxiliary comparable element exists, the relevant coefficients are forced to zero. Consequently, the theorem confines all possible non-zero patterns to the finite boundary situations that generate the extremal biderivations.

The objective of the forthcoming project is to provide evidence that certain components of Equation (18) are equal. In particular, it will be demonstrated that $b_{xy}(e_{xy}, e_{xx}) = b_{uv}(e_{uv}, e_{uu})$ when $x, y, u, v \in P$ satisfy specified conditions. Prior to this, two lemmas will be proven.

Lemma 8. Let $x < y < z \in P$, then

$$b_{xy}(e_{xy}, e_{yy}) = b_{yz}(e_{yy}, e_{yz}) \quad \text{and} \quad b_{xy}(e_{yy}, e_{xy}) = b_{yz}(e_{yz}, e_{yy}). \quad (41)$$

Proof. For any $x < y < z \in P$, applying Lemma 3 to the left or right argument of $b_{xz}(e_{xy}, e_{yz})$ separately, we have

$$b_{yz}(e_{yy}, e_{yz}) = b_{xz}(e_{xy}, e_{yz}) = b_{xy}(e_{xy}, e_{yy}). \quad (42)$$

Thus, we have $b_{xy}(e_{xy}, e_{yy}) = b_{yz}(e_{yy}, e_{yz})$. Using a similar process for $b_{xz}(e_{yz}, e_{xy})$, we obtain the other equality. \square

Lemma 9. Let $x < y < z \in P$, then

$$(1) \quad b_{xy}(e_{xy}, e_{xx}) = b_{yz}(e_{yz}, e_{yy}) = b_{xz}(e_{xz}, e_{xx});$$

$$(2) \quad b_{xy}(e_{xx}, e_{xy}) = b_{yz}(e_{yy}, e_{yz}) = b_{xz}(e_{xx}, e_{xz}).$$

Proof. Let $x < y < z \in P$. For $b_{xz}(e_{xz}, e_{xx})$ and $b_{xz}(e_{xz}, e_{zz})$, using Lemma 5 for its first argument, we obtain

$$\begin{cases} b_{xz}(e_{xz}, e_{xx}) = b_{xy}(e_{xy}, e_{xx}) + b_{yz}(e_{yz}, e_{xx}) = b_{xy}(e_{xy}, e_{xx}); \\ b_{xz}(e_{xz}, e_{zz}) = b_{xy}(e_{xy}, e_{zz}) + b_{yz}(e_{yz}, e_{zz}) = b_{yz}(e_{yz}, e_{zz}) \end{cases} \quad (43)$$

noting that $b_{yz}(e_{yz}, e_{xx}) = b_{xy}(e_{xy}, e_{zz}) = 0$ by Corollary 3. Additionally, we have $b_{xz}(e_{xz}, e_{xx}) = -b_{xz}(e_{xz}, e_{zz})$ by Lemma 4. Plugging the results from (43) into this, we find that

$$b_{xy}(e_{xy}, e_{xx}) = -b_{yz}(e_{yz}, e_{zz}), \quad (44)$$

which implies $b_{xy}(e_{xy}, e_{xx}) = b_{yz}(e_{yz}, e_{yy})$. Then $b_{xy}(e_{xy}, e_{xx}) = b_{yz}(e_{yz}, e_{yy}) = b_{xz}(e_{xz}, e_{xx})$. Similarly, we can obtain the other equality. \square

Corollary 4. Let $x < y \in P$, then $b_{xy}(e_{xy}, e_{xx}) = -b_{xy}(e_{xx}, e_{xy})$.

Proof. Let $x < y \in P$. There exists $z \neq x, y \in P$ that is comparable with x and y by the assumption that any maximal chain in P has at least three elements. We consider three cases.

