Mathematics **2015**, *3*, 727-745; doi:10.3390/math3030727

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

From Classical to Discrete Gravity through Exponential Non-Standard Lagrangians in General Relativity

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Academic Editor: Palle E.T. Jorgensen

Received: 24 May 2015 / Accepted: 4 August 2015 / Published: 14 August 2015

Abstract: Recently, non-standard Lagrangians have gained a growing importance in theoretical physics and in the theory of non-linear differential equations. However, their formulations and implications in general relativity are still in their infancies despite some advances in contemporary cosmology. The main aim of this paper is to fill the gap. Though non-standard Lagrangians may be defined by a multitude form, in this paper, we considered the exponential type. One basic feature of exponential non-standard Lagrangians concerns the modified Euler-Lagrange equation obtained from the standard variational analysis. Accordingly, when applied to spacetime geometries, one unsurprisingly expects modified geodesic equations. However, when taking into account the time-like paths parameterization constraint, remarkably, it was observed that mutually discrete gravity and discrete spacetime emerge in the theory. Two different independent cases were obtained: A geometrical manifold with new spacetime coordinates augmented by a metric signature change and a geometrical manifold characterized by a discretized spacetime metric. Both cases give raise to Einstein's field equations yet the gravity is discretized and originated from "spacetime discreteness". A number of mathematical and physical implications of these results were discussed though this paper and perspectives are given accordingly.

Keywords: non-standard Lagrangians; general relativity; modified geodesic equations; discrete gravity; discrete spacetime; quantized cosmological constant

Mathematics Subject Classification (2000): 70S05; 83C05; 83C27

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1. Introduction

The theory of non-natural or non-standard Lagrangians (NSL) which are characterized by a deformed Lagrangian or deformed kinetic and potential energy terms have recently gained an increasing importance due to their wide applications in applied mathematics [1–22] and theoretical physics [23–29]. However, this topic has not been treated in all science branches due to the difficult apparent meaning of physical quantities, e.g. non-natural kinetic energy. However, in the progress of time, it was realized that the topic deserves careful and serious consideration. Although a number of applications of NSL are well established, their implications in geometrical field theories are still not done in details and further analysis is still required. This will be the main aim of the present work.

NSL may take, in applied mathematics and physical sciences, a large number of forms depending on the problem under study. Well-known examples of NSL are Dirac-Born-Infeld field Lagrangians and p-adic string for tachyon field in theoretical cosmology [30–33], NSL which arise in dissipative dynamical systems with variable coefficients [4,5], NSL which arise in fractional dynamics [7], power-law and exponential Lagrangians introduced in [10] and so on. In this paper, we choose the exponential NSL (ENSL) and we will discuss some of their implications in differential geometry and general relativity (GR). It is well-known that differential geometry concerns the tensor calculus on manifolds and is the mathematical framework of GR. In this paper, a number of properties on differential manifolds will be derived and their implications in GR will be addressed and discussed consequently. In fact, this work will try to answer the following questions: How we can use ENSL for generating solutions in general relativity? What are the effects of ENSL on the particle motion of a particle in a gravitational field? What supplementary advances are possible?

The paper is organized as follows: In Section 2, we setup some basic concepts of ENSL on differential manifolds and we derive the modified geodesic equation on the manifold and the corresponding field equations and we discuss some of their main consequences; in Section 3, we argue about the implications of the modified result on linearized general relativity theory, quantum gravity and Hawking's radiation theory; and finally conclusions are given in Section 4.

2. Exponential Non-Standard Lagrangians in Differential Geometry and General Relativity: Some Basic Consequences

The central concepts in this work are the notion of manifolds and tensors which are the basic mathematical tools in GR. We work in a manifold $\mathcal{M}$ where vectors and tensors are well-defined. For a more methodical presentation, we refer the reader to textbooks in differential geometry and in GR. Through this work, the Einstein summations are used and we work in units $\hbar = c = 1$. Besides, the 4-vector $(t, x, y, z) \to (x^0, x^1, x^2, x^3)$ convention is used, the Roman letters run from one to three and the Greek letters run from zero to three. A subscripted comma denotes the partial derivative with respect to the coordinate associated with the index that follows the comma.

We start by introducing the following definition:

Definition 2.1. [10]: Let $L(\dot{x}(t), x(t), t) \in C^2([a,b]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R})$ be an admissible smooth Lagrangian function with $(\dot{x}(t), x(t), t) \to L(\dot{x}(t), x(t), t) \triangleq L$ assumed to be a C^2 function with respect to all its arguments, *i.e.*, twice continuously differentiable with respect to all of its arguments. The action functional holding the ENSL is defined by $S=\int_a^b e^{\xi L}dt$ where ξ is a free parameter and $x:[a,b]\to\mathbb{R}^n$ are the generalized coordinate with $\dot{x}=dx/dt$ being the temporal derivative assumed to exist and continuous on $[a, b]$.

It should be stressed that in order to obtain a physical and realistic dimension of the ENSL, we must introduce a parameter Ω in the theory, *i.e.*, $S=\int_a^b \Omega e^{\xi L}dt$ with energy dimension $[E] = [ML^{-2}T^{-2}]$ so that we successfully recover the dimensional problem. M = mass, L = length and T = time. However in our approach we set this parameter equal to one for mathematical simplicity. Besides, the NSL in our approach is $L_{NSL}=e^{\xi L}$ and not L. Accordingly, we expect a modification of the Euler-Lagrange equation and any Lagrangian that depart the Euler-Lagrange equation from its standard form is referred to a NSL.

Theorem 2.1. [10] (Modified Euler-Lagrange equations): The function $x = x(t)$ that extremizes the functional $S=\int_a^b e^{\xi L}dt$ necessarily satisfies the following modified Euler-Lagrange equation (MELE) on $[a, b]$:

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\xi\frac{\partial L}{\partial \dot{x}}\left(\frac{\partial L}{\partial t}+\dot{x}\frac{\partial L}{\partial x}+\ddot{x}\frac{\partial L}{\partial \dot{x}}\right) \tag{1}$$

Amazingly, a 2nd-order derivative term appears in the MELE yet the Lagrangian holds only a derivative of the 1st-order. In order to rewrite Equation (1) on a manifold $\mathcal{M}$ assumed of dimension n, we consider a curve $\gamma:[a,b]\to\mathcal{M}$ parameterized with an assigned parameter $\lambda\in[a,b]$ such that the path through the spacetime is specified by writing spacetime coordinates as function of some parameter, *i.e.*, $x^{\mu}(\lambda)$ with a tangent vector $dx^{\mu}/d\lambda$. More explicitly, for an arbitrary differential function $\Phi(x^1,\ldots,x^n)$, the directional derivative is given by

$$\frac{d\Phi}{d\lambda}=\frac{\partial\Phi}{\partial x^i}\frac{dx^i}{d\lambda}, i=1,\ldots,n \tag{2}$$

and since Φ is arbitrary, the directional derivative operator is $d/d\lambda=(dx^i/d\lambda)(\partial/\partial x^i)$ where $(dx^i/d\lambda)$ are the components of the tangent vector. We consider now a path $x^{\mu}(\lambda)$ parameterized by λ where at present $L_{NSL}=e^{\xi L(\lambda,x,dx/d\lambda)}$. Therefore, Equation (1) keeps its general form with $t\to\lambda$, *i.e.*, $\dot{x}=dx/d\lambda$ and $\ddot{x}=d^2x/d\lambda^2$. Equation (1) may be as well extended to any number of phase-space coordinates, *i.e.*, $x\to x^{\mu}$. We let $g_{\mu\nu}$ be the covariant spacetime metric tensor where for the case of a flat Minkowski spacetime $g_{\mu\nu}\to\eta_{\mu\nu}=(-1, +1, +1, +1)$. It is noteworthy that in Euclidian space and Minkowski spacetime the metric is diagonal and constant [34]. We can now derive the modified geodesic equation which corresponds to the extremum of the length between two distinct points A and B. The Lagrangian is given by $L=g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}/2$ and the corresponding NSL is $e^{\xi g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}/2}$.

