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Article

Zeta Function Expression of Spin Partition Functions on Thermal AdS_3

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Abstract: We find a Selberg zeta function expression of certain one-loop spin partition functions on three-dimensional thermal anti-de Sitter space. Of particular interest is the partition function of higher spin fermionic particles. We also set up, in the presence of spin, a Patterson-type formula involving the logarithmic derivative of zeta.

Keywords: thermal AdS; spin partition functions; Patterson–Selberg zeta function; fermionic particles; BTZ black hole

1. Introduction

A symposium in honor of A. Selberg, which commemorated his 70th birthday, was held in Oslo, Norway, during July of 1987. From that event, the volume [1], consisting of 30 invited lectures evolved, included a lecture by S. Patterson in which the construction of Selberg's classic zeta function [2] was extended to cover certain discrete groups (Kleinian groups) Γ acting on real hyperbolic space $\mathbb{H}^n$ whose fundamental domains, however, were of infinite hyperbolic volume. We denote this Patterson–Selberg (P-S) zeta function for a Kleinian group by Z_Γ.

For the special case with $n = 3$ and with Γ generated by a loxodromic element, it was noticed first by the author (in the 2003 paper [3]) that Z_Γ had some relevant connections to the Euclidean BTZ black hole [4]. For example, we were able to express BTZ effective action directly in terms of $\log Z_\Gamma$. Moreover, the zeros of Z_Γ were shown to correspond to BTZ scattering resonances, in a follow up joint work with P. Perry [5], and black hole entropy connections were described in terms of a "deformation" of Z_Γ [6–10]. Application of the BTZ scattering in [5] appears in the work of R. Aros and D. Diaz [11,12], for example, where a holographic formula is derived: a formula that relates the determinant of a scattering operator on a space asymptotically anti-de Sitter (AdS) to a "relative" determinant of a Laplacian in the bulk, a result that one can regard as an entry in the AdS/CFTdictionary.

In some later work, A. Bytsenko and M. Guimarães [13,14] (and the author [15], independently) discovered a formula that expresses the one-loop AdS gravity partition function in terms of a quotient of products involving Z_Γ; see Formula (27) of Section 3. It has also been shown in [16] that modular invariant partition functions for an extreme conformal field theory with central charge divisible by eight have an expression in terms of Z_Γ. These preliminary remarks (where, again, we have focused only on hyperbolic three-space and the case of a single loxodromic generator of Γ) illustrate the natural appearances of the P-S zeta function Z_Γ in the physics literature and its growing applications, of which further details are presented, for example, in the review article [10].

By way of private communication with R. Gopakumar, the author received (among other notes) three pages of "Notes on the heat kernel for AdS_3", by J. David, M. Gaberdiel and R. Gopakumar, in which these authors find a product formula for the one-loop partition function for a massive spin field on thermal AdS_3. They assert that their product "probably has an interpretation as a Selberg zeta function" [17]. We show in this paper (in Section 3, Formula (36)) that this indeed is the case. We use, in fact, the central result of these authors in the extended work [18] (Equation (6.9) there) to express, more generally, various one-loop spin partition functions on thermal AdS_3 in terms of the P-S zeta function Z_Γ. This includes, in particular, the partition function for the gravitino (the super partner of the graviton) in $\mathbb{N} = 1$ supergravity and the more general case of higher spin fermionic (and bosonic) particles.

We also establish a Gangolli–Patterson-type formula (Formula (49) in Section 4) that relates a certain transform of the heat kernel trace to the logarithmic derivative of Z_Γ. The formula offers an interesting contrast to the result known in the spin zero case.

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2. The Action of $SL(2, \mathbb{C})$ on $\mathbb{H}^3$, Thermal AdS_3, and the Patterson–Selberg Zeta Function

$\mathbb{R}, \mathbb{C}$ will denote the field of real, complex numbers, respectively. $\mathbb{H}^3 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ denotes real hyperbolic three-space, and $\tilde{\mathbb{H}} \stackrel{\text{def}}{=} \{\begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix} | u, v \in \mathbb{C}\}$ denotes the ring of quaternions where the bar "$^-$" denotes complex conjugation. There are natural embeddings $\mathbb{C} \to \tilde{\mathbb{H}}$, $\mathbb{H}^3 \to \tilde{\mathbb{H}}$ given by:

$$a \to [a] \stackrel{\text{def}}{=} \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, w = (x, y, z) \to [w] \stackrel{\text{def}}{=} \begin{bmatrix} x + iy & -z \\ \bar{z} & \overline{x + iy} \end{bmatrix} \tag{1}$$

respectively, for $a \in \mathbb{C}, w \in \mathbb{H}^3, i^2 = -1$. Using these embeddings, one can define a standard linear fractional action of the complex special linear group $G = SL(2, \mathbb{C})$ on $\mathbb{H}^3$ as follows. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, $w = (x, y, z) \in \mathbb{H}^3$, $g \cdot w \in \mathbb{H}^3$ is defined by:

$$[g \cdot w] \stackrel{\text{def}}{=} ([a][w] + [b])([c][w] + [d])^{-1} \in \tilde{\mathbb{H}}. \tag{2}$$

One computes that $g \cdot w$ is given explicitly by:

$$g \cdot w = (u, v, \omega) \tag{3}$$

for:

$$\begin{aligned} \omega &= z(|ct+d|^2 + |c|^2 z^2)^{-1}, t \stackrel{\text{def}}{=} x + iy, \\ u + iv &= ((at+b)\overline{(ct+d)} + a\bar{c}z^2)(|ct+d|^2 + |c|^2 z^2)^{-1}. \end{aligned} \tag{4}$$

Elements of $\mathbb{R}^3, \mathbb{H}^3$ can also be written as column vectors. In particular, for:

$$g = \gamma_{(a,b)} \stackrel{\text{def}}{=} \begin{bmatrix} e^{a+ib} & 0 \\ 0 & e^{-a-ib} \end{bmatrix}, a, b \in \mathbb{R}, \tag{5}$$

the equations in (4) reduce to:

$$g \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^{2a}(x \cos 2b - y \sin 2b) \\ e^{2b}(x \sin 2b + y \cos 2b) \\ e^{2a} z \end{bmatrix}. \tag{6}$$