If $x < y < z$, according to Lemma 8, we have

$$b_{xy}(e_{xy}, e_{yy}) = b_{yz}(e_{yy}, e_{yz}). \quad (45)$$

From Lemma 9, we have

$$b_{xy}(e_{xx}, e_{xy}) = b_{yz}(e_{yy}, e_{yz}). \quad (46)$$

Considering (45) and (46), we have

$$b_{xy}(e_{xy}, e_{xx}) = -b_{xy}(e_{xy}, e_{yy}) = -b_{yz}(e_{yy}, e_{yz}) = -b_{xy}(e_{xx}, e_{xy}). \quad (47)$$

For the remaining two cases $x < z < y$ and $z < x < y$, a similar process yields the same result. \square

Theorem 3. Let $x < y$, $u < v$ in P_j , $j \in \mathcal{J}$ where \mathcal{J} is the index set defined in (10), then

$$b_{xy}(e_{xy}, e_{xx}) = b_{uv}(e_{uv}, e_{uu}). \quad (48)$$

Proof. Let $x < y$, $u < v \in P_j$, $j \in \mathcal{J}$. If P_j is a totally ordered set, it is evident from Lemma 9 that $b_{xy}(e_{xy}, e_{xx}) = b_{uv}(e_{uv}, e_{uu})$. If P_j is not a totally ordered set, according to the construction of P_j , there exist chains $l_0, l_1, \dots, l_n \in L_j$ defined in (10) such that $x, y \in l_0$, $u, v \in l_n$, and $l_i \approx l_{i+1}$ for any $i \in \{0, 1, \dots, n-1\}$. Therefore, there exist $x_i < y_i \in l_i \cap l_{i+1}$. Because l_i is a totally ordered set, using Lemma 9, we thus have

$$b_{xy}(e_{xy}, e_{xx}) = b_{x_0 y_0}(e_{x_0 y_0}, e_{x_0 x_0}) = \dots = b_{x_n y_n}(e_{x_n y_n}, e_{x_n x_n}) = b_{uv}(e_{uv}, e_{uu}). \quad (49)$$

This proves (48). \square

Remark 4. The theorem shows that a certain family of coefficients (for instance, $b_{xy}(e_{xy}, e_{xx})$) remains constant along any connected chain component. Intuitively, this follows from propagating local equalities along adjacent links of a chain using Lemmas 8 and 9, so that the local relations, through this chain-wise propagation, ultimately yield a global constancy on the entire component. Consequently, one can associate a single scalar λ_j with each chain component, and these scalars serve as the weights for the inner biderivation contributions.

For any $j \in \mathcal{J}$, let $x_j < y_j \in P_j$, and define

$$\lambda_j = -b_{x_j y_j}(e_{x_j y_j}, e_{x_j x_j}) = b_{x_j y_j}(e_{x_j x_j}, e_{x_j y_j}). \quad (50)$$

A crucial conclusion is that for any pair $u < v$ in P_j , where $j \in \mathcal{J}$, the value $b_{uv}(e_{uv}, e_{uu}) = -\lambda_j$, as demonstrated by the aforementioned theorem.

Now that the requisite preparations have been completed, we may proceed with the proof of the final theorem.

Theorem 4. Let \mathcal{R} be a commutative ring with unity, and let P be a locally finite poset such that any maximal chain in P contains at least three elements. The additive biderivation b of the incidence algebra $I(P, \mathcal{R})$ is the sum of several inner biderivations and extremal biderivations.

Proof. We proceed by first considering the case where P is connected. The general case will follow by extending this result to each connected component of P .

Case 1: P is connected.

By the decomposition (12), we can write P as a union of subsets P_j :

$$P = \bigcup_{j \in \mathcal{J}} P_j.$$

For any $\alpha, \beta \in I(P, \mathcal{R})$, we can decompose them as follows:

$$\alpha = \sum_{j \in \mathcal{J}} \alpha'_j + \alpha^D, \quad \text{where} \quad \alpha'_j = \sum_{x < y \in P_j} \alpha_{xy} e_{xy}, \quad \alpha^D = \sum_{z \in P} \alpha_{zz} e_{zz}, \quad (51)$$

$$\beta = \sum_{j' \in \mathcal{J}} \beta'_{j'} + \beta^D, \quad \text{where} \quad \beta'_{j'} = \sum_{u < v \in P_{j'}} \beta_{uv} e_{uv}, \quad \beta^D = \sum_{w \in P} \beta_{ww} e_{ww}. \quad (52)$$