Lemma 2.1. (Modified Geodesic equations (MGE)): The function $x^{\mu} = x^{\mu}(\lambda)$ that extremizes the functional

$$S=\int_a^b e^{\frac{\xi g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{2}}d\lambda \tag{3}$$

necessarily satisfies the following MGE on $[a, b]$:

$$\frac{d^2x^\mu}{d\lambda^2}+\xi\left(\frac{1}{2}\frac{dg_{\rho\nu}}{d\lambda}\frac{dx^\rho}{d\lambda}+g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}\right)\frac{dx^\sigma}{d\lambda}\frac{dx^\mu}{d\lambda}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}=0 \tag{4}$$

$\Gamma^\nu_{\alpha\beta}$ are the Christoffel connections coefficients of the metric $g_{\alpha\beta}$ defined by [34]:

$$\Gamma^\nu_{\alpha\beta}=\frac{1}{2}g^{\nu\gamma}\left(g_{\alpha\gamma,\beta}+g_{\gamma\beta,\alpha}-g_{\alpha\beta,\gamma}\right) \tag{5}$$

$g_{\alpha\beta,\gamma}=\partial g_{\alpha\beta}/\partial x^\gamma$ and so on.

The proof is classical and the calculation is straightforward: In fact, from $L=\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, we get $\partial L/\partial\dot{x}^\mu=g_{\mu\nu}\dot{x}^\nu$ and hence from the modified geodesic equation we find: $g_{\mu\nu}\ddot{x}^\nu+g_{\mu\nu,\rho}\dot{x}^\rho\dot{x}^\nu-\frac{1}{2}g_{\rho\nu,\mu}\dot{x}^\rho\dot{x}^\nu=0$ Using the chain rule, we calculate $dg_{\mu\nu}/d\tau=g_{\mu\nu,\rho}(dx^\rho/d\tau)$, then by averaging over the interchange of ρ, ν and using the facts that ν $g^{\sigma\mu}g_{\mu\nu}=\delta^\sigma_\nu$, we obtain Equation (4). In Euclidian space and Minkowski spacetime, the derivative of the metric is zero and hence the Christoffel symbol vanishes. Therefore, Equation (4) is reduced to $\ddot{x}^\mu(1+\xi g_{\mu\sigma}\dot{x}^\sigma\dot{x}^\mu)=0$ which gives two independent solutions: $\ddot{x}^\mu=0$ which corresponds for a straight line or $\xi g_{\mu\sigma}\dot{x}^\sigma\dot{x}^\mu=-1$ which gives $\xi\eta_{\sigma\mu}\dot{x}^\sigma\dot{x}^\mu=-1$ which is the modified normalized four-velocity vector.

Remark 2.1. For the case of a constant tensor metric, e.g., $g_{\mu\sigma}=k$, its derivative vanishes and so do all the Christoffel symbols. In that case Equation (4) is reduced to:

$$\frac{d^2x^\mu}{d\lambda^2}\left(1+\xi k\frac{dx^\sigma}{d\lambda}\frac{dx^\mu}{d\lambda}\right)=0$$

Accordingly, we have two possible solutions: $d^2x^\mu/d\lambda^2=0$ or $(dx^\sigma/d\lambda)(dx^\mu/d\lambda)=-1/\xi k$. The first solution gives a straight line and, therefore, the geodesic is the straight line in this case whereas the second solution gives $x^\mu=\pm\sqrt{-1/\xi k}\lambda+C$, C is an integration constant and the geodesic is again a straight line. It is notable that real solutions correspond for $\xi<0$ and $k>0$ or $\xi>0$ and $k<0$. In the static case, any free test particle follows a geodesic line in agreement with the Newton law, yet two different velocities are possible in our agreement.

Remark 2.2. We can always rewrite Equation (4) as:

$$\ddot{x}^m+\frac{\xi}{2}\frac{dg_{im}}{d\lambda}\dot{x}^\rho\dot{x}^j\dot{x}^m+\xi g_{mj}\ddot{x}^m\dot{x}^j\dot{x}^m+\frac{1}{2}g^{nm}\left(g_{in,j}+g_{nj,i}-g_{ij,n}\right)\dot{x}^i\dot{x}^j=0$$

It is notable that, in the standard approach, the geodesics equations for a given manifold are known as they can be calculated using the standard geodesic equation, e.g., on a given plane, the shortest-length path between two points is the line segment which connects the points and on a given sphere the shortest-length path between two points is the shortest great-circle arc which connects the points. Besides, some mathematical theorems state that "*there is only one unique set of geodesics for any given manifold*". Therefore, it is natural to ask—"*What kinds of geodesics are predicted by the MGE and what these geodesics represent when compared to the standard geodesics*?" In order to check rapidly, let us consider the unit sphere centered at the origin in three dimensions, a manifold of two dimensions and let A be a point of the sphere such that a parameterization is given by

$A(x_1,x_2)=(\cos x_1\sin x_2,\sin x_1\sin x_2,\cos x_2)$ such that $x_1\in[0,2\pi)$ and $x_2\in[0,\pi]$ [35]. The corresponding metric tensor and the Christoffel symbols of the 2nd kind are respectively:

$$g_{ij}=\begin{pmatrix}\sin^2 x_2 & 0\\ 0 & 1\end{pmatrix}$$

$$\Gamma^1_{ij}=\begin{pmatrix}0 & \cot x_2\\ \cot x_2 & 0\end{pmatrix}$$

$$\Gamma^1_{ij}=\begin{pmatrix}-\sin x_2\cos x_2 & 0\\ 0 & 0\end{pmatrix}$$

and accordingly the modified geodesic equations are:

$$\frac{d^2x_1}{d\lambda^2}+\xi\sin^2 x_2\frac{d^2x_1}{d\lambda^2}\left(\frac{dx_1}{d\lambda}\right)^2+\xi\sin x_2\frac{dx_2}{d\lambda}\frac{dx_1}{d\lambda}\left(\frac{dx_1}{d\lambda}+\frac{dx_2}{d\lambda}\right)^2+2\cot x_2\frac{dx_1}{d\lambda}\frac{dx_2}{d\lambda}=0 \tag{6}$$

and

$$\frac{d^2x_2}{d\lambda^2}+\xi\left(\frac{dx_2}{d\lambda}\right)^2\frac{d^2x_2}{d\lambda^2}-\sin x_2\cos x_2\left(\frac{dx_1}{d\lambda}\right)^2=0 \tag{7}$$

Equation (6) is satisfied for $x_1 = k$ where k constant and hence Equation (7) is reduced to the 2nd—order nonlinear differential equation:

$$\frac{d^2x_2}{d\lambda^2}\left(1+\xi\left(\frac{dx_2}{d\lambda}\right)^2\right)=0 \tag{8}$$