We have interest in the case $a > 0$. For $Z =$ the ring of integers, let:

$$\Gamma = \Gamma_{(a,b)} \stackrel{\text{def}}{=} \{\gamma_{(a,b)}^n | n \in \mathbb{Z}\} \tag{7}$$

denote the subgroup of G generated by $\gamma_{(a,b)}$. If $\tau = \tau_1 + i\tau_2$ is in the upper 1/2-plane π^+, for example (*i.e.*, $\tau_1, \tau_2 \in \mathbb{R}, Im\tau = \tau_2 > 0$), then $e^{-\pi i \tau} = e^{a+ib}$ for $a = \pi\tau_2 > 0, b = -\pi\tau_1$, and we can form the (Euclidean) thermal AdS_3 quotient $X_\tau \stackrel{\text{def}}{=} \Gamma_\tau \setminus \mathbb{H}^3$ for $\Gamma_\tau \stackrel{\text{def}}{=} \Gamma_{(\pi\tau_2, -\pi\tau_1)}$. Here, the action of Γ_τ on $\mathbb{H}^3$ is given by Equation (6). Equivalently, this action is given by Equation (4): $\gamma_{(\pi\tau_2,-\pi\tau_1)} \cdot w = (u, v, \omega)$ for:

$$\omega = \frac{z}{e^{-2\pi\tau_2}} = |q|^{-1} z, u + iv = e^{-2\pi i\tau}(x+iy) = q^{-1}(x+iy), q \stackrel{\text{def}}{=} e^{2\pi i \tau}, \tag{8}$$

which is of the form given in [19–21], for example, with Z identified with Γ_τ by way of the group isomorphism $n \to \gamma^n_{(\pi\tau_2,-\pi\tau_1)} = \begin{bmatrix} e^{-\pi n i \tau} & 0 \\ 0 & e^{\pi n i \tau} \end{bmatrix}$ (by (5)) for $n \in \mathbb{Z}$.

The Patterson–Selberg (P-S) zeta function corresponding to $\Gamma_{(a,b)}$ in Equation (7) is defined as the Euler product:

$$Z_{\Gamma_{(a,b)}}(z) \stackrel{\text{def}}{=} \prod_{0 \le k_1, k_2 \in \mathbb{Z}} [1 - (e^{2bi})^{k_1}(e^{-2bi})^{k_2} e^{-(k_1+k_2+z)2a}], \tag{9}$$

which is an entire function of $z \in \mathbb{C}$. Again, a more general definition is given in [1], for a discrete group Γ acting on $\mathbb{H}^n$. There is a function L defined on the 1/2-plane $Rez > 0$, such that $Z_{\Gamma_{(a,b)}}(z) = e^{L(z)}$. This function, which we denote by $\log Z_{\Gamma_{(a,b)}}$, is given by:

$$\log Z_{\Gamma_{(a,b)}}(z) = -\sum_{n=1}^{\infty} \frac{e^{-2an(z-1)}}{4n[\sin^2 bn + \sinh^2 an]} \tag{10}$$

for $Rez > 0$. Note that $e^{-2an(z \pm i\pi/2a - 1)} = e^{-2an(z-1)} e^{\mp in\pi} = (-1)^n e^{-2an(z-1)}$, which by Equation (10) gives:

$$\log Z_{\Gamma_{(a,b)}}(z \pm i\pi/2a) = -\sum_{n=1}^{\infty} \frac{(-1)^n e^{-2an(z-1)}}{4n[\sin^2 bn + \sinh^2 an]}. \tag{11}$$

We shall also need the logarithmic derivative:

$$Z'_{\Gamma_{(a,b)}}(z)/Z_{\Gamma_{(a,b)}}(z) = \frac{a}{2}\sum_{n=1}^{\infty}\frac{e^{-2an(z-1)}}{\sin^2 nb + \sinh^2 na}, \tag{12}$$

for $Rez > 0$. Similar to Equation (11), we have:

$$Z'_{\Gamma_{(a,b)}}(z \pm \frac{i\pi}{2a})/Z_{\Gamma_{(a,b)}}(z \pm \frac{i\pi}{2a}) = \frac{a}{2}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-2an(z-1)}}{\sin^2 nb + \sinh^2 na} \tag{13}$$

for $Rez > 0$.

3. The Main Result and Some Examples

The central result of [18] is a formula for the heat kernel trace $K^{(s)}(\tau, \bar{\tau}; t)(t > 0)$ for the spin Laplacian $\Delta_{(s)}$, with spin s, on thermal AdS_3. This is Formula (6.9) there. A slightly modified version of it is the following, which takes into account the anti-periodicity of half-integral spin particles (like fermions, with $s = \frac{1}{2}$ or $\frac{3}{2}$, for example):

$$K^{(s)}(\tau, \bar{\tau}; t) = \frac{(2-\delta_{s,0})}{4\sqrt{\pi t}}\sum_{n=1}^{\infty}\frac{(-1)^{n2s}\tau_2 e^{\frac{-n^2\tau_2^2}{4t}} e^{-(s+1)t}\cos(ns\tau_1)}{\sin^2\frac{n\tau_1}{2} + \sinh^2\frac{n\tau_2}{2}}. \tag{14}$$

$\delta_{s,0}$ is zero for $s \neq 0$ and one for $s = 0$. $\tau = \tau_1 + i\tau_2 \in \pi^+$ ($\tau_1, \tau_2 \in \mathbb{R}, \tau_2 > 0$, as before) is the modulus of the boundary torus T^2 of AdS_3. The notation $q = e^{2\pi i\tau}$ in Equation (8) of Section 2, which is that of [19,21], differs from the notation $q = e^{i\tau}$ in [18]: the τ in [18] is $2\pi\tau$ in [19,21]. Thus, we take here $X_\Gamma = \Gamma \setminus \mathbb{H}^3$ as thermal AdS_3, where $\Gamma \subset SL(2, \mathbb{C})$ is the subgroup generated by the element:

$$\gamma = \begin{bmatrix} e^{-i\tau/2} & 0 \\ 0 & e^{i\tau/2} \end{bmatrix}. \tag{15}$$