Using these decompositions, we expand the biderivation $b(\alpha, \beta)$:

$$b(\alpha, \beta) = b(\alpha^D, \beta^D) + \sum_{j, j' \in \mathcal{J}} b(\alpha'_j, \beta'_{j'}) + \sum_{j \in \mathcal{J}} b(\alpha'_j, \beta^D) + \sum_{j' \in \mathcal{J}} b(\alpha^D, \beta'_{j'}). \quad (53)$$

I. Evaluating $b(\alpha^D, \beta^D)$:

Consider

$$b(\alpha^D, \beta^D) = \sum_{z, w \in P} b(\alpha_{zz} e_{zz}, \beta_{ww} e_{ww}).$$

From Lemma 6 and Theorem 2, we have

$$\begin{aligned} & b(\alpha^D, \beta^D) \\ &= \sum_{z \leq w \in P} b(\alpha_{zz} e_{zz}, \beta_{ww} e_{ww}) \\ &= \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (b(\alpha_{zz} e_{zz}, \beta_{ww} e_{ww}) + b(\alpha_{ww} e_{ww}, \beta_{zz} e_{zz})) + \sum_{\substack{z \text{ is min} \\ \text{or max}}} b(\alpha_{zz} e_{zz}, \beta_{zz} e_{zz}) \end{aligned} \quad (54)$$

By Corollary 3 and Lemma 4, the part of each term in (54) can be expressed as

$$\begin{aligned} & \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (b(\alpha_{zz} e_{zz}, \beta_{ww} e_{ww}) + b(\alpha_{ww} e_{ww}, \beta_{zz} e_{zz})) \\ &= \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (\alpha_{zz} \beta_{ww} b_{zw}(e_{zz}, e_{ww}) + \alpha_{ww} \beta_{zz} b_{zw}(e_{ww}, e_{zz})) e_{zw} \\ &= \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (\alpha_{zz} \beta_{ww} + \alpha_{ww} \beta_{zz}) b_{zw}(e_{zz}, e_{ww}) e_{zw} \end{aligned} \quad (55)$$

By Theorems 1 and 4, the other part of each term in (54) can be expressed as

$$\begin{aligned} & \sum_{\substack{z \text{ is min} \\ \text{or max}}} b(\alpha_{zz} e_{zz}, \beta_{zz} e_{zz}) \\ &= \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (\alpha_{zz} \beta_{zz} b_{zw}(e_{zz}, e_{zz}) + \alpha_{ww} \beta_{ww} b_{zw}(e_{ww}, e_{ww})) \\ &= - \sum_{\substack{z < w \\ z \text{ min}, w \text{ max}}} (\alpha_{zz} \beta_{zz} + \alpha_{ww} \beta_{ww}) b_{zw}(e_{zz}, e_{ww}) e_{zw} \end{aligned} \quad (56)$$

Substituting (55) and (56) back into Equation (54), we obtain

$$\begin{aligned}
 & b(\alpha^D, \beta^D) \\
 &= \sum_{\substack{z < w \in P \\ z \min, w \max}} (\alpha_{zz}\beta_{ww} + \alpha_{ww}\beta_{zz} - \alpha_{zz}\beta_{zz} - \alpha_{ww}\beta_{ww}) b_{zw}(e_{zz}, e_{ww}) e_{zw} \\
 &= \sum_{\substack{z < w \in P \\ z \min, w \max}} [\alpha_{zz}e_{zz} + \alpha_{ww}e_{ww}, [\beta_{zz}e_{zz} + \beta_{ww}e_{ww}, -b_{zw}(e_{zz}, e_{ww})e_{zw}]] \\
 &= \sum_{\substack{z < w \in P \\ z \min, w \max}} [\hat{\alpha}, [\hat{\beta}, T]],
 \end{aligned} \tag{57}$$

where $\hat{\alpha} = \sum_{\substack{z \text{ is min} \\ \text{or max}}} \alpha_{zz}e_{zz}$, $\hat{\beta} = \sum_{\substack{z \text{ is min} \\ \text{or max}}} \beta_{zz}e_{zz}$ and $T = - \sum_{\substack{z < w \\ z \min, w \max}} b_{zw}(e_{zz}, e_{ww})e_{zw}$

