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Article

Photon Localization Revisited

Izumi Ojima [1,†,*] and Hayato Saigo [2,†]

[1] Shimosakamoto, Otsu, Shiga 520-0105, Japan

[2] Nagahama Institute of Bio-Science and Technology, 1266 Tamura-Cho, Nagahama, Shiga, 526-0829, Japan; E-Mail: h_saigoh@nagahama-i-bio.ac.jp

[†] These authors contributed equally to this work.

* Author to whom correspondence should be addressed; E-Mail: ojima@kurims.kyoto-u.ac.jp.

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Abstract: In the light of the Newton–Wigner–Wightman theorem of localizability question, we have proposed before a typical generation mechanism of effective mass for photons to be localized in the form of polaritons owing to photon-media interactions. In this paper, the general essence of this example model is extracted in such a form as quantum field ontology associated with the eventualization principle, which enables us to explain the mutual relations, back and forth, between quantum fields and various forms of particles in the localized form of the former.

Keywords: Photon localization; Eventualization; Tomita decomposition theorem

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1. Introduction

Extending the scope of our joint paper [1], the essence of which is summarized in (1) and (2) below, we discuss in this paper the following points:

(1) The localizability of massless photons has been examined historically starting from a joint paper of Newton and Wigner [2] and one due to Wightman [3], the essential contents of which can be summarized in the form of the Newton–Wigner–Wightman theorem (Section 2). In the light of this theorem concerning a specific problem involving the photon, we try here to clarify the mathematical and conceptual relations among spatial points, localization processes of physical systems into restricted regions in space (and time), in contrast to the usual formulation dependent directly on the concepts of

particles and their masses (in a spacetime structure given in an *a priori* way). In this context, Wightman's mathematical formulation of the Newton–Wigner paper plays an important role: on the basis of an imprimitivity system on the three-dimensional space, the absence of position observables is shown to follow from the vanishing mass $m = 0$ of a free photon.

(2) We encounter here a sharp conflict between the mathematically clear-cut negative result and the actual existence of experimental devices for detecting photons in quantum optics, which is impossible without the spatial localization of detected photons. Fortunately, we have found in our joint paper [1] an affirmative answer to this long-standing problem on the basis of the group-theoretical concept of imprimitivity systems utilized in [3]: the above-mentioned conflict is resolved by the presence of coupled modes of photons with material media, which generate non-trivial deviations of refractive index n from one or equivalently generate the mass $m > 0$, in such typical example cases as "polaritons", as will be shown later (Section 3.3).

(3) Through the model example of polaritons, we learn that such fundamental issues concerning a mass and its carrier particles should be viewed as something variable dependent on the contexts and situations surrounding them. Thus, we need and can elaborate on highly philosophical abstract questions like "what is a mass?" or "what are particles as mass points?", in mathematically accessible contexts. For this purpose, we certainly need to set up suitable theoretical and/or mathematical frameworks and models so that they allow us to systematically control the dynamics of our object systems coupled with their external systems. Once this coupling scheme is established, the external systems can be seen to serve as reference frames for the purpose of describing the object systems and the processes carried out by them. Such a framework and methodology are available in the form of Tomita's integral decomposition theorem (Section 4.3) viewed from the standpoint of the "quadrality scheme" based on "Micro-Macro duality" (Sections 3.2 and Section 4.2).

(4) Whenever we try to observe and describe a physical system, the necessity is evident for a coupling between the target physical system and a measuring system, without which any physical quantities belonging to the former cannot be measured, visualized nor described. For the sake of a satisfactory understanding of the connection between the above group-theoretical context and the description of the actual physical processes, we need to explain the latter aspect in close relation to the imprimitivity systems, which is one of the main purposes of the present paper.

For this purpose, a universal method is indispensable for constructing a model system consisting of the target system to be described coupled with a reference system to register and to describe the results of a measurement process: this purpose is successfully accomplished by means of Tomita's theorem of the integral decomposition of a state.

A satisfactory understanding of fundamental concepts of space (and time) coordinates and velocities is attainable in the scheme, and at the same time, a crucial premise underlying such comprehension is the understanding that these concepts are never among pre-existing attributes inherent in the object system, but are epigenetic properties emerging through what is to be called the "eventualization processes", as will be explained in Section 5. These epigenetic aspects are closely related to the choices of different contexts of placing an object system and the boundary conditions specifying various different choices of subalgebras of central observables, reflected in the choices of sub-central (or central) measures appearing in Tomita's theorem (Section 4.4).

(5) While the above explanation guarantees the naturality and genericity of the polariton picture mentioned in (2), as one of the typical explicit examples for making photons localizable, the freedom in choices of sub-central measures clarifies their specialty in the spatial homogeneity of mass generation. In fact, under such conditions that the spatial homogeneity is not required, many such forms of photon localizations are allowed as Debye shielding, various forms of dressed photons, among which cavity QED can equally be understandable.

(6) Along this line of thought, it becomes also possible to compare and unify various other forms of localizations and their "leakages" at the same time: for instance, the presence of non-vanishing mass m can be viewed as an index of timelike and spacetime-homogeneous parameter of leakage from spatial localization as exhibited by the decay rate $\propto e^{-mr}$ of correlation functions in the clustering limit. On the other hand, the decay width Γ in the energy spectrum can also be interpreted as a time-homogeneous parameter of leakage from chronological localizations of resonance modes (as exhibited through the decay rate $\propto e^{-\Gamma t/2}$ of the relaxation of correlations; to be precise, it is more appropriate to regard the inverse of m and Γ as leakages). The tunneling rate $\propto \sqrt{|E-V|}$ can be interpreted as the leakage rate of spatial localization materialized by the potential barrier V.