In Formula (14), the infrared divergent term due to the infinite hyperbolic volume of X_Γ is, by convention, neglected. $2\pi\tau_1, 2\pi\tau_2$ correspond respectively to the angular potential θ and inverse temperature β. The heat kernel on X_Γ is obtained, of course, by averaging over Γ the heat kernel on $\mathbb{H}^3$ (the method of images), and one obtains the trace $K^{(s)}(\tau, \bar{\tau}; t)$ by integration (over X_Γ) along the diagonal. In the spin zero case ($s = 0$), a proof of Formula (14) is given in [23], for example; also, see [24–27], for example.

We apply Formula (14) in this section to set up a general formula (once and for all) that expresses the effective energy $-\log(-\Delta_{(s)} + M)$ in terms of the P-S zeta function. See Theorem 1, which we use to show that various zeta function expressions of one-loop partition functions can be derived, uniformly, by varying the spin and mass parameters s, M.

Start with the standard formula:

$$\int_0^\infty t^{-\frac{3}{2}} e^{-A/4t} e^{-Bt} dt = 2\sqrt{\frac{\pi}{A}} e^{-\sqrt{AB}}, \quad A > 0, B \geq 0, \tag{16}$$

to compute:

$$-\log\det(-\Delta_{(s)} + M) \overset{\text{def}}{=} \int_0^\infty K^{(s)}(\tau, \bar{\tau}; t) e^{-Mt}\frac{dt}{t}. \tag{17}$$

Plug in Formula (14) and commute the integration with the summation, as one would usually do. In the present case (for the choices $A = n^2\tau_2^2, B = s + 1 + M$, with $n \geq 1$), Formula (16) gives:

$$- \log \det(-\Delta_{(s)} + M) = \frac{(2 - \delta_{(s,0)})}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n2s} e^{-n\tau_2\sqrt{s+1+M}} \cos(sn\tau_1)}{n[\sin^2 \frac{n\tau_1}{2} + \sinh^2 \frac{n\tau_2}{2}]} \tag{18}$$

for $s + 1 + M \geq 0$ (since $\tau_2 > 0$ by definition of τ). Thus, the main point is to express the sum in Equation (18) in terms of Z_Γ, for a suitable choice of Γ.

For this, define $z = 1 + \sqrt{s + 1 + M}$ and write $2\cos(sn\tau_1) = e^{isn\tau_1} + e^{-isn\tau_1}$:

$$2e^{-n\tau_2\sqrt{s+1+M}} \cos(sn\tau_1) = e^{-n\tau_2(z-1)}\left(e^{isn\tau_1} + e^{-isn\tau_1}\right) = e^{-n\tau_2(z-i\tau_1 s/\tau_2-1)} + e^{-n\tau_2(z+i\tau_1 s/\tau_2-1)} \tag{19}$$

which permits Equation (18) to be written as:

$$- \log \det(-\Delta_{(s)} + M) = \frac{(2 - \delta_{(s,0)})}{2} \sum_{n=1}^{\infty} (-1)^{n2s} \frac{[e^{-2n\frac{\tau_2}{2}(z-i\tau_1 s/\tau_2-1)} + e^{-2n\frac{\tau_2}{2}(z+i\tau_1 s/\tau_2-1)}]}{2n[\sin^2 \frac{n\tau_1}{2} + \sinh^2 \frac{n\tau_2}{2}]}. \tag{20}$$

We separate out the integral and half-integral spin cases. If $s = m \in \mathbb{Z}$, then for $n \geq 1$, $(-1)^{n2s} = 1$ in Equation (20), and if $s = (2m+1)/2$, then $(-1)^{n2s} = (-1)^n$. In the latter case, the sum in Equation (20) is the sum:

$$- 2\log Z_{\Gamma_{(\tau_2/2,\tau_1/2)}}(z - i\tau_1 s/\tau_2 + i\pi/\tau_2) - 2\log Z_{\Gamma_{(\tau_2/2,\tau_1/2)}}(z + i\tau_1 s/\tau_2 - i\pi/\tau_2) \tag{21}$$

by Equation (11). In the former case $s = m$, we have by Equation (10) a similar sum of logs, but without the term $-i\pi/\tau_2$. Going back to the definition of z and definition Equation (17), we have established:

Theorem 1. *Let $M \in \mathbb{R}$ (possibly $M < 0$) with $s + 1 + M \geq 0$. The following formula holds:*

$$\log \det(-\Delta_{(s)} + M) = 2[\log Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} + i\tau_1 s/\tau_2) + \log Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} - i\tau_1 s/\tau_2)] \tag{22}$$

for integral spin $s = 1, 2, 3, 4, \cdots$, or:

$$2[\log Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} + i\tau_1 s/\tau_2 + i\pi/\tau_2) + \log Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} - i\tau_1 s/\tau_2 - i\pi/\tau_2)], \tag{23}$$

for half-integral spin $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$, where the Patterson–Selberg zeta function Z_{Γ_τ} is given by definition Equation (9) for the choice $(a, b) = (\tau_2/2, \tau_1/2)$ there. That is $\Gamma_\tau = \Gamma_{(\tau_2/2,\tau_1/2)} \stackrel{\text{def}}{=} \{\gamma_\tau^n | n \in \mathbb{Z}\}$ is the subgroup of $SL(2, \mathbb{C})$ generated by the element:

$$\gamma_\tau \stackrel{\text{def}}{=} \begin{bmatrix} e^{(\tau_2+i\tau_1)/2} & 0 \\ 0 & e^{-(\tau_2+i\tau_1)/2} \end{bmatrix} = \begin{bmatrix} e^{i\bar{\tau}/2} & 0 \\ 0 & e^{-i\bar{\tau}/2} \end{bmatrix} \tag{24}$$

as in Equations (5) and (7). Consequently, we also have (since $\overline{Z_\Gamma(z)} = Z_\Gamma(\bar{z})$):

$$\begin{aligned} \det(-\Delta_{(s)} + M) &= Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} + i\tau_1 s/\tau_2 + i\pi/\tau_2)^2 Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} - i\tau_1 s/\tau_2 - i\pi/\tau_2)^2 \\ &= |Z_{\Gamma_\tau}(1 + \sqrt{s + 1 + M} + i\tau_1 s/\tau_2 + i\pi/\tau_2)|^4 \end{aligned} \tag{25}$$

for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$, where the term $\pm i\pi/\tau_2$ is neglected for $s = 1, 2, 3, 4, \cdots$.