II. Evaluating $\sum_{j \neq j' \in \mathcal{J}} b(\alpha'_j, \beta'_{j'})$,

we have

$$\sum_{j \neq j' \in \mathcal{J}} b(\alpha'_j, \beta'_{j'}) = \sum_{j \neq j' \in \mathcal{J}} \sum_{x < y \in P_j} \sum_{u < v \in P_{j'}} b(\alpha_{xy}e_{xy}, \beta_{uv}e_{uv}).$$

Consider elements $x < y \in P_j$ and $u < v \in P_{j'}$, where each pair in x, y, u, v is comparable. A maximal chain, denoted by $l = \{x, y, u, v\}^+$, exists in P that contains x, y, u, v . There exist $l' \in L_i$ and $l'' \in L_j$, where L_i, L_j are defined in (10), such that $x, y \in l'$ and $u, v \in l''$. It is obvious that $l' \approx l \approx l''$, then l would belong to both L_j and $L_{j'}$. Thus, we get $x, y, u, v \in l \in L_j \cap L_{j'}$, which contradicts Lemma 1.

Hence, the existing pair of elements among x, y, u, v is not comparable, and by Lemma 6, it follows that

$$b(\alpha_{xy}e_{xy}, \beta_{uv}e_{uv}) = 0.$$

Therefore,

$$\sum_{j \neq j' \in \mathcal{J}} b(\alpha'_j, \beta'_{j'}) = 0. \tag{58}$$

III. Evaluating $\sum_{j \in \mathcal{J}} b(\alpha'_j, \beta'_j)$,

we have

$$\sum_{j \in \mathcal{J}} b(\alpha'_j, \beta'_j) = \sum_{j \in \mathcal{J}} \sum_{x < y, u < v \in P_j} b(\alpha_{xy}e_{xy}, \beta_{uv}e_{uv}).$$

From Lemmas 2 and 6, the $b(\alpha_{xy}e_{xy}, \beta_{uv}e_{uv})$ part in above equation is non-zero only when any pair in x, y, u, v is compared and $[e_{xy}, e_{uv}] \neq 0$, which implies either $x < y = u < v$ or $u < v = x < y$. Thus, the sum simplifies to

$$\begin{aligned}
 \sum_{j \in \mathcal{J}} b(\alpha'_j, \beta'_j) &= \sum_{j \in \mathcal{J}} \sum_{x < y < z \in P_j} (b(\alpha_{xy}e_{xy}, \beta_{yz}e_{yz}) + b(\alpha_{yz}e_{yz}, \beta_{xy}e_{xy})) \\
 &= \sum_{j \in \mathcal{J}} \sum_{x < y < z \in P_j} (b_{xz}(\alpha_{xy}e_{xy}, \beta_{yz}e_{yz}) + b_{xz}(\alpha_{yz}e_{yz}, \beta_{xy}e_{xy})) e_{xz}.
 \end{aligned} \tag{59}$$

Applying Lemmas 3 and 8, we obtain

$$\begin{aligned}
 & b_{xz}(\alpha_{xy}e_{xy}, \beta_{yz}e_{yz})e_{xz} + b_{xz}(\alpha_{yz}e_{yz}, \beta_{xy}e_{xy})e_{xz} \\
 &= (\alpha_{xy}\beta_{yz}b_{xy}(e_{xy}, e_{yz}) + \alpha_{yz}\beta_{xy}b_{yz}(e_{yz}, e_{xy}))e_{xz} \\
 &= \lambda_j(\alpha_{xy}\beta_{yz} - \alpha_{yz}\beta_{xy})e_{xz},
 \end{aligned} \tag{60}$$

where λ_j is defined by (50). Consequently, Equation (59) becomes

$$\sum_{j \in \mathcal{J}} b(\alpha'_j, \beta'_j) = \sum_{j \in \mathcal{J}} \lambda_j \sum_{x < y < z \in P_j} (\alpha_{xy} \beta_{yz} - \alpha_{yz} \beta_{xy}) e_{xz}. \quad (61)$$