(7) The universality, naturality and the necessity of the present standpoint is verified by the above considerations in terms of sub-central measures and of the corresponding commutative algebras $\mathcal{B}$. On the basis of the bi-directionality between quantum fields and particles, moreover, such a unified viewpoint will be meaningful that the microscopic quantum systems consisting of quantum fields can be controlled and designed from the macro side via the control of quantum fields.

(8) To make the above possibility certain, it would be important to recognize the constitution of the macroscopic levels in close relation to the microscopic quantum regimes. This question is answered in terms of "eventualization processes", which can be mathematically described as the filtered "cones" to amplify the connections between macro and micro (which is analogous to the forcing method in the context of the foundations of mathematics), with micro ends given by the dynamics of quantum fields and macro ones by the point-like events as the apices of cones of eventualization.

2. Newton–Wigner–Wightman Theorem

In 1949, Newton and Wigner [2] raised the question of the localizability of single free particles. They attempted to formulate the properties of the localized states on the basis of the natural requirements of relativistic covariance.

Physical quantities available in this formulation admitting direct physical meaning are restricted inevitably to the generators of Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes L_+^\uparrow$ (with $L_+^\uparrow$ the orthochronous proper Lorentz group), which is locally isomorphic to the semi-direct product $\mathcal{H}_2(\mathbb{C}) \rtimes SL(2,\mathbb{C})$ of the Jordan algebra $\mathcal{H}_2(\mathbb{C})$ of Hermitian (2×2)-matrices and $SL(2,\mathbb{C})$, consisting of the energy-momentum vector P_μ and of the Lorentz generators $M_{\mu\nu}$ (composed of angular momenta M_{ij} and of Lorentz boosts M_{0i}). The problem is then to find conditions under which "position operators" can naturally be derived from the Poincaré generators $(P_\mu, M_{\mu\nu})$. In [2], position operators have been shown to exist in massive cases in an essentially unique way for "elementary" systems in the sense of the irreducibility of the corresponding representations of $\mathcal{P}_+^\uparrow$, so that the localizability of a state can be defined in terms of such position operators. In massless cases, however, no localized states are found to exist in the above sense. That was the beginning of the story.

Wightman [3] clarified the situation by recapturing the concept of "localization" in quite a general form as follows. In place of the usual approaches with unbounded generators of position operators, he has formulated the problem in terms of their spectral resolution in the form of Axioms (i)–(iii):

(i) The spectral resolution of position operators: This is defined by a family $\mathcal{B}(\mathbb{R}^3) \ni \Delta \longmapsto E(\Delta) \in Proj(\mathfrak{H})$ of projection-valued measures $E(\Delta)$ in a Hilbert space $\mathfrak{H}$ defined for each Borel subset Δ of $\mathbb{R}^3$, characterized by the following Properties (ia), (ib) and (ic):
   (a) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$;
   (b) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$, if $\Delta_1 \cap \Delta_2 = \phi$;
   (c) $E(\mathbb{R}^3) = 1$;
(ii) Physical interpretation of $E(\Delta)$: When the system is prepared in a state ω, the expectation value $\omega(E(\Delta))$ of a spectral measure $E(\Delta)$ gives the probability for the system to be found in a localized region Δ;
(iii) Covariance of the spectral resolution: Under a transformation $(\mathbf{a}, \mathcal{R})$ with a spatial rotation $\mathcal{R}$ followed by a spatial translation $\mathbf{a}$, a Borel subset Δ is transformed into $\mathcal{R}\Delta + \mathbf{a}$. The corresponding unitary implementer is given in $\mathfrak{H}$ by $U(\mathbf{a}, \mathcal{R})$, which represents $(\mathbf{a}, \mathcal{R})$ covariantly on E in such a way that:

$$E(\Delta) \to E(\mathcal{R}\Delta + \mathbf{a}) = U(\mathbf{a}, \mathcal{R})E(\Delta)U(\mathbf{a}, \mathcal{R})^{-1}$$

Note that, in spite of the relevance of the relativistic covariance, the localizability discussed above is the localization of states in space at a given time formulated in terms of spatial translations $\mathbf{a}$ and rotations $\mathcal{R}$, respectively. To understand the reason, one should imagine the situation with Axioms (i)–(iii) replaced with those for the whole spacetime; then, the CCR relations hold between four-momenta p_μ and space-time coordinates x^ν, which implies the Lebesgue spectrum covering the whole $\mathbb{R}^4$ for both observables $\hat{p}_\mu$ and $\hat{x}^\nu$. Therefore, any such physical requirement as the spectrum condition or as the mass spectrum cannot be imposed on the energy-momentum spectrum $\hat{p}_\mu$, and hence, the concept of the localizability in space-time does not make sense.