Note that the Γ_τ here differs from the Γ_τ in Section 2, which is not an issue. We have focused on the case $s \neq 0$ for the applications we have in mind.

Since various one-loop partition functions are given in terms of $\det(-\Delta_{(s)} + M)$, for suitable s, M, we can use the preceding formula to express these functions in terms of Z_{Γ_τ}. In what follows are some examples (that draw freely from the results and notation in the literature) where, for convenience, we shall write Γ for $\Gamma_\tau = \Gamma_{(\tau_2/2, \tau_1/2)}$.

From (7.7) of [18], for example,

$$Z_{1-loop}^{graviton} = [\det(-\Delta_{(1)} + 2)]^{1/2} / [\det(-\Delta_{(2)} - 2)]^{1/2} \tag{26}$$

Thus, by Equation (25):

$$\begin{aligned} Z_{1-loop}^{graviton} &= \left| Z_\Gamma(3 + i\frac{\tau_1}{\tau_2}) \right|^2 / \left| Z_\Gamma(2 + 2i\frac{\tau_1}{\tau_2}) \right|^2 \\ &= \frac{Z_\Gamma(3 + i\frac{\tau_1}{\tau_2}) Z_\Gamma(3 - i\frac{\tau_1}{\tau_2})}{Z_\Gamma(2 + 2i\frac{\tau_1}{\tau_2}) Z_\Gamma(2 - 2i\frac{\tau_1}{\tau_2})} \end{aligned}, \tag{27}$$

where, again, we use that $\overline{Z_\Gamma(z)} = Z_\Gamma(\bar{z})$. As indicated in the Introduction, Formula (27) was found by A. Bytsenko and M. Guimarães [13], as well as (independently) by the author; see Theorem 3.26 on page 337 of [15]. Equation (27) corrects the slight error in the version given on page 337 in [13].

For the Majorana gravitino of $\mathbb{N} = 1$ supergravity, one has by (7.27) of [18], for example,

$$Z_{1-loop}^{gravitino} = \left[\frac{\det(-\Delta_{(3/2)} - 9/4)}{\det(-\Delta_{(1/2)} + 3/4)} \right]^{\frac{1}{2}} \tag{28}$$

which, by Equation (25), can be expressed as:

$$\begin{aligned} Z_{1-loop}^{gravitino} &= \frac{\left| Z_\Gamma(\frac{3}{2} + i\frac{3}{2}\frac{\tau_1}{\tau_2} + \frac{i\pi}{\tau_2}) \right|^2}{\left| Z_\Gamma(\frac{5}{2} + \frac{i}{2}\frac{\tau_1}{\tau_2} + \frac{i\pi}{\tau_2}) \right|^2} \\ &= \frac{Z_\Gamma(\frac{3}{2} + i\frac{3}{2}\frac{\tau_1}{\tau_2} + \frac{i\pi}{\tau_2}) Z_\Gamma(\frac{3}{2} - i\frac{3}{2}\frac{\tau_1}{\tau_2} - \frac{i\pi}{\tau_2})}{Z_\Gamma(\frac{5}{2} + \frac{i}{2}\frac{\tau_1}{\tau_2} + \frac{i\pi}{\tau_2}) Z_\Gamma(\frac{5}{2} - \frac{i}{2}\frac{\tau_1}{\tau_2} - \frac{i\pi}{\tau_2})}. \end{aligned} \tag{29}$$

Furthermore, compare Formula (3.11) of [14].

$Z_{1-loop}^{gravitino}$ is also computed "from scratch" in [28].

Regarding some remarks in the Introduction, we show, affirmatively, in the next example that the one-loop partition function Z_{1-loop} on AdS_3 computed in [17] for a spin s field of mass m_0 indeed has an interpretation in terms of a suitable Selberg-type zeta function, as these authors suspected. Their Formula (14) is:

$$Z_{1-loop} = \prod_{k,l \geq 0}^{\infty} (1 - q^{h_s + k} \bar{q}^{\bar{h}_s + l})^{-1} (1 - q^{\bar{h}_s + k} \bar{q}^{h_s + l})^{-1} \tag{30}$$

where $k, l \in \mathbb{Z}$, $q = e^{i\tau}$, $h_s \overset{\text{def}}{=} (z+s)/2$, $\bar{h}_s \overset{\text{def}}{=} (z-s)/2$ for $z \overset{\text{def}}{=} 1 + \sqrt{(s-1)^2 + m_o^2}$. We show therefore that the product in Equation (30) is expressible in terms of the P-S zeta function Z_Γ in Equation (9) for

$\Gamma = \Gamma_{(a,b)}$, where in fact, we take $a = \tau_2/2, b = \tau_1/2 \cdot \tau = \tau_1 + i\tau_2 \Rightarrow \tau/2 = b + ia, \bar{\tau}/2 = b - ia$. Then:

$$\begin{aligned} q^{h_s+k}\bar{q}^{\bar{h}_s+l} &= e^{i\tau(h_s+k)}e^{-i\tau(\bar{h}_s+l)} \\ &\overset{\text{def}}{=} e^{i\frac{\tau}{2}(z+s+2k)}e^{-i\frac{\bar{\tau}}{2}(z-s+l)} \\ &= e^{i(b+ia)(z+s+2k)}e^{-i(b-ia)(z-s+2l)}, \end{aligned} \tag{31}$$

which simplifies to:

$$e^{2ibs+2ibk-2ibl-2az-2ak-2al} = e^{2ibs}(e^{2bi})^k(e^{-2bi})^l e^{-(k+l+z)2a}. \tag{32}$$

