Mathematics **2015**, *3*, 1222-1240; doi:10.3390/math3041222

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

Robust Finite-Time Anti-Synchronization of Chaotic Systems with Different Dimensions

Israr Ahmad [1,2,*], Azizan Bin Saaban [1], Adyda Binti Ibrahim [1] and Mohammad Shahzad [2]

[1] School of Quantitative Sciences, College of Arts & Sciences, University Utara Malaysia, Sintok 06010, Kedah, Malaysia; E-Mails: azizan.s@uum.edu.my (A.B.S.); adyda@uum.edu.my (A.B.I.)

[2] Nizwa College of Applied Sciences, Ministry of Higher Education, Nizwa 611, Oman; E-Mail: dmsinfinite@gmail.com

* Author to whom correspondence should be addressed; E-Mail: iak_2000plus@yahoo.com; Tel.: +968-6049286304; Fax: +968-6049286309.

Academic Editor: Palle E. T. Jorgensen

Received: 14 July 2015 / Accepted: 30 November 2015 / Published: 8 December 2015

Abstract: In this paper, we demonstrate that anti-synchronization (AS) phenomena of chaotic systems with different dimensions can coexist in the finite-time with under the effect of both unknown model uncertainty and external disturbance. Based on the finite-time stability theory and using the master-slave system AS scheme, a generalized approach for the finite-time AS is proposed that guarantee the global stability of the closed-loop for reduced order and increased order AS in the finite time. Numerical simulation results further verify the robustness and effectiveness of the proposed finite-time reduced order and increased order AS schemes.

Keywords: anti-synchronization; finite-time stability theory; chaotic Lu system; hyperchaotic Li system

1. Introduction

In the last two decades, synchronization and anti-synchronization (AS) of chaotic systems have attracted considerable attention due to the potential applications in different scientific areas, such as secure communications [1], image encryption [2], laser technology [3], physical systems [4], and artificial neural networks [5]. As a result, a wide range of synchronization and AS control techniques and methods have been proposed. These include sliding mode control [6], active control [7], non-linear control [8], periodically intermittent control [9], adaptive control [10], H-∞ synchronization [11], projective synchronization [12], and generalized synchronization [13], *etc.* Anti-synchronization (AS) is a special form of generalized synchronization. AS is a mechanism in which the state vectors of the two synchronized chaotic systems have the same amplitude but opposite signs. That is, the sum of two signals converges to zero when the AS phenomenon appears. AS has successful applications in different scientific disciplines [14,15]. It has been experimentally, as well as numerically confirmed, that the two coupled chaotic systems can achieve AS [16,17]. The AS phenomenon in non-equilibrium systems suggests that it can be used as a technique for particle separation in a mixture of interacting particles [17]. A current study of using AS control of lasers, one can generate not only drop-outs in intensity but also short high-intensity pulses, and this results in pulses of particular shapes [18].

In recent years, anti-synchronization of chaotic systems with different dimensions has been reported in the literature [19–21]. Two types of AS have been discussed, namely reduced order AS [20] and increased order AS [21]. Mossa and Noorani [20] presented the idea of reduced order AS of chaotic systems with uncertain parameters. Using the adaptive control strategy based on the Lyapunov stability theory [22], the same authors [21] studied the increased order AS of uncertain hyperchaotic Lu and chaotic Lu systems.

However, the finding of these results [20,21] are limited to the asymptotic stability of the resulting AS behavior. This means that the corresponding state trajectories of the two coupled chaotic systems are anti-synchronized in an infinite settling time. In addition, the proposed reduced (increased) order AS of chaotic systems has been achieved without considering the effect of both model uncertainties and external disturbances. These results [20,21] would have been more interesting if it had included the finite-time AS behavior rather than merely asymptotic stability under the effect of both model uncertainties and external disturbances.

In real applications, it has been reported [22] that the finite-time AS stability is important to chaotic systems as the systems are required for AS quickly as possible. In case of chaotic systems with different dimensions, both systems have different topological properties and the traces changes in their respective trajectories with time are different. These properties of chaotic systems with different dimensions increase security in the communication channel and, therefore, require an effective approach to anti-synchronize chaotic systems with different dimensions in a finite-time.