IV. Evaluating $\sum_{j \in \mathcal{J}} b(\alpha'_j, \beta^D) + \sum_{j \in \mathcal{J}} b(\alpha^D, \beta'_j)$:

First, consider $b(\alpha'_j, \beta^D)$:

$$b(\alpha'_j, \beta^D) = \sum_{x < z \in P_j} b(\alpha_{xz} e_{xz}, \beta^D).$$

From Lemma 2, $b(\alpha_{xz} e_{xz}, \beta_{ww} e_{ww}) = 0$ unless $w = x$ or $w = z$. Therefore,

$$\begin{aligned} b(\alpha'_j, \beta^D) &= \sum_{x < z \in P_j} (\alpha_{xz} \beta_{xx} b_{xz}(e_{xz}, e_{xx}) + \alpha_{xz} \beta_{zz} b_{xz}(e_{xz}, e_{zz})) e_{xz} \\ &= \sum_{x < z \in P_j} (\alpha_{xz} \beta_{xx} - \alpha_{xz} \beta_{zz}) b_{xz}(e_{xz}, e_{xx}) e_{xz} \\ &= \lambda_j \sum_{x < z \in P_j} (\alpha_{xz} \beta_{zz} - \alpha_{xz} \beta_{xx}) e_{xz}. \end{aligned} \quad (62)$$

Similarly, for $b(\alpha^D, \beta'_j)$, we obtain

$$b(\alpha^D, \beta'_j) = \lambda_j \sum_{x < z \in P_j} (\alpha_{xx} \beta_{xz} - \alpha_{zz} \beta_{xz}) e_{xz}.$$

Combining these results, we have

$$\sum_{j \in \mathcal{J}} b(\alpha'_j, \beta^D) + \sum_{j \in \mathcal{J}} b(\alpha^D, \beta'_j) = \sum_{j \in \mathcal{J}} \lambda_j \sum_{x < z \in P_j} (\alpha_{xz} \beta_{zz} - \alpha_{xz} \beta_{xx} + \alpha_{xx} \beta_{xz} - \alpha_{zz} \beta_{xz}) e_{xz}. \quad (63)$$

Combining All Components:

Substituting Equations (57), (58), (61), and (63) into Equation (53), we obtain

$$\begin{aligned} b(\alpha, \beta) - b(\alpha^D, \beta^D) &= \sum_{j \in \mathcal{J}} \lambda_j \left(\sum_{x < y < z \in P_j} (\alpha_{xy} \beta_{yz} - \alpha_{yz} \beta_{xy}) + \sum_{x < z \in P_j} (\alpha_{xz} \beta_{zz} - \alpha_{xz} \beta_{xx} + \alpha_{xx} \beta_{xz} - \alpha_{zz} \beta_{xz}) \right) e_{xz} \\ &= \sum_{j \in \mathcal{J}} \lambda_j \sum_{x \leq y \leq z \in P_j} (\alpha_{xy} \beta_{yz} - \alpha_{yz} \beta_{xy}) e_{xz} \\ &= \sum_{j \in \mathcal{J}} \lambda_j [\alpha_j, \beta_j], \end{aligned}$$

where $\alpha_j = \sum_{x \leq y \in P_j} \alpha_{xy} e_{xy}$, $\beta_j = \sum_{x \leq y \in P_j} \beta_{xy} e_{xy}$.

Consider the aforementioned points; we thus deduce the following result:

$$b(\alpha, \beta) = \sum_{j \in \mathcal{J}} \lambda_j [\alpha_j, \beta_j] + [\hat{\alpha}, [\hat{\beta}, T]]. \quad (64)$$

Case 2: P is not connected.