As stated in Remark 2.1, two possible solutions exist: $d^2x^\mu/d\lambda^2 = 0$ or $dx_2/d\lambda=\pm\sqrt{-1/\xi}$. Both solutions yield a straight line geodesic in the real plane mainly for $\xi < 0$. The first solution gives $x_2=\lambda\in[-\pi,\pi]$ whereas the second physical solution gives $x_2=\sqrt{-1/\xi}\lambda$ assuming x_2 ($\lambda = 0$) = 0. The solution $x_1 = k$ defines a plane through the origin of space intersecting the unit sphere in a great circle. The 1st solution $x_2 = \lambda$ corresponds for a motion around the great circle with unit speed whereas the 2nd solution $x_2=\sqrt{-1/\xi}\lambda$ corresponds for a motion around the great circle with a velocity $\sqrt{-1/\xi}$. Equation (7) is satisfied for $x_2 = \pi/2$ and therefore Equation (6) is reduced to the 2nd—order nonlinear differential equation:

$$\frac{d^2x_1}{d\lambda^2}\left(1+\xi\frac{d^2x_1}{d\lambda^2}\right)=0 \tag{9}$$

Two possible solutions exist: $d^2x_1/d\lambda^2 = 0$ or $dx_1/d\lambda=\pm\sqrt{-1/\xi}$ ($\xi < 0$). The first solution gives $x_1=\lambda\in[0,2\pi]$ whereas the second physical solution gives $x_1=\sqrt{-1/\xi}\lambda$ assuming x_1 ($\lambda = 0$) = 0. The solution $x_2 = \pi/2$ defines a plane through the origin of space intersecting the unit sphere in a great circle. The 1st solution $x_1 = \lambda$ corresponds for a motion around the great circle with unit speed whereas the 2nd solution $x_1=\sqrt{-1/\xi}\lambda$ corresponds for a motion around the great circle with a velocity $\sqrt{-1/\xi}$. This may suggest that the geodesic dynamics in the ENSL approach are not characterized just by a universal speed, but possibly by other universal velocities, which are proportional to $\sqrt{-1/\xi}$. A number of studies were proposed in literature to resolve various celestial anomalies that consider some anomalous acceleration and anomalous velocities in order to modify the law of inertia [36,37]. It should be stressed that modified geodesics equations in general relativity were found in literature through different aspects and frameworks, mainly within the framework of quantum gravity effects which lead to violation of the equivalence principle [38]. One therefore expects that the modified geodesics equations obtained in the ENSL approach will be motivating to use in quantum gravity effects since these later modify the geodesic motion of a particle. Besides, in Newtonian physics any violation in the equivalence principle is manifested by the modification of the geodesic equation [39]. In addition, modified geodesic equations are discussed with the Modified Newtonian Dynamics (MOND) [40–42] as a possible explanation of the galactic rotation curves. It was also discussed at the relativistic level within the linearized gravity limits [43], in dark energy length scale [44], in gravitons fractional physics [45] and fractional actionlike dark energy cosmology [46,47].

Remark 2.3. A path moving from A to B through a spacetime is specified by giving the four spacetime coordinates as a function of some parameter $x_\mu(\lambda)$. For timelike paths, we can use the proper time τ which we can calculate along an arbitrary timelike path by $\tau = \int_a^b e^{\xi\mathcal{F}} d\lambda$. Here $\lambda(A) = a$, $\lambda(B) = b$ and $\mathcal{F} \triangleq (-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda))^{1/2}$. Therefore we have $d\tau = e^{\xi(-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda))^{1/2}} d\lambda = e^{\xi\mathcal{F}} d\lambda$ [34].

In such a case, the modified Euler-Lagrange equation is written as:

$$\frac{\partial\mathcal{F}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\right) = \xi\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\left(\dot{x}^\mu\frac{\partial\mathcal{F}}{\partial x^\mu} + \ddot{x}^\mu\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\right) \equiv \xi\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\frac{d\mathcal{F}}{d\lambda} \tag{10}$$

However, one can always choose the scalar parameter of motion $\lambda = \tau$ such that $e^{\xi\mathcal{F}} = 1$, *i.e.*, $\mathcal{F} = 2i\pi n/\xi, \xi \neq 0, n \in \mathbb{Z}$ and then we find:

$$\frac{\xi^2}{4\pi^2 n^2} g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 \tag{11}$$

One can select $\xi = \pm\pi i$ which gives $g_{\mu\nu}dx^\mu dx^\nu = -n^2 d\tau^2$ and therefore the interval between two closed points is $ds^2 \triangleq g_{\mu\nu}dx^\mu dx^\nu = -n^2 d\tau^2$, a result which can be interpreted in two different ways as stated in the following lemma:

Lemma 2.2. In the set $\mathbb{Z}$ of integers, in order to preserve the regular form of the first integral of the standard geodesic equation, we can either introduce new spacetime coordinates $X^\mu \triangleq x^\mu/n$ and changing metric signature $g_{\mu\nu} \to \mathcal{G}_{\mu\nu} \triangleq -g_{\mu\nu}$ so that $ds^2 = \mathcal{G}_{\mu\nu}dX^\mu dX^\nu$ or define the discretized spacetime metric $\mathfrak{g}_{\mu\nu} \triangleq -g_{\mu\nu}/n^2$ with negative sign which yields a new discretized interval between two closed points $d\mathfrak{s}^2 \triangleq \mathfrak{g}_{\mu\nu}dx^\mu dx^\nu$.

Lemma 2.3. For $\xi = \pm 2\pi i$, the action functional $S = \int_a^b e^{\xi L} dt$ is complexified and the modified geodesic Equation (4) or (6) is reduced to its standard form.

Lemma 2.4. For $\xi = \pm 2\pi i$, the usual spacetime $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ changes its signature and all its spacetime metric components are discretized and take one of the follows forms: $d\mathfrak{s}^2 = \mathfrak{g}_{\mu\nu}dx^\mu dx^\nu$ or $d\mathcal{S}^2 = \mathcal{G}_{\mu\nu}dX^\mu dX^\nu$.

In reality, complexification in general relativity was discussed in literature through different contexts [48–52]. Spacetime complexification has many advantages in general relativity (see [50] and references therein). Besides, the signature change in general relativity is discussed in literature and has many interesting cosmological and astrophysical consequences (see [53] and references therein).

Remark 2.4. If we choose $\xi = \pm 2\pi$ then we find $ds^2 \triangleq g_{\mu\nu} dx^\mu dx^\nu = n^2 d\tau^2$ and accordingly in the set $\mathbb{Z}$, in order to preserve the regular form of the first integral of the standard geodesic equation, we can either introduce new spacetime coordinates $X^\mu \triangleq x^\mu / n$ without changing metric signature which gives $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ or define the discretized spacetime metric $\mathfrak{g}_{\mu\nu} \triangleq g_{\mu\nu} / n^2$ with positive sign which yields a new discretized interval between two closed points $d\mathfrak{s}^2 \triangleq \mathfrak{g}_{\mu\nu} dx^\mu dx^\nu$.