Motivated by the aforesaid discussion, the purpose of this article is to make an innovative contribution in this direction. In this article, the authors study finite-time reduced (increased)-order AS behavior between two chaotic systems under the effect of both unknown model uncertainties and external disturbances. Using the master-slave system AS scheme, a generalized feedback control scheme will be proposed that would guarantee finite-time reduced (increased) order AS globally. Two illustrative examples are given to verify the robustness and performance of the proposed finite-time AS approach: finite-time reduced order AS between the hyperchaotic Li [23] and the chaotic Lu [24] systems and finite-time increased order AS between the chaotic Lu and the hyperchaotic Li systems. To the best of the authors' knowledge, no attempt has been made for the robust finite-time AS scheme for chaotic systems with different dimensions and this has remained an open problem.

The remainder of the paper is organized as follows: Section 2 presents a brief descriptions of the hyperchaotic Li and the chaotic Lu systems. In Section 3, the problem of finite-time reduced-order AS between the hyperchaotic Li and the chaotic Lu systems is solved. Accordingly, Section 4 is devoted to solve the finite-time increased-order AS problem between the chaotic Lu and the hyperchaotic Li systems. The paper concludes in Section 5.

2. Systems Descriptions

2.1. Hyperchaotic Li System

The vector form of the hyperchaotic Li system [23] which is described as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} -a & a & 0 & 0 \\ b & 0 & 0 & -1 \\ 0 & 0 & -c & 1 \\ d & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ x(t)z(t) \\ -x(t)y(t) \\ 0 \end{bmatrix} \tag{1}$$

where, $\left[x(t), y(t), z(t), w(t)\right]^T \in R^4$ are the state variables and $a > 0,\ b > 0,\ c > 0,\ d > 0,$ are the corresponding parameters of the system (1). The hyperchaotic Li system (1) exhibits a chaotic attractor for the parameters values: $a = 10,\ b = 35,\ c = 1.4$ and $d = 5$ as shown in the following Figure 1.

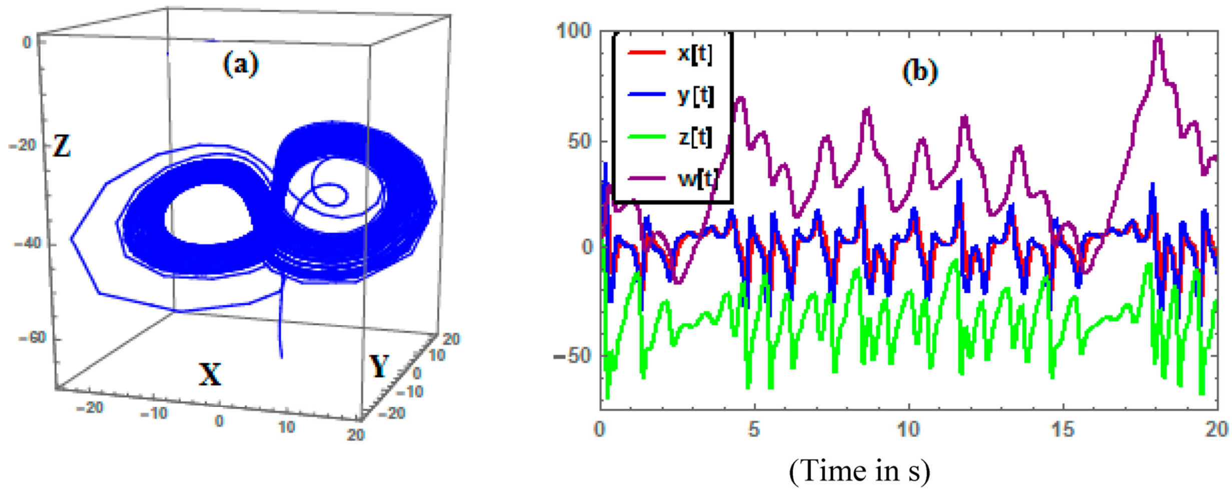


Figure 1. (**a**) 3D projection of the hyperchaotic Li system, and (**b**) time series of the state trajectories of the hyperchaotic Li system.

2.2. Chaotic Lu System

The vector form of the chaotic Lu system [24] which is described as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -x(t)z(t) \\ x(t)y(t) \end{bmatrix} \tag{2}$$

where, $\left[x(t), y(t), z(t)\right] \in R^3$ are the state variables and $\alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\delta > 0$ are the corresponding parameters of the system Equation (2). The chaotic Lu system exhibits a chaotic attractor with the parameter values: $\alpha = 36,\ \beta = 3,\ \gamma = 20$ as shown in the following Figure 2.