Similarly:

$$q^{\bar{h}_s+k}\bar{q}^{h_s+l} = e^{-2ibs}(e^{2bi})^k(e^{-2bi})^l e^{-(k+l+z)2a}. \tag{33}$$

By definition Equation (9) (for any $z \in \mathbb{C}$):

$$Z_{\Gamma_{(a,b)}}(z \pm i\frac{b}{a}s) = \prod_{0 \le k,l \in \mathbb{Z}} \left[1 - (e^{2bi})^k(e^{-2bi})^l e^{-(k+l+z)2a}e^{\mp 2bsi}\right], \tag{34}$$

which, by Equations (32) and (33), gives for $\Gamma = \Gamma_{(a,b)}$:

$$\begin{aligned} Z_\Gamma(z + i\frac{b}{a}s) &= \prod_{k,l}\left[1 - q^{\bar{h}_s+k}\bar{q}^{h_s+l}\right], \\ Z_\Gamma(z - i\frac{b}{a}s) &= \overline{\prod_{k,l}\left[1 - q^{h_s+k}\bar{q}^{\bar{h}_s+l}\right]} \\ &= \overline{Z_\Gamma(z + i\frac{b}{a}s)}. \end{aligned} \tag{35}$$

Therefore, by definition of z, a, b, we can express Formula (30), for $\Gamma = \Gamma_{(\tau_2/2,\tau_1/2)}$, as:

$$\begin{aligned} Z_{1-loop} &= \left|Z_\Gamma(1 + \sqrt{(s-1)^2 + m_0^2} + is\tau_1/\tau_2)\right|^{-2} \\ &= Z_\Gamma(1 + \sqrt{(s-1)^2 + m_0^2} + is\tau_1/\tau_2)^{-1} Z_\Gamma(1 + \sqrt{(s-1)^2 + m_0^2} - is\tau_1/\tau_2)^{-1}. \end{aligned} \tag{36}$$

Formula (29), which involves fermions of spin $\frac{1}{2}, \frac{2}{3}$, can be completely generalized, which we now show. In their study of M. Vasiliev's higher spin supergravity on AdS_3, T. Creutzig, Y. Hikida and P. Rϕnne [29] obtain in their Equation (3.7) one of their main results: the computation of the one-loop determinant $Z_F^{(s)}$ (in their notation) for higher spin fermionic particles:

$$\begin{aligned} Z_F^{(s)} &= \frac{\det^{\frac{1}{2}}\left(-\Delta + (s+\frac{1}{2})(s-\frac{5}{2})\right)^{TT}_{(s+\frac{1}{2})}}{\det^{\frac{1}{2}}\left(-\Delta + (s-\frac{1}{2})(s+\frac{1}{2})\right)^{TT}_{(s-\frac{1}{2})}} \\ &= \prod_{n=s}^{\infty}|1 + q^{n+1/2}|^2 \end{aligned} \tag{37}$$

for $q = e^{2\pi i\tau}$. In this notation, which we shall translate to our earlier notation, it is emphasized that the Laplacian Δ on AdS_3 is restricted to act on transverse, traceless components of spin $s + \frac{1}{2}, s - \frac{1}{2}$ gauge fields.

For our purpose, and for consistency with earlier notation, we start with an arbitrary half-integral spin $s = (2m+1)/2$, $m = 1, 2, 3, 4, \cdots$ Then, for us, the determinant quotient in Equation (37), which we shall also denote by $Z_F^{(m)}$, means:

$$Z_F^{(s)} = Z_F^{(m)} = \left[\frac{\det\left(-\Delta_{(m+\frac{1}{2})} + (m+\frac{1}{2})(m-\frac{5}{2})\right)}{\det\left(-\Delta_{(m-\frac{1}{2})} + (m-\frac{1}{2})(m+\frac{1}{2})\right)} \right]^{\frac{1}{2}}. \tag{38}$$

Thus, for $m = 1$, for example, $Z_F^{(1)}$ is exactly the one-loop partition function for the gravitino given in Equation (28) and Equation (29). We generalize Equation (29) by expressing $Z_F^{(m)}$, similarly, in terms of the P-S zeta function Z_Γ, again for $\Gamma = \Gamma_\tau$. Note first that for $M = (m+\frac{1}{2})(m-\frac{5}{2}), (m-\frac{1}{2})(m+\frac{1}{2})$, respectively, $s + 1 + M = (m+\frac{1}{2}) + 1 + M = (2m-1)^2/4$, $(s-1) + 1 + M = (m-\frac{1}{2}) + 1 + (m-\frac{1}{2})(m+\frac{1}{2}) = (2m+1)^2/4$, respectively. Therefore, $1 + \sqrt{s+1+M}, 1 + \sqrt{(s-1)+1+M} = (1+2m)/2, (3+2m)/2$, respectively, so that Theorem 1 gives:

$$\begin{aligned} Z_F^{(m)} &= \frac{\left|Z_{\Gamma_\tau}((1+2m)/2 + i(m+\frac{1}{2})\tau_1/\tau_2 + i\pi/\tau_2)\right|^2}{\left|Z_{\Gamma_\tau}((3+2m)/2 + i(m-\frac{1}{2})\tau_1/\tau_2 + i\pi/\tau_2)\right|^2} \\ &= \frac{|Z_{\Gamma_\tau}(s + is\tau_1/\tau_2 + i\pi/\tau_2)|^2}{|Z_{\Gamma_\tau}(s+1+i(s-1)\tau_1/\tau_2 + i\pi/\tau_2)|^2} \\ &= \frac{Z_{\Gamma_\tau}(s + is\tau_1/\tau_2 + i\pi/\tau_2) Z_{\Gamma_\tau}(s - is\tau_1/\tau_2 - i\pi/\tau_2)}{Z_{\Gamma_\tau}(s+1+i(s-1)\tau_1/\tau_2 + i\pi/\tau_2) Z_{\Gamma_\tau}(s+1-i(s-1)\tau_1/\tau_2 - i\pi/\tau_2)} \end{aligned} \tag{39}$$

for $s = (2m+1)/2, m = 1, 2, 3, 4, \cdots$, which is the desired generalization of Formula (29) and which covers the higher spin fermionic particles considered in [29].