According to Mackey's theory of induced representations, Wightman's formulation can easily be seen as the condition for the family of operators $\{E(\Delta)\}$ to constitute a system of imprimitivity [4] under the action of the unitary representation $U(\mathbf{a}, \mathcal{R})$ in $\mathfrak{H}$ of the three-dimensional Euclidean group $SE(3) := \mathbb{R}^3 \rtimes SO(3)$ given by the semi-direct product of the spatial translations $\mathbb{R}^3$ and the rotation group $SO(3)$. In a more algebraic form, the pair (E, U) can also be viewed as a covariant W*-dynamical system $L^\infty(\mathbb{R}^3) \underset{\tau}{\curvearrowleft} SE(3), [\tau_{(\mathbf{a},\mathcal{R})}(f)](\mathbf{x}) := f(\mathcal{R}^{-1}(\mathbf{x} - \mathbf{a}))$, given by the covariant *-representation $E : L^\infty(\mathbb{R}^3) \ni f \longmapsto E(f) = \int f(\mathbf{x})dE(\mathbf{x}) \in B(\mathfrak{H})$, s.t. $E(\chi_\Delta) = E(\Delta)$, of the commutative algebra $L^\infty(\mathbb{R}^3)$ generated by the position operators acted on by $SE(3)$ characterized by the covariance condition:

$$E(\tau_{(\mathbf{a},\mathcal{R})}(f)) = U(a, \mathcal{R})E(f)U(a, \mathcal{R})^{-1} \tag{1}$$
$$\text{for } f \in L^\infty(\mathbb{R}^3), (\mathbf{a}, \mathcal{R}) \in SE(3)$$

As will be seen later, this algebraic reformulation turns out to be useful for constructing coupled systems of photon degrees of freedom with matter systems, which play crucial roles in observing or measuring the former in the actual situations. Thus, Wightman's formulation of the Newton–Wigner localizability problem is just to examine whether the Hilbert space $\mathfrak{H}$ of the representation $(U, \mathfrak{H})$ of $SE(3)$ can accommodate a representation E of the algebra $L^\infty(\mathbb{R}^3)$ consisting of position operators, covariant under the action of $SE(3)$ in the sense of Equation (1). If this condition holds, then the abelian components $\mathbb{R}^3$ in the semi-direct product group $SE(3)$ provide us with the position operators; in the case of a massless spinless field, Equation (1) trivially holds, owing to the triviality of the representation of the spin rotation group and of the commutativity between $\mathbb{R}^3$ and the function f on $\mathbb{R}^3$.

Applying Mackey's general theory to the case of the three-dimensional Euclidean group $SE(3)$, Wightman proved the following fundamental result as a purely kinematical consequence:

Theorem 1 ([3], excerpt from Theorems 6 and 7)**.** *A Lorentz covariant massive system is always localizable. The only localizable massless elementary system (*i.e.*, irreducible representation) has spin zero.*

Corollary 2. *A free photon is not localizable.*

The essential mechanism causing (non-)localizability in the sense of Newton–Wigner–Wightman can be found in the structure of Wigner's little groups, the stabilizer groups of standard four-momenta on each type of $\mathcal{P}_+^\uparrow$-orbits in p-space.

When $m \neq 0$, the little group corresponding to the residual degrees of freedom in a rest frame is the group $SO(3)$ of spatial rotations. As a consequence, "the space of rest frames" becomes $SO(1,3)/SO(3) \cong \mathbb{R}^3$. The physical meaning of this homeomorphism is just a correspondence between a rest frame $r \in SO(1,3)/SO(3)$ for registering positions and a boost $k \in SO(1,3)$ required for transforming a fixed rest frame r_0 to the chosen one $r = kr_0$. The universality (or independence for the choice of the frame) of positions is recovered up to Compton wavelength $h/(mc)$, again due to massiveness.

Remark 3. *Here, the coordinates of rest frames just play the role of the order parameters (or "sector parameters") on each $\mathcal{P}_+^\uparrow$-orbit as the space of "degenerate vacua" associated with certain symmetry breaking, which should play the roles of position operators appearing in the imprimitivity system.*

In sharp contrast, there is no rest frame for a massless particle if it has a non-trivial spin: its little group is isomorphic to the two-dimensional Euclidean group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ (locally isomorphic to $\mathbb{C} \rtimes U(1)$), whose rotational generator corresponds to the helicity. Since the other two translation generators corresponding to gauge transformations span non-compact directions in contrast to the massive cases with a compact $SO(3)$, the allowed representation (without an indefinite inner product) is only the trivial one, which leaves the transverse modes invariant, and hence, the little group cannot provide position operators in the massless case. Here, we note again the special situation with massless spinless particles: in this case, the little group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ does not cause any obstructions, since it reduces to the abelian $\mathbb{R}^2$ owing to the absence of helicity components.

After the papers by Newton and Wigner and by Wightman, many discussions have been developed around the photon localization problem. As far as we know, the arguments seem to be divided into two opposite directions, one relying on purely dynamical bases [5] and another on pure kinematics [6], where it is almost impossible to find any meaningful agreements. Below, we propose an alternative strategy based on the concept of "effective mass", which can provide a reasonable reconciliation between these conflicting ideas because of its "kinematical" nature arising from some dynamical origin.

3. Polariton as a Typical Model of Effective Mass Generation

3.1. Physical Roles Played by a Coupled External System

In spite of the above theoretical difficulty in the localizability of photons, however, it is a plain fact that almost no experiments can be performed in quantum optics where photons must be registered by localized detectors. To elaborate on this problem, we will see that it is indispensable to reexamine the behavior of a photon in composite systems coupled with some external system, such as a material media constituting apparatus without which any kind of measurement processes cannot make sense. For this purpose, the above group-theoretical analysis of the localizability of kinematical nature should be extended to incorporate algebraic aspects involved in the formation of a coupled dynamics between photons to be detected and the measuring devices consisting of matters.