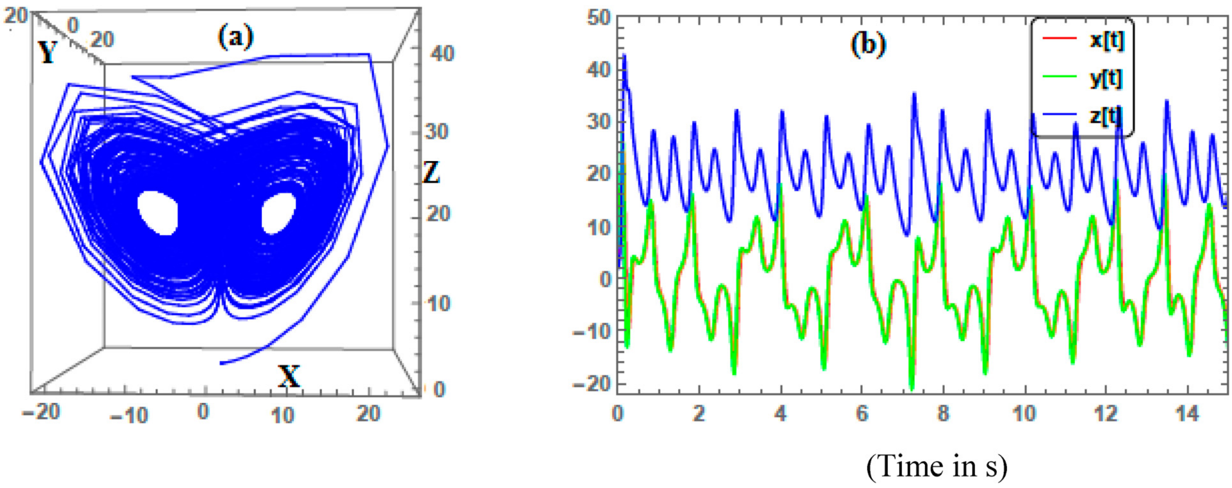


Figure 2. (**a**) 3D projection of the chaotic Lu attractor, and (**b**) time series of the state trajectories for the chaotic Lu system.

3. Finite-Time Reduced Order Anti-Synchronization Scheme

3.1. Some Basic Preliminaries and Lemmas

Lemma 1 [25]. For any $\Omega \in R^+$ and $X,\ Y \in R$, the following inequality holds true:

$$2|X||Y| \le \Omega X^2 + \Omega^{-1} Y^2$$

Proof of Lemma 1. Consider the following inequality:

$$2\sqrt{M_A M_G} \le M_A + M_G,\ \forall\ M_A \ge 0,\ M_G \ge 0 \tag{3}$$

Which holds true for M_A and M_G being the arithmetic and geometric means respectively. Now substituting $M_A = \Omega X^2$ and $M_G = \Omega^{-1} Y^2$ to the above inequality Equation (3) that yields the following form:

$$2|X||Y| = 2\sqrt{X^2 Y^2} \le \Omega X^2 + \Omega^{-1} Y^2$$

Lemma 2. If $p \in (0,\ 1)$ and $a,\ b \ge 0$, then, the following inequality holds true:

$$(a+b)^p \le a^p + b^p \tag{4}$$

Proof of Lemma 2. Let us assumed the following function defined as:

$$f(t) = (1+t)^p - 1 - t^p,\ \text{ for } t \ge 0 \tag{5}$$

then,

$$f'(t) = p(1+t)^{p-1} - pt^{p-1} < 0, \text{ for all } t \in (0,\ \infty) \tag{6}$$

Since $f(0)=0$, it follows that $f'(t)<0$ on $(0,\ \infty)$.