We discuss accordingly both cases:

2.1. New Spacetime Coordinates with a Metric Signature Change (*NSTC + MC*)

In fact, working in the new coordinates system $X^\mu \triangleq x^\mu / n$ result on the modification of many differential operators like the gradient ∇, the Laplacian Δ and the d'Alembertian $\square$. More explicitly, we have:

$$\nabla = \frac{\partial}{\partial x^\alpha} \to \blacktriangledown = \frac{1}{n}\frac{\partial}{\partial X^\alpha} \equiv \frac{1}{n}\nabla \tag{12}$$

$$\Delta = \nabla^2 = -\delta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} \to \blacktriangle = \blacktriangledown^2 = -\frac{1}{n^2}\delta^{\alpha\beta}\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial X^\beta} = \frac{1}{n^2}\Delta \tag{13}$$

$$\square = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} \to \blacksquare = -\frac{1}{n^2}\eta^{\alpha\beta}\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial X^\beta} = -\frac{1}{n^2}\square \tag{14}$$

$\delta^{\alpha\beta}$ is the Kronecker symbol and $\blacktriangledown$, $\blacktriangle$ and $\blacksquare$ are respectively the gradient, Laplacian and d'Alembertian operators in the new complexified coordinates system. The components $\vec{x} \triangleq (x^0, x^i) \equiv (\tau, x, y, z), i = 1,2,3$ are now complexified and the complexified coordinates system is $\vec{X} \triangleq (X^0, X^i) \equiv (\tau, X, Y, Z) = \vec{x}/n$. According to Einstein postulates, the geometry should manifest itself as spacetime curvature through Einstein's field equations (EFE): $R_{\alpha\beta} - g^{\alpha\beta} R/2 = kT_{\alpha\beta}$ and in an empty spacetime the field equations are $R_{\alpha\beta} = 0$. Here $R_{\alpha\beta}$ is the Ricci tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $T_{\alpha\beta}$ is the energy-momentum tensor and k is a constant to be determined [54]. In the new system of coordinates, Christoffel symbols Equation (5) are $\mathbf{\Gamma}^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} / n$. The Riemann tensor is written directly in terms of the new spacetime metric:

$$\mathbf{R}^\alpha_{\beta\gamma\delta} \triangleq \mathbf{\Gamma}^\alpha_{\beta\delta,\gamma} - \mathbf{\Gamma}^\alpha_{\beta\gamma,\delta} + \mathbf{\Gamma}^\nu_{\alpha\gamma}\mathbf{\Gamma}^\mu_{\beta\delta} - \mathbf{\Gamma}^\nu_{\alpha\delta}\mathbf{\Gamma}^\mu_{\beta\gamma} = \frac{1}{n^2}\left(\Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\beta\delta} - \Gamma^\nu_{\alpha\delta}\Gamma^\mu_{\beta\gamma}\right) = \frac{1}{n^2}R^\alpha_{\beta\gamma\delta} \tag{15}$$

and therefore

$$\mathbf{R}_{\alpha\beta\gamma\delta} \triangleq \frac{1}{2}\left(\mathcal{G}_{\beta\gamma,\alpha\delta} + \mathcal{G}_{\alpha\delta,\beta\gamma} - \mathcal{G}_{\beta\delta,\alpha\gamma} - \mathcal{G}_{\alpha\gamma,\beta\delta}\right) + \mathcal{G}_{\mu\nu}\mathbf{\Gamma}^\nu_{\alpha\gamma}\mathbf{\Gamma}^\mu_{\beta\delta} - \mathcal{G}_{\mu\nu}\mathbf{\Gamma}^\nu_{\alpha\delta}\mathbf{\Gamma}^\mu_{\beta\gamma} \tag{16}$$

$$= -\frac{1}{n^2}\left(\frac{1}{2}\left(g_{\beta\gamma,\alpha\delta} + g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta}\right) + g_{\mu\nu}\mathbf{\Gamma}^\nu_{\alpha\gamma}\mathbf{\Gamma}^\mu_{\beta\delta} - g_{\mu\nu}\mathbf{\Gamma}^\nu_{\alpha\delta}\mathbf{\Gamma}^\mu_{\beta\gamma}\right) = -\frac{1}{n^2}R_{\alpha\beta\gamma\delta} \tag{17}$$

It is then obvious that in the new system $R^{\alpha}_{\beta\gamma\delta} = n^2\mathbf{R}^{\alpha}_{\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta} = -n^2\mathbf{R}_{\alpha\beta\gamma\delta}$. The Ricci tensor is obtained from the Riemann tensor by simply contracting over two of the indices which gives $R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = n^2 g^{\alpha\beta}\mathbf{R}^{\gamma}_{\alpha\gamma\beta} = -n^2\mathcal{G}^{\alpha\beta}\mathbf{R}^{\gamma}_{\alpha\gamma\beta} = -n^2\mathbf{R}$. It is easy to check that the EFE is now $\mathbf{R}_{\alpha\beta} - \mathcal{G}_{\alpha\beta}\mathbf{R}/2 = k\mathbb{T}_{\alpha\beta}/n^2$, $\mathbb{T}_{\alpha\beta}$ being the stress energy-momentum tensor in the new system. To find *k*, we follow the standard arguments and we consider a weak gravitational flat and static spacetime field, e.g., the solar system and we perturb the metric $\mathcal{G}_{\mu\nu}$ around the new Minkowski metric $\mathfrak{H}_{\mu\nu} = -\eta_{\mu\nu}$ as $\mathcal{G}_{\mu\nu} = \mathfrak{H}_{\mu\nu} + \mathcal{E}_{\mu\nu}, |\mathcal{E}| << 1$. We suppose now that we have a particle moving freely with a slow velocity with respect to the celerity of light and moving along a geodesic. Since then, the velocity product is too small, the only terms contributing at this level are those with subscripts 00 with $\Gamma^0_{00} = 0$ since it contains only time derivatives which are assumed to vanish. The time component is therefore not interesting and we are left with spatial components, which are $d^2\vec{X}/d\tau^2 + \Gamma^i_{00} = 0$. As we have slow motion, we can replace τ by $\mathsf{t} = t/n$ and then $d^2x^i/dt^2 + \Gamma^i_{00} = 0$. We evaluate Γ^i_{00} and we find $\Gamma^i_{00} = \nabla\mathcal{E}_{00}/2$ and therefore the geodesic equation become $d^2x^i/dt^2 + \nabla\mathcal{E}_{00}/2 = 0$ and finally we get $d^2\vec{X}/d\tau^2 + (1/n)(\partial\phi/\partial\vec{X}) = 0$, $\Phi = \varepsilon_{00}/2$ being the potential field. We can introduce the new discretized potential field $\phi = \phi/n$ in order to recover the standard form of the Poisson equation $d^2\vec{X}/d\tau^2 + \partial\phi/\partial\vec{X} = 0$. In the weak field approximation, and in particular for the 00-component, we find $\mathrm{R}_{00} = -\blacktriangle\phi$ which gives $\mathbf{R}_{00} = -(1/n^2)\partial^2\phi/\partial X^2$. For the case of a dust with a density ρ and pressure-free (Newtonian case), $T_{\alpha\beta} = \rho u^{\alpha}u^{\beta}$ where $u = (1, \mathsf{u})$ is a unit timelike vector tangent to the worldlines of the dust particles. In the new system $X^{\mu} \triangleq x^{\mu}/n$, we have $U^{\mu} \triangleq u^{\mu}/n$ and then the discrete stress energy-momentum tensor for the case of a dust is now $\mathbb{T}_{\alpha\beta} = \rho n U_{\alpha}U_{\beta}$. After substituting in the field equations $\mathbf{R}_{\alpha\beta} - \mathcal{G}_{\alpha\beta}\mathbf{R}/2 = k\mathbb{T}_{\alpha\beta}/n^2$ we obtain in the Newtonian limit $\mathbf{R}_{\alpha\beta} = (k\rho U^{\alpha}U^{\beta} - k\rho\mathcal{G}_{\alpha\beta}/2)/n^2$. Accordingly, $\mathbf{R}_{00} = k\rho/2n^2$ and using $\mathbf{R}_{00} = -(1/n^2)\partial^2\phi/\partial X^2$ we find $\partial^2\phi/\partial X^2 = -k\rho/2n^2 = 4\pi G\rho n^2$ which gives $k = -8\pi G n^4$ since the Poisson equation in our framework is $\partial^2\phi/\partial X^2 = 4\pi G\rho n^2$ and the EFE are now written as $\mathbf{R}_{\alpha\beta} - \mathcal{G}_{\alpha\beta}\mathbf{R}/2 = -8\pi G n^2\mathbb{T}_{\alpha\beta}$. Nevertheless, we can introduce a discrete gravitational field $\mathsf{G} = Gn^2$ so that both the Poisson equation and the EFE are recovered normally. For *n* = 1 all the previous arguments are reduced to their standard forms. These results hold for ξ = ±2π. However, for ξ = ±2π, then the EFE is $\mathbf{R}_{\alpha\beta} - g_{\alpha\beta}\mathbf{R}/2 = -8\pi G n^2\mathbb{T}_{\alpha\beta}$ where $R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = n^2 g^{\alpha\beta}\mathbf{R}^{\gamma}_{\alpha\gamma\beta} = n^2\mathbf{R}$ and $\mathsf{G} = Gn^2$.