When P is disconnected, we utilize the decomposition (7):

$$P = \bigcup_{i \in I} P^i,$$

where each P^i is a connected component of P . Further, each connected component P^i can be decomposed as

$$P^i = \bigcup_{j \in \mathcal{J}_i} P_j^i$$

using decomposition (11).

For any $\alpha, \beta \in I(P, \mathcal{R})$, write

$$\alpha = \sum_{i \in I} \alpha^i \quad \text{and} \quad \beta = \sum_{i \in I} \beta^i,$$

where $\alpha^i, \beta^i \in I(P^i, \mathcal{R})$, as per decomposition (8).

By Equation (24), the biderivation b satisfies

$$b(\alpha, \beta) = \sum_{i \in I} b(\alpha^i, \beta^i).$$

Since each P^i is connected, applying the result from Case 1, we obtain

$$b(\alpha^i, \beta^i) = \sum_{j \in \mathcal{J}_i} \lambda_j^i [\alpha_j^i, \beta_j^i] + [\hat{\alpha}^i, [\hat{\beta}^i, T^i]].$$

Therefore, combining all components, we conclude that

$$b(\alpha, \beta) = \sum_{i \in I} \left(\sum_{j \in \mathcal{J}_i} \lambda_j^i [\alpha_j^i, \beta_j^i] + [\hat{\alpha}^i, [\hat{\beta}^i, T^i]] \right), \quad (65)$$

as desired. It is evident that b is the sum of several inner biderivations and extremal biderivations. \square

Remark 5. The decomposition in the main theorem can be understood in three intuitive steps. First, Lemma 6 reduces the problem to comparable pairs, so the analysis localizes to connected chain components of the poset. Second, on each chain component, the constancy result (Theorem 3) produces scalar weights that result in the inner biderivation terms. Third, Theorems 1 and 2 show that only finite maximal chains produce additional terms that cannot be absorbed into inner parts; these survive as $[\hat{\alpha}^i, [\hat{\beta}^i, T^i]]$, i.e., the extremal biderivations. Thus, the "inner + extremal" decomposition reflects a structural constraint imposed by the chain decomposition of the poset.

It is evident that if there are no $x < y \in P$ where x is the minimal element and y is the maximal element, $\hat{\alpha}^i = \hat{\beta}^i = T^i = 0$. Consequently, the next corollary holds.

Corollary 5. Let P be a poset that has at least three elements, and let \mathcal{R} be a commutative ring with unity. In the incidence algebra of P over \mathcal{R} , if the number of elements in any maximal chain in P is infinite, every additive biderivation is the sum of several inner biderivations.

Example 4. We conclude by describing the structure of an arbitrary additive biderivation b on $I(S^1, \mathcal{R})$, where S^1 is the poset introduced in Example 3. Recall that

$$S^1 = \bigcup_{j \in \mathcal{J}^1} S_j^1, \quad \mathcal{J}^1 = \{1, 2\}, \quad S_1^1 = \{a, b, d\}, \quad S_2^1 = \{a, c, e, f\}.$$

For any $\alpha, \beta \in I(S^1, \mathcal{R})$, the biderivation takes the form

$$b(\alpha, \beta) = \lambda_1^1 [\alpha_1^1, \beta_1^1] + \lambda_2^2 [\alpha_2^1, \beta_2^1] + [\hat{\alpha}^1, [\hat{\beta}^1, T^1]],$$

$$\alpha_j^1 = \sum_{x < y \in S_j^1} \alpha_{xy} e_{xy}, \quad \beta_j^1 = \sum_{x < y \in S_j^1} \beta_{xy} e_{xy},$$

$$\hat{\alpha}^1 = \sum_{z \in S^1 \text{ min or max}} \alpha_{zz} e_{zz}, \quad \hat{\beta}^1 = \sum_{z \in S^1 \text{ min or max}} \beta_{zz} e_{zz},$$

$$T^1 = - \sum_{\substack{z < w \in S^1, \\ z \text{ min}, w \text{ max}}} b_{zw}(e_{zz}, e_{ww}) e_{zw}.$$

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