Our scheme of the localization for photons can be summarized as follows:

- Photons are coupled with external system into a composite system with a coupled dynamics.
- Positive effective mass emerges in the composite system.
- Once a positive effective mass appears, Wightman's theorem itself provides the "kinematical basis" for the localization of a photon.

From our point of view, therefore, this theorem of Wightman's interpreted traditionally as a no-go theorem against localizability becomes actually an affirmative support for it. It conveys such a strongly selective meaning (which will be discussed in detail in Section 4) that, whenever a photon is localized, it should carry a non-zero effective mass.

In the next subsection, we explain the meaning of our scheme from a physical point of view.

3.2. How to Define the Effective Mass of a Photon

As a typical example of our scheme, we focus first on a photon interacting with homogeneous medium, in the case of monochromatic light with angular frequency ω as a classical light wave. For simplicity, we neglect here the effect of absorption, that is the imaginary part of the refractive index. When a photon interacting with matter can be treated as a single particle, it is natural to identify its velocity $\mathbf{v}$ with the "signal velocity" of light in the medium. The relativistic total energy E of the particle should be related to $v := \sqrt{\mathbf{v} \cdot \mathbf{v}}$ by its mass m_{eff}:

$$E = \frac{m_{\mathsf{eff}} c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

Since v is well known to be smaller than the light velocity c (theoretically or experimentally), m_{eff} is positive (when the particle picture above is valid). Then, we may consider m_{eff} as the relativistic "effective (rest) mass of a photon", and identify its momentum $\mathbf{p}$ with:

$$\mathbf{p} = \frac{m_{\text{eff}}\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

Hence, as long as "an interacting photon" can be well approximated by a single particle, it should be massive, according to which its "localization problem" is resolved. The validity of this picture will be confirmed later in the next subsection.

The concrete forms of energy/momentum are related to the Abraham–Minkowski controversy [7–9] and modified versions of Einstein/de Broglie formulae [1].

Our argument itself, however, does not depend on the energy/momentum formulae. The only essential point is that a massless particle can be made massive through some interactions. That is, while a free photon satisfies:

$$E_{\text{free}}^2 - c^2 p_{\text{free}}^2 = 0 \tag{4}$$

an interacting photon satisfies:

$$E^2 - c^2 p^2 = m_{\text{eff}}^2 c^4 > 0 \tag{5}$$

To sum up, an "interacting photon" can gain a positive effective mass, while a "free photon" remains massless! This is the key we have sought for. We note, however, that the present argument is based on the assumption that "a photon dressed with interactions" can be viewed as a single particle. We proceed to consolidate the validity of this picture, especially the existence of particles whose effective mass is produced by the interactions, analogous to Higgs mechanism: such a universal model for photon localization certainly exists, which is based on the concept of polariton, well known in optical and solid physics.

3.3. Polariton Picture

In these areas of physics, the propagation of light in a medium is viewed as follows: by the interaction between light and matter, the creation of an "exciton (an excited state of a polarization field above the Fermi surface)" and annihilation of a photon will be followed by annihilation of an exciton and creation of a photon, $\cdots$, and so on. This chain of processes itself is often considered as the motion of particles called *polaritons* (in this case, "exciton-polaritons"), which constitute particles associated with the coupled wave of the polarization wave and the electromagnetic wave.

The concept of polariton has been introduced to develop a microscopic theory of electromagnetic interactions in materials ([10,11]). Injected photons become polaritons by the interaction with matter. As the exciton-phonon interaction is dissipative, the polariton picture gives a scenario of absorption. It has provided an approximation better than the scenarios without it. Moreover, the group velocity of polaritons discussed below gives another confirmation of the presence of an effective mass.

As is well known, permittivity $\epsilon(\omega)$ is given by the following equality,

$$\epsilon(\omega) = n^2 = \frac{c^2 k^2}{\omega^2} \tag{6}$$

and hence, we can determine the dispersion relation (between frequency and wave number) of the polariton once the formula of permittivity is specified. In general, this dispersion relation implies branching, analogous to the Higgs mechanism. The signal pulse corresponding to each branch can also be detected in many experiments, for example in [12] cited below.

In the simple case, the permittivity is given by the transverse frequency ω_T of the exciton (lattice vibration) as follows:

$$\epsilon(\omega) = \epsilon_\infty + \frac{\omega_T^2(\epsilon_{st} - \epsilon_\infty)}{\omega_T^2 - \omega^2} \tag{7}$$

where ϵ_∞ denotes $\lim_{\omega\to\infty} \epsilon(\omega)$ and $\epsilon_{st} = \epsilon(0)$ (static permittivity). With a slight improvement through the wavenumber dependence of the exciton energy, the theoretical result of polariton group velocity $\partial\omega/\partial\mathbf{k} < c$ based on the above dispersion relation can explain satisfactorily the experimental data of the passing time of light in materials (for example, [12]). This strongly supports the validity of the polariton picture.

From the above arguments, polaritons can be considered as a universal model of the "interacting photons in a medium" in the previous section. The positive mass of a polariton gives a solution to its "localization problem". Conversely, as the "consequence" of Wightman's theorem, it follows that "all" physically accessible photons as particles, which can be localized, are more or less polaritons (or similar particles) because only the interaction can give a photon its effective mass, if it does not violate the particle picture.