2.2. A Discretized Spacetime Metric (DSTM)

If we choose the discrete metric $\mathfrak{g}_{\mu\nu} \triangleq -g_{\mu\nu}/n^2$ then the gradient ∇ and the Laplacian ∆ operators keep their standard forms whereas the d'Alembertian operator is $\square \to \blacksquare$. The perturbation of the weak gravitational flat and static spacetime field is now $\mathfrak{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h| << 1$ where $\eta_{\mu\nu} \triangleq -\eta_{\mu\nu}/n^2$ and $h_{\mu\nu} \triangleq -h_{\mu\nu}/n^2$. It should be pointed out that in that case that we require that $\mathfrak{g}_{\delta\beta}\mathfrak{g}^{\alpha\beta} = \delta^{\alpha}{}_{\delta}$, Christoffel symbols are not affected, since raising and then lowering the same index are inverse operations. The Riemann tensor written directly in terms of the spacetime metric:

$$R_{\alpha\beta\gamma\delta}=\frac{1}{2}\left(g_{\beta\gamma,\alpha\delta}+g_{\alpha\delta,\beta\gamma}-g_{\beta\delta,\alpha\gamma}-g_{\alpha\gamma,\beta\delta}\right)+g_{\mu\nu}\Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\beta\delta}-g_{\mu\nu}\Gamma^{\nu}_{\alpha\delta}\Gamma^{\mu}_{\beta\gamma} \tag{18}$$

is now written in terms of $\mathfrak{g}_{\mu\nu}\triangleq -g_{\mu\nu}/n^2$ as:

$$\mathcal{R}_{\alpha\beta\gamma\delta}=-n^2\left(\frac{1}{2}\left(\mathfrak{g}_{\beta\gamma,\alpha\delta}+\mathfrak{g}_{\alpha\delta,\beta\gamma}-\mathfrak{g}_{\beta\delta,\alpha\gamma}-\mathfrak{g}_{\alpha\gamma,\beta\delta}\right)+\mathfrak{g}_{\mu\nu}\Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\beta\delta}-\mathfrak{g}_{\mu\nu}\Gamma^{\nu}_{\alpha\delta}\Gamma^{\mu}_{\beta\gamma}\right) \tag{19}$$

and therefore $\mathcal{R}^{\alpha}_{\beta\gamma\delta}=-n^2\mathcal{R}_{\alpha\beta\gamma\delta}$. The Ricci tensor is accordingly $\mathcal{R}_{\alpha\beta}=\mathcal{R}^{\gamma}_{\alpha\gamma\beta}=-n^2\mathcal{R}_{\gamma\alpha\gamma\beta}$ and the resulting Ricci scalar is $\mathcal{R}=\mathfrak{g}^{\alpha\beta}\mathcal{R}_{\alpha\beta}$. From Bianchi identities $\mathcal{R}_{\alpha\beta\gamma\delta;\nu}+\mathcal{R}_{\beta\alpha\nu\gamma;\delta}+\mathcal{R}_{\alpha\beta\delta\nu;\gamma}=0$ we find $\mathcal{R}^{\alpha\beta}_{;\alpha}=\mathfrak{g}^{\alpha\beta}\mathcal{R}_{;\alpha}/2$ and hence the Einstein tensor is $\mathcal{R}_{\alpha\beta}-\mathfrak{g}_{\alpha\beta}\mathcal{R}/2$. The new EFE is $\mathcal{R}_{\alpha\beta}-\mathfrak{g}_{\alpha\beta}\mathcal{R}/2=k\mathcal{T}_{\alpha\beta}$ where $\mathcal{T}_{\alpha\beta}=\rho u^{\alpha}u^{\beta}$ in the vacuum. Then, in the Newtonian limit we have $\mathcal{R}_{\alpha\beta}=k\rho u^{\alpha}u^{\beta}+k\rho\mathfrak{g}_{\alpha\beta}/2$ which gives $\mathcal{R}_{00}=k\rho/2n^2$ and using $\mathcal{R}_{00}=-\partial^2\phi/\partial x^2$ we find $\partial^2\phi/\partial x^2=-k\rho/2n^2=4\pi G\rho$ which gives $k=-8\pi Gn^2$ since the Poisson equation is $\partial^2\phi/\partial x^2=4\pi G\rho$. Accordingly, the Einstein field equations are $\mathcal{R}_{\alpha\beta}-\mathfrak{g}_{\alpha\beta}\mathcal{R}/2=-8\pi Gn^2\mathcal{T}_{\alpha\beta}$. One can introduce a discrete gravitational field $\mathrm{G}=Gn^2$ so that the standard form of the EFE is recovered, *i.e.*, $\mathcal{R}_{\alpha\beta}-\mathfrak{g}_{\alpha\beta}\mathcal{R}/2=-8\pi\mathrm{G}\mathcal{T}_{\alpha\beta}$. These results hold once more for $\xi=\pm 2\pi i$. However, for $\xi=\pm 2\pi$, then the EFE is $\mathrm{R}_{\alpha\beta}-g_{\alpha\beta}\mathrm{R}/2=-8\pi Gn^2\mathbb{T}_{\alpha\beta}$ and again we have $R=g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta}=n^2g^{\alpha\beta}\mathcal{R}^{\gamma}_{\alpha\gamma\beta}=n^2\mathcal{R}$ and $\mathrm{G}=Gn^2$.