Similarly, M. Gaberdiel, R. Gopakumar and A. Saha in [30] compute the one-loop partition function $Z_B^{(s)}$ (B for bosons) for quadratic fluctuations of fields about a thermal AdS_3 background:

$$Z_B^{(s)} = \left[\frac{\det\left(-\Delta_{(s-1)} + s(s-1)\right)}{\det\left(-\Delta_{(s)} + s(s-3)\right)} \right]^{\frac{1}{2}} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \tag{40}$$

for $s = 2, 3, 4, 5, \cdots, q = e^{i\tau}$. Noting that $\sqrt{(s-1)+1+s(s-1)}, \sqrt{s+1+s(s-3)} = s, s-1$, respectively, we obtain from Theorem 1:

$$\begin{aligned} Z_B^{(s)} &= \frac{|Z_{\Gamma_\tau}(1+s+i(s-1)\tau_1/\tau_2)|^2}{|Z_{\Gamma_\tau}(s+is\tau_1/\tau_2)|^2} \\ &= \frac{Z_{\Gamma_\tau}(1+s+i(s-1)\tau_1/\tau_2) Z_{\Gamma_\tau}(1+s-i(s-1)\tau_1/\tau_2)}{Z_{\Gamma_\tau}(s+is\tau_1/\tau_2) Z_{\Gamma_\tau}(s-is\tau_1/\tau_2)}. \end{aligned} \tag{41}$$

Thus, in the pure gravity case, with $s = 2$ in particular, Formula (41) reduces to Formula (27) for the one-loop partition function of the graviton.

4. A Gangolli–Patterson-Type Formula in the Presence of Spin

In this section, we present a new formula (in Theorem 2) that directly relates a particular integral transform of the heat kernel trace $K^{(s)}(\tau, \bar{\tau}; t)$ in Equation (14) to the logarithmic derivative of Z_{Γ_τ}. In the spin zero case, the formula reduces to Equations (50) and (51) below, which is that of [9,23], for example. Furthermore, the formula is, in fact, an exact analogue of R. Gangolli's Formula (2.39) in [22] for the case of the generalized Selberg zeta function attached to a space form $X_{\Gamma_0} = \Gamma_0 \setminus G_0/K_0$ of a rank one symmetric space G_0/K_0, where K_0 is a maximal compact subgroup of a non-compact semisimple Lie group G_0 and where $\Gamma_0 \subset G_0$ is a discrete subgroup, such that Γ_0/G_0 is compact. Here, we note that for $G_0 = SL(2,\mathbb{C})$, $K_0 = SU(2)$, $\mathbb{H}^3 \overset{(i)}{=} G_0/K_0$ is indeed a rank one symmetric space, but for $\Gamma_0 = \Gamma_\tau, \Gamma_0/G_0$, however, is not compact. A proof of the quotient structure (i) relies on the action of $SL(2,\mathbb{C})$ on $\mathbb{H}^3$ discussed in Section 1 and can be found in Appendix A2 of [9], for example.

The first step is to apply the Laplace transform formula:

$$\int_0^\infty e^{-\alpha t} t^{-1/2} e^{-\beta/4t}\, dt = \sqrt{\frac{\pi}{\alpha}} e^{-\sqrt{\alpha\beta}}, Re\alpha > 0, Re\beta \geq 0 \tag{42}$$

with the choices $\alpha = z(z-2)+s+1 = (z-1)^2+s, \beta = n^2\tau_2^2 > 0$ for $z \in \mathbb{C}, n \geq 1$. Thus, we assume that $Re(z-1)^2 + s \overset{(ii)}{>} 0$:

$$\int_0^\infty e^{-z(z-2)t} e^{-n^2\tau_2^2/4t} e^{-(s+1)t} t^{-1/2}\, dt = \frac{\sqrt{\pi}}{\sqrt{(z-1)^2+s}} e^{-n\tau_2\sqrt{(z-1)^2+s}}. \tag{43}$$

Again, commute the integration with the summation in Equation (14), assuming (ii). Equation (43) gives:

$$\int_0^\infty e^{-z(z-2)t} K^{(s)}(\tau,\bar{\tau};t)\, dt = \frac{(2-\delta_{s,0})\tau_2}{4\sqrt{(z-1)^2+s}} \sum_{n=1}^\infty \frac{(-1)^{2ns}(\cos(sn\tau_1))e^{-n\tau_2\sqrt{(z-1)^2+s}}}{\sin^2\frac{n\tau_1}{2} + \sinh^2\frac{n\tau_2}{2}}. \tag{44}$$

The second step, which is the main step (as in the argument to establish Theorem 1), is to relate the sum in Equation (44) to Z_{Γ_τ}, *i.e.*, specifically to the logarithmic derivative of Z_{Γ_τ}. For this, let:

$$w^\pm \overset{\text{def}}{=} 1 + \sqrt{(z-1)^2+s} \pm is\frac{\tau_1}{\tau_2}. \tag{45}$$

Then, for $\Gamma = \Gamma_{(\tau_2/2,\tau_1/2)}$, Equation (13) gives:

$$Z'_\Gamma(w^\pm \pm \frac{i\pi}{\tau_2})/Z_\Gamma(w^\pm \pm \frac{i\pi}{\tau_2}) = \frac{\tau_2}{4}\sum_{n=1}^\infty \frac{(-1)^n e^{-n\tau_2\left[\sqrt{(z-1)^2+s}\pm is\frac{\tau_1}{\tau_2}\right]}}{\sin^2\frac{n\tau_1}{2}+\sinh^2\frac{n\tau_2}{2}}. \tag{46}$$