4. Effective Mass Generation in General

4.1. Toward General Situations

In the last subsection, we have discussed that the interaction of photons with media can cause their localization by giving effective masses to them. Then a natural question arises: is the existence of media a necessary condition for the emergence of effective photon mass? The answer is no: in fact, light beams with a finite transverse size have group velocities less than c.

In a recent publication [13], Giovannni *et al.* show experimentally that even in a vacuum, photons (in the optical regime) travel at a speed less than c when it is transversally structured, such as Bessel beams or Gaussian beams, by measuring a change in the arrival time of time-correlated photon pairs. They show a reduction in the velocity of photons in both a Bessel beam and a focused Gaussian one. Their work highlights that, even in free space, the invariance of the speed of light only applies to plane waves, *i.e.*, free photons.

From our viewpoint, this result can be understood quite naturally in the light of the Newton–Wigner–Wightman theorem. As we have seen, the theorem states that every localizable elementary system (particle) with spin must be massive. It implies that photons in the real world should travel less than c, in any conditions, which makes the probability distribution of its position well defined without contradicting the presence of spin. Hence, transversally-structured photons should become slow.

The scenario also applies to more general settings. Any kinds of boundary conditions with finite volume (like cavity) or even nanoparticles in the context of dressed photons [14] will make photons heavier and slower, even without medium!

4.2. Wightman's Theorem Reinterpreted as the "Basis" for Localization

Our general scheme of the localization for photons can be depicted as follows, the essence of which can be understood in accordance to the basic formulation of the "quadrality scheme" [15] underlying the Micro-Macro duality [16–19]:

$$
\begin{array}{ccc}
 & & \text{Localization of photons} \\
 & & \Uparrow \\
\text{Effective mass of photons} & \Longrightarrow & \text{Change in kinematics} \\
\Uparrow & & \\
\text{Dynamical interaction} & & \\
\text{between photons and the external system} & &
\end{array}
$$

In order to actualize the physical properties of a given system, such as photons driven by an invisible microscopic dynamics, we need to couple it with some external measuring system through which a composite system is formed. According to this formation of coupled dynamics, the kinematics controlling the observed photons is modified, and what can be actually observed is a result of this changed kinematics, realized in our case in the form of localized photons.

4.3. Araki–Borchers–Arveson Theorem on Space-Time Shifts

Before going into the details of mass generation mechanisms, we examine here the theoretical framework relevant to our context. For this purpose, we first need the qualification of the unitary operators $U(x)$ and their generators of space-time translations acting on a von Neumann algebra $\mathcal{X}$. Among several versions of such results due to Borchers [20,21], Arveson [22], *etc.*, the most relevant one to us can be found in the following theorem due to Araki [23]:

Theorem 4 (Araki [23])**.** *Let $(\mathcal{X}, \tau_a)$ be a W*-dynamical system consisting of a von Neumann algebra $\mathcal{X}$ in a Hilbert space $\mathfrak{H}$ acted upon by the group of space-time translations $a \in \mathbb{R}^4$ via automorphisms $\tau_a(A) = U(a)AU(a)^{-1}$ ($A \in \mathcal{X}$) implemented by a strongly continuous unitary representation $U(a) \in \mathcal{U}(\mathfrak{H})$. If $U(a)$ satisfies the spectrum condition with the spectrum of its generators contained in the forward light cone, then:*

(i) *each element of the center $\mathfrak{Z} \equiv \mathcal{X} \cap \mathcal{X}' = (\mathcal{X} \vee \mathcal{X}')'$ commutes with all $U(a)$: i.e., $U(a) \in (\mathcal{X} \vee \mathcal{X}')'' = \mathcal{X} \vee \mathcal{X}'$;*

(ii) *if there is a vector $\Omega \in \mathfrak{H}$ cyclic for $\mathcal{X}$ and invariant under $U(a)$, then $U(a)$ belongs to $\mathcal{X}$: $U(a) \in \mathcal{X}$.*

Putting the Newton–Wigner localizability in this context, we examine whether we can find some such operators in $\mathcal{X}$ as playing similar roles to the position operators $\hat{x}$ (affiliated with $\mathcal{X}$) behaving like $U(a)\hat{x}U(a)^{-1} = \hat{x} + a$. From this viewpoint, the problem of the localizability of quantum fields just reduces to examining whether a quantum-mechanical CCR $(\hat{x}, \hat{p})$ can be embedded in a von Neumann algebra $\mathcal{X}$ describing a system of quantum fields in the vacuum situation characterized by the spectrum condition. While this problem can be answered affirmatively for quantum fields with mass $m > 0$, it is not in the case with mass $m = 0$ and with spin $s > 0$.