Remark 2.5.: General covariance which is defined as "the ability to change system coordinates while not affecting the validity of the field equations" is an indispensable property in Einstein's general relativity as well in any modern field theory. It is notable that in Einstein's general relativity, the spacetime is considered semi-Riemannian, the gravity is represented by the spacetime curvature and the metric tensor governed the EFE [55]. Besides, general relativity is characterized by its invariance under active diffeomorphisms and background independence [56]. However, that theory is non-discrete, whereas in our approach, either we have new spacetime coordinates with a metric signature change or the spacetime metric is discretized. The question then arises is the covariance principle as well as the underlying diffeomorphism structure are affected in the ENSL approach. In fact, if the condition $e^{\xi\mathcal{F}}=1$ does not hold for $\lambda=\tau$ then the problem seems to be complicated due to the presence of the RHS term in Equation (10). One expects accordingly that the covariance principle is violated due to the presence of time-dependent terms in the geodesic equation. However, there are many arguments which states that the violation of the general covariance may serve for the existence of the dark matter of the gravitational origin [57,58]. Breaking diffeomorphism invariance was also discussed in emergent gravity theory [59] and in quantum gravity [60]. Whatever is the case, we argue that the constraint $e^{\xi\mathcal{F}}=1$, which yields to discreteness, may be relevant for more general theories in which Einstein's general relativity is an emergent theory from a more fundamental theory that requires a primary version of diffeomorphism invariance and general covariance. This is an open problem that deserves a careful analysis. It should be stressed at the end that the in general changing coordinates may lead to new interesting results in relativistic astrophysics and cosmology. Nevertheless, in our approach, we have a passage from continuous to discrete coordinates. Though the simplicity of such a transition, it may be helpful to explore in a future work if such a passage may lead to a more fundamental symmetry principle and to a generalized covariance principle in nature which in their turns lead to a generalization of relativity theory. There are some arguments which state that a number of present cosmological problem are raised since the Einstein's theory of gravitation is not a perfect theory [61]. One hopes that our discrete approach will be able to solve some of these tricky contemporary problems. There still seems to be interest in discussing discrete static and Reissner-Nordstrom rotating black holes based on our discrete approach in a future work. One expects that important consequences in particular when dealing with black-holes quantum gravity fluctuations, their discrete energy spectrum [62] and with quantum micro-black holes [63] which are predicted in extra-spatial dimensions theories. At the end, let us add that in the NSTC + MC case, the scalar curvature is given by $R = -n^2\mathbf{R}$. Using this result in de-Sitter space with the standard metric $ds^2 = -dx_0^2 + \sum_{i=1}^{N} dx_i^2$ defined as a submanifold of a Minkowski space of one higher-dimension [64] and which is characterized by the scalar curvature $R = 2N\Lambda/(N-2)$, Λ being the cosmological constant, we find that for $\xi = \pm 2\pi i$, $\mathbf{R} = -2N\Lambda/n^2(N-2)$. One then can define a quantized cosmological constant $\boldsymbol{\Lambda} = \Lambda/n^2$ with $\mathbf{R} < 0$ or $\boldsymbol{\Lambda} = -\Lambda/n^2$ with $\mathbf{R} > 0$ which in our opinion can have extreme consequences on the physics of the early universe [65] and in quantum geometry [66]. However, for $\xi = \pm 2\pi$, the quantized or discretized cosmological constant is defined by $\boldsymbol{\Lambda} = \Lambda/n^2$ yet $\mathbf{R} > 0$.

As a simple illustration, we consider a particle moving under the effect of the gravitational field merely. In the presence of a spherically symmetric massive body, the flat spacetime metric is modified and according to the Birkhoff's theorem, the Schwarzschild solution is the unique spherically symmetric vacuum solution to general relativity [54]. For the Case 2.1., using Einstein's field equations $\mathbf{R}_{\alpha\beta} = 0$ in the vacuum and merely in the new system $\vec{X} \triangleq (X^0, X^i) \equiv (\tau, X, Y, Z) = \vec{x}/n$, it can be seen that the solution with $\mathsf{G} = Gn^2$, $r = n\mathsf{r}$, $t = n\mathsf{t}$ is:

$$d\mathsf{s}^2 = \left(1 - \frac{2M\mathsf{G}}{\mathsf{r}n^3}\right) n^2 d\mathsf{t}^2 - \left(1 - \frac{2M\mathsf{G}}{\mathsf{r}n^3}\right)^{-1} n^2 d\mathsf{r}^2 - n^4 \mathsf{r}^2 d\Omega^2 \tag{20}$$

$$\equiv \left(1 - \frac{2M\mathsf{G}}{rn^2}\right) dt^2 - \left(1 - \frac{2M\mathsf{G}}{rn^2}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{21}$$

where $(\mathsf{t}, \mathsf{r}, \Omega)$ are the Schwarzschild coordinates in the new system. Here $d\Omega$ is the line element in the unit sphere. However, in the case of a discretized metric $\mathbf{g}_{\mu\nu} \triangleq -g_{\mu\nu}/n^2$, it is easy to check that the Schwarzschild metric takes the form:

$$ds^2 = \left(1 - \frac{2M\mathsf{G}}{rn^2}\right) n^2 dt^2 - \left(1 - \frac{2M\mathsf{G}}{rn^2}\right)^{-1} n^2 dr^2 - n^4 r^2 d\Omega^2 \tag{22}$$

It is easy now to revisit the gravitational time dilatation and the gravitational redshift for both cases. First, let us consider two observers placed in a stationary gravitational field approximated by the Newton potential, which are supposed static with respect to the center of mass [67]. Suppose a light ray passes by the two observers. For the Case 2.1, it is well-known that the time lapses between two subsequent light crests $\delta\mathsf{t}_1$ and $\delta\mathsf{t}_2$ measured by both observers are related by $\delta\mathsf{t}_1\sqrt{g_{00}(\mathsf{r}_1)} = \delta\mathsf{t}_2\sqrt{g_{00}(\mathsf{r}_2)}$ which in a weak gravitational field reduces to:

$$\frac{\delta \mathrm{t}_1}{\delta \mathrm{t}_2} \equiv \frac{\delta t_1}{\delta t_2} \approx 1 + \frac{M\mathsf{G}}{n^3}\left(\frac{1}{\mathrm{r}_1} - \frac{1}{\mathrm{r}_2}\right) = 1 + \frac{M\mathsf{G}}{n^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{23}$$

For an observer located on the surface of the Sun and another one located on the surface of the Earth, we find $\delta t_{Earth} \approx \delta t_{Sun} - 2.12\times 10^{-6} n^{-2}\delta t_{Sun}$. In other words, for $n = 1$, the time lapse between the two crests measured on the Sun is larger than what would be measured on the surface of the Earth. Similarly, one finds $\delta t_{Sun} \approx \delta t_{Earth} + 2.12\times 10^{-6} n^{-3}\delta t_{Earth}$ (gravitational time contraction). Similarly, for the gravitational redshift of light, we find in our approach:

$$\nu_1 \approx \nu_2\left(1 + \frac{M\mathsf{G}}{n^3}\left(\frac{1}{\mathrm{r}_1} - \frac{1}{\mathrm{r}_2}\right)\right) = \nu_2\left(1 + \frac{M\mathsf{G}}{n^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right) \tag{24}$$

where ν denotes the frequency of the photon. In addition, in the Schwarzschild metric Equation (21) two singularities appear: one at the origin and another one at $r = 2M\mathsf{G}/n^2$ which is discrete and hence for very large discrete value of *n*, the metric Equation (21) is reduced to the discrete Minkowski spacetime. It is notable that for very large discrete value of *n*, the gravity in the new system is larger than the one in the old frame, *i.e.*, $\mathsf{G} = Gn^2 > G$. Then, the singularity radius $r = 2M\mathsf{G}/n^2 \to 0$ for very large positive discrete value of *n*. For the Case 2.1, we find similar results.

These results hold once more for $\xi = \pm 2\pi i$. On the other hand, for $\xi = \pm 2\pi$, $X^\mu \triangleq x^\mu/n$ and the metric signature is not altered yet the previous results are not altered.

We summarize our results in Tables 1–3:

Table 1. Main differences between standard general relativity (GR) and GR arising from exponential non-standard Lagrangians (ENSL).