In other words, we see that by Equation (46):

$$\begin{aligned} Z'_\Gamma(w^+ \pm \frac{i\pi}{\tau_2})/Z_\Gamma(w^+ \pm \frac{i\pi}{\tau_2}) + Z'_\Gamma(w^- \pm \frac{i\pi}{\tau_2})/Z_\Gamma(w^- \pm \frac{i\pi}{\tau_2}) &= \frac{\tau_2}{4}\sum_{n=1}^\infty \frac{(-1)^n e^{-n\tau_2\sqrt{(z-1)^2+s}} 2\cos ns\tau_1}{\sin^2\frac{n\tau_1}{2}+\sinh^2\frac{n\tau_2}{2}} \\ &= \sqrt{(z-1)^2+s}\int_0^\infty e^{-z(z-2)t}K^{(s)}(\tau,\bar{\tau};t)\, dt \end{aligned} \tag{47}$$

for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$. In the integral case $s = 0, 1, 2, 3, \cdots$, use Equation (11) (instead of Equation (13)) to conclude, similarly, that:

$$\begin{aligned} Z'_\Gamma(w^+)/Z_\Gamma(w^+) + Z'_\Gamma(w^-)/Z_\Gamma(w^-) &= \frac{\tau_2}{4}\sum_{n=1}^\infty \frac{e^{-n\tau_2\sqrt{(z-1)^2+s}} 2\cos ns\tau_1}{\sin^2\frac{n\tau_1}{2}+\sinh^2\frac{n\tau_2}{2}} \\ &= \frac{2\sqrt{(z-1)^2+s}}{2-\delta_{s,0}}\int_0^\infty e^{-z(z-2)t}K^{(s)}(\tau,\bar{\tau};t)\, dt, \end{aligned} \tag{48}$$

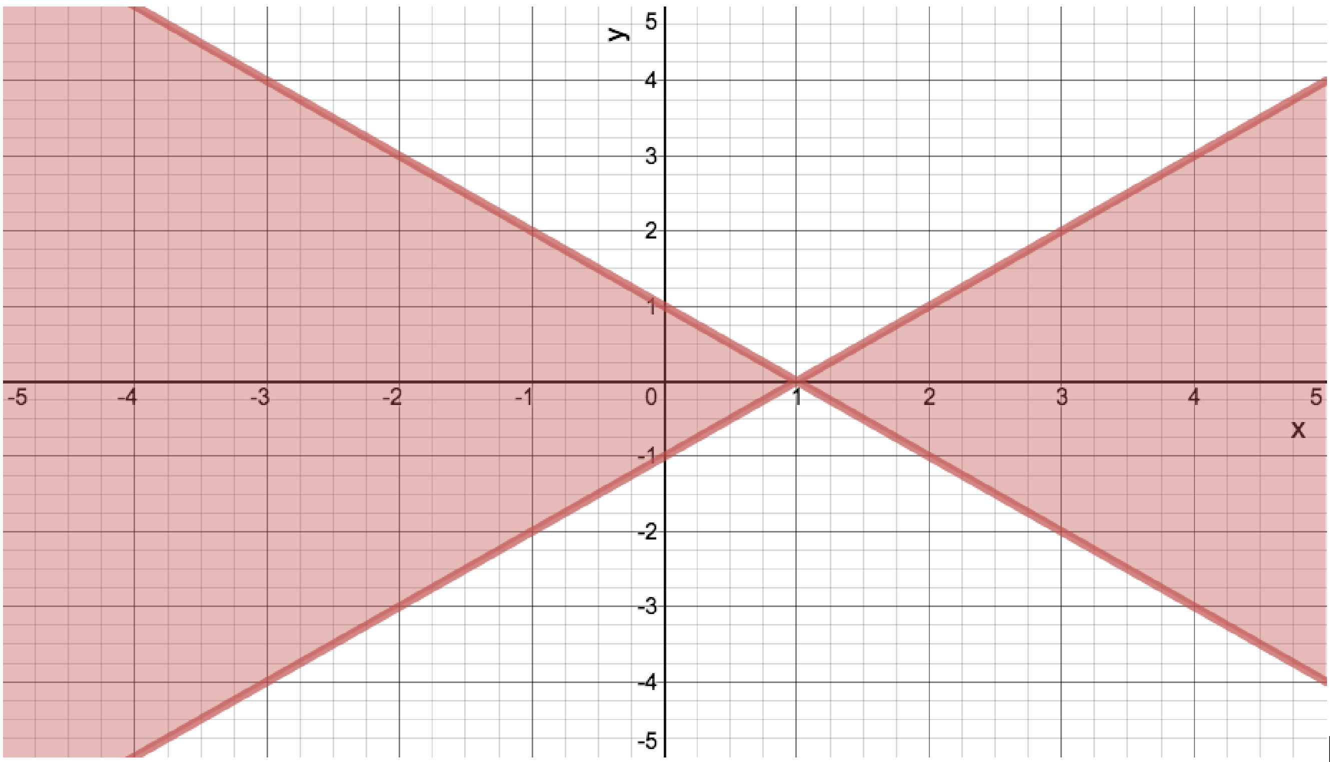
again by Equation (44). All of this gives:

Theorem 2. *For $z \in \mathbb{C}$ with $Re(z-1)^2 + s > 0$, for $\Gamma = \Gamma_{(\tau_2/2, \tau_1/2)}$, as in Theorem 1, and for the heat kernel trace $K^{(s)}(\tau, \bar{\tau}; t)$ in Equation (14), one has that:*

$$\int_0^\infty e^{-z(z-2)t} K^{(s)}(\tau,\bar{\tau};t)\,dt = \frac{1}{\sqrt{(z-1)^2+s}} \left[\frac{Z'_\Gamma(1+\sqrt{(z-1)^2+s}+is\frac{\tau_1}{\tau_2}+\frac{i\pi}{\tau_2})}{Z_\Gamma(1+\sqrt{(z-1)^2+s}+is\frac{\tau_1}{\tau_2}+\frac{i\pi}{\tau_2})} + \frac{Z'_\Gamma(1+\sqrt{(z-1)^2+s}-is\frac{\tau_1}{\tau_2}+\frac{i\pi}{\tau_2})}{Z_\Gamma(1+\sqrt{(z-1)^2+s}-is\frac{\tau_1}{\tau_2}+\frac{i\pi}{\tau_2})} \right] \tag{49}$$

for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$. The term $+\frac{i\pi}{\tau_2}$ could be replaced by $-\frac{i\pi}{\tau_2}$. For $s = 0, 1, 2, 3, \cdots$, the formula in (49) holds if we neglect the term $+\frac{i\pi}{\tau_2}$ and multiply the right-hand side there by $(2-\delta_{s,0})/2$, which is one for $s \neq 0$ and is $1/2$ for $s = 0$. Thus, for integral spin $s = 0, 1, 2, 3, \cdots$, Equation (49) is replaced by Formula (48), again with $w^\pm$ defined in Equation (45).

In Theorem 2, we are mainly interested in the case $s \neq 0 : s > 0$. In particular, the hypothesis $Re(z-1)^2 + s > 0$ holds if $Re(z-1)^2 \geq 0$. For $z = x + iy$, the latter condition is $(x-1)^2 \geq y$, the graph of this inequality being the set of points $(x, y) \in \mathbb{R}^2$ in the two shaded areas below (including the boundaries $y = \pm(x-1)$):



[H]

In case $s = 0$, we can use that for the branch of the square root with $-\pi \leq arg w < \pi$, and one has $\sqrt{w^2} = w$ for $-\frac{\pi}{2} \leq arg w < \frac{\pi}{2}$: $\sqrt{(z-1)^2} = z - 1$ for $-\frac{\pi}{2} \leq arg(z-1) < \frac{\pi}{2}$. We also assume $Re(z-1)^2 > 0$ (for $s = 0$). Both of these conditions hold for z in the right side of the shaded area above (with the boundaries $y = \pm(x-1)$ not included):

$$\int_0^\infty e^{-z(z-2)t} K^{(0)}(\tau,\bar{\tau};t)\,dt = \frac{1}{(z-1)} Z'_\Gamma(z)/Z_\Gamma(z). \tag{50}$$

Here, by Equation (14),

$$K^{(0)}(\tau,\bar{\tau};t) = \frac{\tau_2}{4\sqrt{\pi t}} \sum_{n=1}^{\infty} \frac{e^{-n^2\tau_2^2/4t}e^{-t}}{\sin^2\frac{n\tau_1}{2} + \sinh^2\frac{n\tau_2}{2}} \tag{51}$$

and as we have indicated earlier, with references given, Formula (50) was already known for $s = 0$. In [9], for example, $K^{(0)}(\tau,\bar{\tau};t)$ is denoted by $trK_t^{*\Gamma}$, and Formula (50) is proven for $Rez > 1$ (a domain of z larger than the right side of the shaded area above). Moreover, in that reference (again, where the notation a, b here and there corresponds to $\tau_2/2, \tau_1/2$, respectively), we show that the transform of $K^{(0)}(\tau,\bar{\tau};t)$ in Equation (50) coincides with the trace of the resolvent kernel G_0^Γ (or Green's function) of the Laplacian $\Delta_{(0)}$:

$$\int_0^\infty e^{-z(z-2)t}K^{(0)}(\tau,\bar{\tau};t)dt = \iiint_{\Gamma/\mathbb{H}^3} G_0^\Gamma(p,p;z)\,dv(p) \tag{52}$$

for $Rez > 1$, where $dv = dx\,dy\,dz/z^3$ is the hyperbolic volume element. This means that one can think of Formula (49) as a type of "spin version" of Patterson's formula: Proposition 3.3 in [1]; our normalization of Z_Γ differs from his by a factor of two. A version of Patterson's formula is also developed in [9] for the BTZ black hole with a conical singularity [4,6].

Conflicts of Interest

The author declares no conflict of interest.

Errata

The review article [10] was referenced in the Introduction. We take this opportunity to correct some minor errors therein.

1. The statement $X_a \overset{\text{def}}{=} \Gamma_a \setminus \mathbb{H}^2 \overset{\text{def}}{=} \{\begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z}\}$ in Equation (40) should read $X_a \overset{\text{def}}{=} \Gamma_a \setminus \mathbb{H}^2, \Gamma_a \overset{\text{def}}{=} \{\begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z}\}$.
2. In Equation (58), a parenthesis should be added: $I_{s-\frac{1}{2}}(2\pi|n|\frac{y_2}{a}$ should read $I_{s-\frac{1}{2}}(2\pi|n|\frac{y_2}{a})$.
3. In the third sentence following Equation (63), the phrase "extended to deformation (14)" should read "extended to the deformation (14)". Thus, in the article, "the" should be added.
4. In the sentence that concludes Section 4, the phrase "single zero $s = \frac{1}{2}$ of z" should read "single non-trivial zero $s = \frac{1}{2}$ of z", since clearly, the gamma function used in Definition (72) has trivial zeros at $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$.
5. The phrase "By the second Formula in (46)" that precede Equation (82) should read (simply) "By Formula (47)".
6. The phrase "Fourier transform of $\hat{f}_t(x)$" that follows Equation (86) should read "Fourier transform of $f_t(x)$".
7. In Equation (89), we should have parentheses: $K_{ir}\frac{2\pi ny}{a}$ should read $K_{ir}(\frac{2\pi ny}{a})$.
8. In Equation (50), we need another parenthesis: $G_a(x_1 + n_1a, y_1), (x_2 + n_2a, y); s)$ should read $G_a((x_1 + n_1a, y_1), (x_2 + n_2a, y); s)$.

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