The essence of the obstruction in the latter case is interpreted from Wightman's viewpoint in the above as the violation of the covariance condition $E(\tau_{(\mathbf{a},\mathcal{R})}(f)) = U(a,\mathcal{R})E(f)U(a,\mathcal{R})^{-1}$ to characterize the imprimitivity. From a more naive intuitive angle, the absence of localization operator $\hat{x}$ may also be taken tentatively as the absence of such operators in $\mathcal{X}$ (of massless theory) as non-commutative with $U(a)$, which means $U(a) \in \mathcal{X}'$, and hence, $U(a) \in \mathcal{X} \cap \mathcal{X}' = \mathfrak{Z}$. This means the non-triviality of the center $\mathfrak{Z} = \mathcal{X} \cap \mathcal{X}' \neq \mathbb{C}1$, as is avoided as much as possible in the usual approach to quantum theory sticking to the irreducibility of the algebra $\mathcal{X}$. Once one accepts the plain fact of the presence of superselection rules with non-trivial sector structures, however, the appearance of the non-trivial center is nothing mysterious. As has been elaborated in the context of Micro-Macro duality [16–19], the non-triviality of the center can be nicely and universally controlled by the concept of central decomposition, which provides us with the canonical path from a given expectation value ω to the actual measurement situations to exhibit the precise contents of ω according to its probabilistic interpretation based on the Born rule [24]. To see the precise contents of this statement, we should clarify the interplay between the group-theoretical context of the imprimitivity system and the algebraic aspect of the same situation formulated by Tomita's theorem of the integral decomposition of a state.

4.4. Tomita's Theorem of the Integral Decomposition of a State

From the mathematical viewpoint, an idealized form of constructing a coupled system of the object system with an external reference one can be found conveniently in Tomita's theorem of the integral decomposition of a state as follows:

Theorem 5 (Tomita [25]). *For a state ω of a unital C*-algebra $\mathcal{A}$, the following three sets are in a one-to-one correspondence:*

1. *sub-central measures μ (pseudo-)supported by the space $F_{\mathcal{A}}$ of factor states on $\mathcal{A}$;*
2. *abelian von Neumann subalgebras $\mathcal{B}$ of the center $\mathfrak{Z}_{\pi_\omega}(\mathcal{A}) = \pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})'$;*
3. *central projections C on $\mathfrak{H}_\omega$, such that:*

$$C\Omega_\omega = \Omega_\omega, \quad C\pi_\omega(\mathcal{A})C \subset \{C\pi_\omega(\mathcal{A})C\}' \tag{8}$$

If μ, $\mathcal{B}$ and C are in the above correspondence, then the following relations hold:

(i) $\mathcal{B} = \{\pi_\omega(\mathcal{A}) \cup \{C\}\}'$;

(ii) $C = [\mathcal{B}\Omega_\omega]$*: projection operator onto the subspace spanned by* $\mathcal{B}\Omega_\omega$;

(iii) $\mu(\hat{A}_1\hat{A}_2\cdots\hat{A}_n) = \langle\Omega_\omega|\, \pi_\omega(A_1)C\, \pi_\omega(A_2)C\cdots C\pi_\omega(A_n)\Omega_\omega\rangle$ *for* $A_1, A_2, \cdots, A_n \in \mathcal{A}$;

(iv) The map $\kappa_\mu : L^\infty(E_{\mathcal{A}}, \mu) \to \mathcal{B}$ *defined by:*

$$\langle\Omega_\omega|\, \kappa_\mu(f)\pi_\omega(A)\Omega_\omega\rangle = \int d\mu(\omega')f(\omega')\omega'(A) \tag{9}$$

for $f \in L^\infty(E_{\mathcal{A}}, \mu)$ *and* $A \in \mathcal{A}$ *is a *-isomorphism, satisfying the following equality for* $A, B \in \mathcal{A}$:

$$\kappa_\mu(\hat{A})\pi_\omega(B)\Omega_\omega = \pi_\omega(B)C\pi_\omega(A)\Omega_\omega \tag{10}$$

Some vocabulary in the above need be explained: the space $F_{\mathcal{A}}$ of factor states on $\mathcal{A}$ is the set of all of the factor states φ whose (GNS) representations π_φ have trivial centers: $\pi_\varphi(\mathcal{A})'' \cap \pi_\varphi(\mathcal{A})' = \mathbb{C}1_{\mathfrak{H}_\varphi}$. This $F_{\mathcal{A}}$ divided by the quasi-equivalence relation $\approx$ defined by the unitary equivalence up to multiplicity $F_{\mathcal{A}}/\approx$ plays the role of sector-classifying space (or sector space for short) whose elements we call "sectors" mathematically or "pure phases" physically. Then, Tomita's theorem plays a crucial role in verifying mathematically the so-called Born rule [24] postulated in quantum theory in physics.