Details	**Standard General Relativity**	**Discrete General Relativity from ENSL**
Geodesic equation (GE)	$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$	$\frac{d^2x^\mu}{d\lambda^2} + \xi\left(\frac{1}{2}\frac{dg_{\rho\nu}}{d\lambda}\frac{dx^\rho}{d\lambda} + g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}\right)\frac{dx^\sigma}{d\lambda}\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$
Parameterization (Para)	$\tau = \int_0^1 \mathcal{F} d\lambda$ $\mathcal{F} \triangleq (-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda))^{1/2}$	$\tau = \int_0^1 e^{\xi\mathcal{F}} d\lambda$ $\mathcal{F} \triangleq (-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda))^{1/2}$
GE after Para	$\frac{\partial\mathcal{F}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\right) = 0$	$\frac{\partial\mathcal{F}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\right) = \xi\frac{\partial\mathcal{F}}{\partial\dot{x}^\mu}\frac{d\mathcal{F}}{d\lambda}$
Condition for $\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} = 1$	$\mathcal{F} = 1 \to$ real action	$\mathcal{F} = \frac{2i\pi n}{\xi}, \xi \neq 0, n \in \mathbb{Z} \to$ complexified action
Discrete solutions	Do not exist	Exist in two different forms (Table 2)

Table 2. Main differences between the two independent discrete solutions obtained for $\xi = \pm 2\pi i$.

Details	New Spacetime Coordinates $X^{\mu} \triangleq \frac{x^{\mu}}{n}$ with a Metric Signature Change $g_{\mu\nu} \to \mathbb{G}_{\mu\nu} \triangleq -g_{\mu\nu}$	A Discretized Spacetime Metric $\mathfrak{g}_{\mu\nu} \triangleq \frac{g_{\mu\nu}}{n^2}$
EFE	$\mathsf{R}_{\mu\nu} - \mathbb{G}_{\mu\nu}\frac{\mathsf{R}}{2} = -8\pi \mathsf{G}\mathbb{T}_{\mu\nu}$	$\mathcal{R}_{\mu\nu} - \mathfrak{g}_{\mu\nu}\frac{\mathcal{R}}{2} = -8\pi \mathsf{G}\mathcal{T}_{\mu\nu}$
Scalar curvature	$R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = -n^2\mathsf{R}$	$R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = -n^2\mathcal{R}$
Discrete gravity	$G = \mathsf{G}/n^2$	$G = \mathsf{G}/n^2$

The above results suggest that gravity originates from "spacetime discreteness" if the standard action is replaced by the ENSL. If we set by $\mathsf{G}_n \triangleq Gn^2$, there are infinite numbers of quanta of gravity, *i.e.*, gravity can take on only discrete values and then it is quantized. It is interesting to obtain discrete scalar curvature in all previous cases and may have important consequences in Riemann geometry [68,69] and it is an open problem.

Table 3. Main differences between the two independent discrete solutions obtained for $\xi = \pm 2\pi$.

Details	New Spacetime Coordinates $X^{\mu} \triangleq \frac{x^{\mu}}{n}$ Free from a Metric Signature Change	A Discretized Spacetime Metric $\mathfrak{g}_{\mu\nu} \triangleq \frac{g_{\mu\nu}}{n^2}$
EFE	$\mathsf{R}_{\mu\nu} - g_{\mu\nu}\frac{\mathsf{R}}{2} = -8\pi \mathsf{G}\mathbb{T}_{\mu\nu}$	$\mathcal{R}_{\mu\nu} - \mathfrak{g}_{\mu\nu}\frac{\mathcal{R}}{2} = -8\pi \mathsf{G}\mathcal{T}_{\mu\nu}$
Scalar curvature	$R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = n^2\mathsf{R}$	$R = g^{\alpha\beta}R^{\gamma}_{\alpha\gamma\beta} = -n^2\mathcal{R}$
Discrete gravity	$G = \mathsf{G}/n^2$	$G = \mathsf{G}/n^2$

3. Some Applications of Discrete Gravity, Discrete Metric and New Spacetime Coordinates

The aim of this section is to give some implications of NSTC + MC and DSTM in theoretical physics, yet their details will be addressed carefully in future works.

3.1. The Linearized Theory

We start by the "mathematics of the linearized theory" in discrete gravity approach. In the standard linearized theory, the spacetime is almost flat and takes the form $ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta}), |h| << 1)$. If we adopt the trace-reversed metric perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \eta_{\alpha\beta}\eta^{\mu\nu}h_{\mu\nu}/2$ and the Lorentz gauge $\bar{h}^{\alpha\beta}_{,\beta} = 0$, then the Einstein field equations are reduced to a set of decoupled wave equations written as $\Box\bar{h}^{\alpha\beta} = 16\pi G T^{\alpha\beta}$ [70]. The solution of the equations is given by $\bar{h}^{\alpha\beta} = A^{\alpha\beta}e^{2\pi i k_{\mu}x^{\mu}}$ and describes gravitational waves propagating at the celerity of light with null wave vector $k_{\alpha}\, k^{\alpha} = 0$. Outside a sphere surrounding the source, we have $A^{0\beta} = 0$, *i.e.*, transverse wave since $A^{ik}k_j = 0$ and $A^j{}_j = 0$, *i.e.*, traceless wave amplitude. Therefore, for the so-called transverse-traceless gauge, $\Box h^{\alpha\beta} = 0$. This is the wave equation for massless gravitons. For the case of massive gravitons, it was observed in [71] that the gravitational waves may be absorbed by a background cosmic fluid with density ρ characterized by a negative pressure $p = -\rho$, and described by an energy–momentum density tensor of the form $T^{\alpha\beta} = -pg^{\alpha\beta}$. In a conformally flat spacetime described by the metric tensor $g_{\alpha\beta} = e^{\psi}\eta_{\alpha\beta}$ with ψ being a some function of the spacetime coordinates, the gravitational waves are described by the massive wave equation $\Box\psi + m_g^2\psi = -S$ where $m_g^2 = 64\pi G\rho/3$ and $S = 32\pi G\rho/3$ [72].

For the case of NSTC + MC, the massless graviton wave equation is not modified, whereas for the massive graviton wave equation we find $\Box\psi + \mathsf{m}_g^2\psi = -\mathsf{S}$ where $\mathsf{m}_g^2 = m_g^2/n^2$ and $\mathsf{S} = S/n^2$ are respectively the discrete graviton mass and the discrete gravitational source. For the case of DSTM, we find as well $\Box\psi + \mathsf{m}_g^2\psi = -\mathsf{S}$. This also implies that the scalar curvature $\mathbf{R}$ or $\mathcal{R}$ which is proportional to the energy density (and hence to m_g^2) by means of the EFE is also discrete, *i.e.*, $\mathbf{R} \propto m_g^2/n^2$ and $\mathcal{R} \propto m_g^2/n^2$. Therefore we conclude that in our approach we can obtain discrete graviton spectrum and the graviton boson may possess a discrete mass which tends to zero for very large discrete values of n. It is notable that there are many arguments which state that the square of the graviton mass is closely related to the cosmological constant Λ, *i.e.*, $\Lambda \approx \mathsf{m}_g^2$ [73]. If it is the true case, then we can associate for the cosmological constant discrete values $\mathsf{\Lambda} = \Lambda/n^2$. For large values of *n*, the discrete cosmological constant tends to zero. This may have important cosmological consequences that will be discussed in a future research work.

3.2. The Gravitational Bohr Atom

Let us look to the problem of the gravitational Bohr atom. In fact we assume a small mass *m* orbiting a mass $M >> m$ with a velocity v in a $1/r$ potential. The kinetic energy is $mv^2/2$ and the quantized gravitational potential in our approach will be $-mM\mathsf{G}/r \equiv -mMGn^2/r$ and accordingly the total energy is given by $mv^2/2 - mMGn^2/r$. However, one can define the quantized radius $\boldsymbol{r} = r/n^2$ and subsequently the angular momentum is $\mathbb{L} = mv\boldsymbol{r} = mvr/n^2 = L/n^2$. However, the centripetal acceleration of the planet is given by $mv^2/\boldsymbol{r} = mMG/\boldsymbol{r}^2$ which gives $v^2 = MG/\boldsymbol{r}$. In normal units where $\hbar, c \neq 1$, we can write $\mathbb{L} = (m\sqrt{MG\boldsymbol{r}}) \equiv n\mathfrak{h}$ where $\mathfrak{h}$ is the effective Planck's constant. Then $\boldsymbol{r} = n^2\mathfrak{h}^2/m^2MG \equiv n^4\mathfrak{h}^2/m^2M\mathrm{G}$. This result coincides with [74,75] but from a completely different approach. The total energy is then given by $E = -GmM/2\boldsymbol{r} \equiv -\mathsf{G}mM/2\boldsymbol{r}n^2 = -m^3M^2G^2/2n^2\hbar^2$.