Via the definition $\widehat{A}(\rho) := \rho(A)$, $\rho \in E_{\mathcal{A}}$, any element $A \in \mathcal{A}$ can be expressed by a continuous function $\widehat{A} : E_{\mathcal{A}} \to \mathbb{C}$ on the state space $E_{\mathcal{A}}$. Among measures on $E_{\mathcal{A}}$, a measure μ is called barycentric for a state $\omega \in E_{\mathcal{A}}$ if it satisfies $\omega = \int_{E_{\mathcal{A}}} \rho d\mu(\rho) \in E_{\mathcal{A}}$ and is said to be sub-central if linear functionals $\int_\Delta \rho d\mu(\rho)$ and $\int_{E_{\mathcal{A}}\setminus\Delta} \sigma d\mu(\sigma)$ on $\mathcal{A}$ are disjoint for any Borel set $\Delta \subset E_{\mathcal{A}}$, having no non-vanishing intertwiners between them: *i.e.*, $T \int_\Delta \pi_\rho(A) d\mu(\rho) = \int_{E_{\mathcal{A}}\setminus\Delta} \pi_\sigma(A) d\mu(\sigma) T$ for $\forall A \in \mathcal{A}$ implies $T = 0$ (by close analysis, the basic mechanism can be traced back to the very definition of disjointness of states in terms of the barycentric measure, by which the algebra $\mathcal{B}$ is abelianized and by which the entangled coupling is established between the spectrum of $\mathcal{B}$ and the states of $\mathcal{A}$). If the abelian subalgebra $\mathcal{B}$ in the above theorem is equal to the center $\mathcal{B} = \mathfrak{Z}_{\pi_\omega}(\mathcal{A})$, the measure μ is called the central measure of ω, determined uniquely $\mu = \mu_\omega$ by the state ω, and the corresponding barycentric decomposition $\omega = \int_{F_{\mathcal{A}}} \rho d\mu_\omega(\rho) = b(\mu_\omega)$ is called the central decomposition of ω. This last concept plays crucial roles in establishing precisely the bi-directional relations between microscopic and macroscopic aspects in quantum theory, as has been exhibited by the examples of Micro-Macro duality (see, for instance, [16–19]). In what follows, the connection to the previous subsection can be clarified through such notation that the von Neumann algebra $\pi_\omega(\mathcal{A})''$ corresponding via the representation π_ω to the C*-algebra $\mathcal{A}$ is denoted by $\mathcal{X} = \pi_\omega(\mathcal{A})''$, and hence, $\mathfrak{Z}_{\pi_\omega}(\mathcal{A}) = \pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})' = \mathcal{X} \cap \mathcal{X}' = \mathfrak{Z}$.

Now, we focus on the parallelism between Tomita's theorem of the integral decomposition of a state and the imprimitivity theorem, on the basis of the common features in the presence of commutative directions, $\mathcal{B} = \kappa_\mu(C(E_{\mathcal{X}}))''$ in the former and N or $\hat{N}$ in the latter: namely, we embed the group W* algebra $W^*(N) \simeq L^\infty(\hat{N})$ of the latter in $\mathcal{B} = \kappa_\mu(L^\infty(E_{\mathcal{X}}))$ of the former, which is consistent with their spectra, $\hat{N} \subset F_{\mathcal{X}}(\subset E_{\mathcal{X}})$. From the Lie algebraic viewpoint, the Lie algebra $\mathfrak{n}$ of space(-time) translations N is spanned by the (energy-)momentum $\hat{p}$, and hence, its dual $\hat{N}$ has an abelian Lie algebra $\hat{\mathfrak{n}}$ generated by $\hat{x}$ dual to $\hat{p}$ and which is embedded in the spectrum $Spec(\mathcal{B}) \subset F_{\mathcal{X}}$. To justify this correspondence, we need to supplement it by specifying how these group-theoretical entities act on the relevant mappings acting on the relevant algebraic structures constituting Tomita's theorem. In terms of the automorphic action $\tau_g \in Aut(\mathcal{X})$ of $g \in G = N \rtimes L$ on $\mathcal{X}$, we have $\omega(\tau_g(A)) = (\omega \circ \tau_g)(A)$ for $A \in \mathcal{X}$, and hence,

$$\omega \circ \tau_g = \int (\rho \circ \tau_g) d\mu_\omega(\rho) = \int \rho d\mu_{\omega\circ\tau_g}(\rho)$$

which implies $\tau_g^r(f)(\rho) := f(\rho \circ \tau_g)$ for $f \in C(E_{\mathcal{X}})$, because of:

$$\begin{aligned}\mu_{\omega\circ\tau_g}(f) &= \int f(\rho) d\mu_{\omega\circ\tau_g}(\rho) = \int f(\rho) d\mu_\omega(\rho \circ \tau_g^{-1}) \\ &= \int f(\rho' \circ \tau_g) d\mu_\omega(\rho') = \mu_\omega(\tau_g^r(f))\end{aligned}$$

In this way, the action $\rho \longmapsto \rho \circ \tau_g = \tau_g^*(\rho)$ of $g \in G = N \rtimes L$ on $\rho \in Spec(\mathcal{B}) \subset F_{\mathcal{X}}$ is just contravariant to τ_g. Thus, from the assumption on the symmetry corresponding to the subgroup L to be unbroken, the relation $\omega \circ \tau_g = \omega \circ \tau_{(x,l)} = \omega \circ \tau_x = \tau_x^*(\omega)$ follows for $g = (x, l) \in G = N \rtimes L$, which justifies $\hat{N} \subset F_{\mathcal{X}} = supp(\mu_\omega)$.

At first sight, the above distinction between central and sub-central may look too subtle, but it plays important roles in different treatments, for instance between spatial and spacetime degrees of freedom in Wightman's theorem concerning the localizability, as mentioned already after the theorem. In this connection, we consider the problem as to how classically visible configurations of electromagnetic field can be specified in close relation to its microscopic quantum behavior, for the purpose of which the most convenient concept seems to be the coherent state and the Segal–Bargmann transform associated with it. Since coherent states are usually treated within the framework of quantum mechanics for systems with finite degrees of freedom, the aspect commonly discussed is the so-called overcompleteness relations due to the non-orthogonality, $\langle\alpha|\beta\rangle \neq 0$, between coherent states $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with different coherence parameters $\alpha \neq \beta$.

We note that, in connection with Tomita's theorem, a composite system arises in such a form as $\mathcal{A} \otimes C(\Sigma)$ consisting of the object system $\mathcal{A}$ and of the external system $\Sigma(\subset F_{\mathcal{A}})$ to which measured data are to be registered through measurement processes involving $\mathcal{A}$. In this scheme, the universal reference system Σ can be viewed as naturally emergent from the object system $\mathcal{A}$ itself just as the classifying space of its sector structure. Then, via the logical extension [26] to parametrize the object system $\mathcal{A}$ by its sectors in Σ, an abstract model of quantum fields $\varphi : \Sigma \to \mathcal{A}$ can be created, constituting a crossed product $\varphi \in \mathcal{A} \rtimes \widehat{\mathcal{U}(\Sigma)}$ (via the co-action of the structure group $\mathcal{U}(\Sigma)$ of Σ). Thus, the above non-orthogonality can be resolved by the effects of the classifying parameters of sectors Σ in $F_{\mathcal{A}}$. As a result, we arrive at the quantum-probabilistic realization of coherent states in such a form as the "exponential vectors" treated by Obata [27] in the context of "Fock expansions" of white noises. What is important conceptually in this framework is the analyticity due to the Segal–Bargmann transform and the associated reproducing kernel (RK) to be identified through the projection operator P in $L^2(\Sigma, d\mu)$ onto its subspace $\mathcal{H}L^2(\Sigma, d\mu)$ of coherent states expressed by holomorphic functions on Σ [28], where $d\mu$ denotes the Gaussian measure.

As commented on briefly above, we can find various useful relations and connections of quantum theory in terms of the concept of "quantum fields". From this viewpoint, we elaborate on its roles in attaining a transparent understanding of the mutual relations among fields, particles and mass in the next section.

5. Quantum Field Ontology

5.1. From Particles to Fields

As we have discussed in Section 4, the scenario of generating effective mass applies to general settings. Any kinds of boundary conditions with finite volume (like cavity) will make photons heavier and slower, even without medium. This fact itself leads to a paradoxical physical question: how can the boundary condition affect a particle traveling in vacua? What is a spooky action through vacua?

Our answer is quite simple: in fact, a photon is not "a particle traveling in vacua". It is just a field filling the space time, before it "becomes" a particle, or more rigorously, before it appears in a particle-like event caused via the interaction (energy-momentum exchange with the external system). As we will discuss in this section, it is quite unreasonable to imagine a photon as a traveling particle unless any kinds of interaction are there.

On the basis of the above arguments, we discuss the limitation of the particle concept in connection with a new physical interpretation of Newton–Wigner–Wightman analysis. To begin with, we note the inconsistency between the concept of spin and that of a particle, which seems to have been forgotten at some stage in history. In fact, the concept of a classical massless point particle with non-zero spin cannot survive special relativity with the world line of such a particle obscured by the spin: instead of being a purely "internal" degree of freedom, the spin causes kinematical extensivity of the particle, which is exhibited in a boost transformation, as is pointed out by Bacry in [29].

The result of Newton–Wigner–Wightman analysis can be understood to show that this inconsistency cannot be eliminated by generalizing the problem in the context of quantum theory: a massless particle cannot be localized unless the spin is zero. Even in the massive case, the concept of localization is not independent of the choice of reference frames. There is no well-defined concept of "spacetime localization", as we have mentioned.

These facts are consistent with the idea that the position is not a clear cut *a priori* concept, but an emergent property. Instead of a point particle, therefore, we should find something else having a spacetime structure to accommodate events in point-like forms, which is nothing but the quantum field. In other words, the Newton–Wigner–Wightman analysis should be re-interpreted as "the existence proof of a quantum field", showing its inevitability.

5.2. From Fields to Particles: The Principle of Eventualization

This does not mean that the particle-like property is artificial nor fictional. On the contrary, point-like events do take place in any kind of elementary process of quantum measurement, such as exposure on a film, photon counting, and so on.

This apparent contradiction is solved if we adopt the universality of the indeterminate processes emerging as point-like events (energy-momentum exchanges) from quantum fields via formation of a composite system with external systems (like media or systems giving boundary conditions), even the latter coming from the part of the degrees of freedom of quantum fields. Let us call these fundamental processes eventualization. From our viewpoint, the most radical implication of Newton–Wigner–Wightman analysis is that we should abandon the ontology based on the naive particle picture and replace it by the one based on quantum fields with their eventualization.

At first sight, the idea of eventualization may look like just a palliative to avoid the contradiction between abstract theory of localization and the concrete localization phenomena. Actually, however, it opens the door to the quite natural formulation of quantum physics. In fact, the notion of the measurement process can be considered as a special kind of eventualization process with amplification. As we will discuss in a forthcoming paper [30], a glossary of "quantum paradoxes" is solved by just imposing an axiom that we call the "eventualization principle".

> Eventualization principle: Quantum fields can affect macroscopic systems only through eventualization.

In other words, we hypothesize that the notion of "macroscopic systems", including a Schrödinger cat, can be characterized, or defined, by the collection of events, formed by perpetual eventualization.

6. Conclusions

In the present article, we have tried to put the physical mechanism of photon localization in general and wide contexts, on the mathematical basis of Tomita's theorem of integral decompositions of states and on the conceptual one formulated as the eventualization. Owing to the latter one, invisible microscopic quantum processes are visualized at the macroscopic levels in the form of various types of "events".

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Author Contributions

Both authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

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