However, if we make the substitution $m \to e, M \to e, G \to 1/4\pi\varepsilon_0$, then we recover the Bohr's quantum results [76]. In the new system, we have $\boldsymbol{r} = n^4\mathfrak{h}^2/m^2M\mathrm{G}$ and $E = -\mathsf{G}mM/2\boldsymbol{r}n^2$. If we introduce the discrete Planck's constant $\mathfrak{h} = n\hbar$, the discretized masses $m = \mathfrak{m}n^2$ and $M = \mathbb{M}/n^2$, we find $\boldsymbol{r} = \mathfrak{h}^2n^2/\mathfrak{m}^2\mathbb{M}\mathrm{G}$ and $E = -\mathrm{G}\mathfrak{m}\mathbb{M}/2\boldsymbol{r}n^2$. Hence, we recover the general form of the quantized radius in the gravitational Bohr atom problem. Notice that the microscopic discrete mass (mdm) is $\mathfrak{m} = m/n^2$ whereas the macroscopic discrete mass (Mdm) is $\mathbb{M} = Mn^2$. For very large values of the integer *n*, mdm become smaller whereas Mdm become larger yet the product $\mathfrak{m}\mathbb{M} = k$ where *k* is a constant. Therefore, mdm times Mdm is a universal constant. We expect that discrete masses and discrete Planck's constant have motivating consequences in quantum mechanics and quantum gravity.

3.3. Black Hole Hawking's Radiation

As a 3rd illustration, we discuss the implication of the discrete gravitational constant on Hawking's radiation emitted by a black hole. This fact represents one of the most amazing predictions in theoretical physics. According to Hawking, a black hole emits continuous thermal radiation whose temperature is given in units $\hbar = c = 1$ by $T = 1/8\pi GM$ [77]. Here, M is the mass of the Schwarzschild black hole. In our framework, this temperature must be discrete and replaced by $T = n^2/8\pi \mathrm{G}M$ which gives a discrete temperature given by $\mathbb{T} = n^2 T$. Now, from the discrete black hole temperature, we can calculate the discrete entropy $d\mathbb{S} = dQ/\mathbb{T} = dQ/n^2 T = dS/n^2$ where dQ is the quantity of heat added. Accordingly, we find $\mathbb{S} = S/n^2$ and since in Hawking's radiation theory $S = A/4$ with A being the surface area of the black hole, then $\mathbb{S} = A/4n^2 \equiv \mathbb{A}/4$ where $\mathbb{A} = A/n^2$ is the discrete surface area. We therefore conclude that, in NSTC + MC and DSTM approaches, a Schwarzschild black hole has discrete entropy/discrete surface area. In the particular case of a Schwarzschild hole, the area of such a black hole with mass M is $A = 16\pi G^2 M^2$ and therefore $\mathbb{A} = 16\pi G^2 M^2/n^2$. As a result, we conjecture that the eigenvalues of the black hole event horizon area are of the form $A \propto n^2 L_P^2$ where L_P is the Planck's length. This result modifies the Bekenstein's black hole discrete spectrum arguments where $A \propto n L_P^2$ [78]. The angular frequencies of the quanta of the Hawking radiation are then given by $\omega_0 \propto n^2/32\pi GM$ [79]. Hence, for large values of n, ω_0 in NSTC + MC and DSTM approaches is much larger than the value obtained by Bekenstein.

4. Conclusions

In this paper, we have discussed some implications of NSL, in particular ENSL, in general relativity. A number of attractive features are obtained which confirm that NSL deserves careful attention and serious consideration. On a given geometrical manifold $\mathcal{M}$, ENSL are characterized by modified geodesic equations where some of their physical properties were discussed. It was observed that the geodesic dynamics in the ENSL approach is not characterized merely by a universal speed, but probably by additional general velocities that are proportional to $\sqrt{-1/\xi}$. When taking into account the timelike paths parameterization constraint, it was observed that discrete gravity and discrete spacetime emerge surprisingly in the theory. For $\xi = \pm 2\pi i$ and in the set $\mathbb{Z}$, two different independent cases were obtained in order to preserve the regular form of the first integral of the standard geodesic equation: a geometrical manifold with new spacetime coordinates augmented by a metric signature change and a geometrical manifold characterized by a negative discretized spacetime metric. Both cases give raise to Einstein's field equations yet the gravity is discretized and originated from "spacetime discreteness". Besides the gradient ∇, the Laplacian Δ and the d'Alembertian $\Box$ operators are as well modified and took discrete shapes. However, for $\xi = \pm 2\pi$, then in the set $\mathbb{Z}$, in order to preserve the regular form of the first integral of the standard geodesic equation, one can either define a new discrete spacetime coordinates $X^\mu \triangleq x^\mu/n$ without changing metric signature or define a positive discretized spacetime metric which yields a new discretized interval between two closed points. Both cases $\xi = \pm 2\pi i$ and $\xi = \pm 2\pi$ result on a discretized gravity whereas discretized scalar curvature is negative for the 1st case and is positive for the 2nd case. We have discussed three different implications of discrete gravity:

first, in the linearized theory of general relativity, it was observed that graviton holds a discrete spectrum with a discrete mass as well; second, in the theory of gravitational Bohr atom, the Planck's constant is discretized and besides a balance equation between the macroscopic mass and the microscopic mass is obtained; third, in the Hawking's black holes radiation theory, the entropy and the surface area of the black are discretized, and we conjectured that the eigenvalues of the black hole event horizon area are of the form $A \propto n^2 L_P^2$ where L_P is the Planck's length and that the angular frequencies of the quanta of the Hawking radiation in NSTC + MC and DSTM approaches are much larger than values obtained by Bekenstein. It was also observed that by considering a de-Sitter space and dealing with the NSTC + MC case, the cosmological constant may be quantized or discretized which may have drastic consequences on the physics of the early universe and quantum geometry. Besides, the scalar curvature is negative for a positive cosmological constant and is positive for a negative cosmological constant. It is noteworthy that, in general relativity, a symmetric vacuum solution with a negative cosmological constant and a negative scalar curvature is known as the anti-de-Sitter space (AdS). Therefore, we have a deviation from the standard geometrical result. However, for $\xi = \pm 2\pi$, the de Sitter space is characterized by a quantized cosmological constant and a positive scalar curvature. All these results require more analysis; nevertheless, they prove that ENSL may reveal many interesting properties which could not be derived from the standard Lagrangians approach. It will be of interest to reformulate the geometrical ENSL approach for the case of time-dependent metric tensor [80], the energy problem in the present theory of gravity, and some of their cosmological implications in the early universe. A number of details and applications are in progress.

Acknowledgment

I would like to thank Leo Jiang for inviting my part to publish a work in Mathematics-Special Issue: Mathematical Physics. I would like as well to thank the anonymous referees for their valuable comments and suggestions.

Conflicts of Interest

The author declares no conflict of interest.

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