If $a,\ b\neq 0,$ then, substituting $t=\dfrac{a}{b}$ in Equation (6) that yields:

$$\left(1+\frac{a}{b}\right)^{p}-1-\left(\frac{a}{b}\right)^{p}<0$$

$$\Leftrightarrow\left(\frac{a+b}{b}\right)^{p}-1-\left(\frac{a}{b}\right)^{p}<0$$

$$\Leftrightarrow(a+b)^{p}-\left(a^{p}+b^{p}\right)<0$$

$$\Leftrightarrow(a+b)^{p}<a^{p}+b^{p}$$

Lemma 3 [26]. Assume that there exists a continuous positive definite function $V(t):R^{n}\to R^{n}$ such that $V(t)$ is radially unbounded and satisfies the following differential inequality:

$$\dot{V}(t)\leq-\psi\left(V(t)\right)^{\rho},\ \forall\ \ V(t_0)\geq 0,\ t\geq t_0 \tag{7}$$

where $\psi>0$ and $0<\rho<1$ are two constant numbers. Then, for any t_0, $V(t)$ satisfies the following inequality:

$$V(t)^{1-\rho}\leq V(t_0)^{1-\rho}-\psi(1-\rho)(t-t_0),\quad t_0\leq t\leq T_1 \tag{8}$$

and

$$V(t)\equiv 0,\ \ \forall\ t\geq T_1$$

Then, the origin is globally stable in the finite-time T_1. The settling time T_1 is given as follows:

$$T_1=t_0+\frac{1}{\psi(1-\rho)}V(t_0)^{1-\rho} \tag{9}$$

3.2. Problem Statement

To achieve finite-time reduced-order AS between the hyperchaotic Li and chaotic Lu systems, it is assumed that the projection part of the hyperchaotic Li system is considered as the master system and is described as follows:

$$\left.\begin{array}{l}(\text{Master system})\\ \dot{x}_1(t)=a\left(y_1(t)-x_1(t)\right)+D_1\left(x_1(t)\right)\\ \dot{y}_1(t)=bx_1(t)-w_1(t)+x_1(t)z_1(t)+D_2\left(y_1(t)\right)\\ \dot{z}_1(t)=-cz_1(t)+w_1(t)-x_1(t)y_1(t)+D_3\left(z_1(t)\right)\end{array}\right\} \tag{10}$$

where $\left[x_1(t),\ y_1(t),\ z_1(t),\ w_1(t)\right]^{T}\in R^{4}$ are the state variables, $a,\ b,\ c$ and d are the corresponding control parameters of the master system Equation (10), respectively. $D_i\ \ (i=1,\ 2,\ 3)$ are the unknown model uncertainties and external disturbances present in the master system. Likewise, the Lu chaotic system is considered as the slave system and is described as follows:

$$\left.\begin{array}{l}(\text{Slave system})\\ \dot{x}_2(t)=\alpha\left(y_2(t)-x_2(t)\right)+d_1\left(x_2(t)\right)+u_1(t)\\ \dot{y}_2(t)=\gamma y_2(t)-x_2(t)z_2(t)+d_2\left(y_2(t)\right)+u_2(t)\\ \dot{z}_2(t)=-\beta z_2(t)+x_2(t)y_2(t)+d_3\left(z_2(t)\right)+u_3(t)\end{array}\right\} \tag{11}$$

where $\left[x_2(t),\ y_2(t),\ z_2(t)\right]^T \in R^3$ are the state variables, $\alpha,\ \beta,\ \gamma$ are the corresponding control parameters of the slave system respectively. $d_i\left(i=1,\ 2,\ 3\right)$ are the unknown model uncertainties and external disturbances present in the slave system, and $\boldsymbol{u}(t)\in R^3$ is the control input that is yet to be deigned.

Let $\boldsymbol{e}(t)=X(t)+Y(t),?\ \boldsymbol{e}(t)\in R^3$ be the AS error vector. Then, the time varying AS error system of the master (10) and slave (11) systems is described as below:

$$\left.\begin{array}{l}\dot{e}_1(t)=\alpha\left(e_2(t)-e_1(t)\right)+(a-\alpha)\left(y_1(t)-x_1(t)\right)+d_1\left(x_2(t)\right)+D_1\left(x_1(t)\right)+u_1(t)\\ \dot{e}_2(t)=be_1(t)+\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)-bx_2(t)-\gamma y_1(t)-w_1(t)+d_2\left(y_2(t)\right)+\\ \qquad D_2\left(y_1(t)\right)+u_2(t)\\ \dot{e}_3(t)=-\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+(\beta-c)z_1(t)+w_1(t)+d_3\left(z_2(t)\right)+D_3\left(z_1(t)\right)+u_3(t)\end{array}\right\} \tag{12}$$

Under these circumstances, it is desired to design a feedback controller that synthesis a smooth control input $\boldsymbol{u}(t)$. This control inputs accomplishes the reduced order AS within finite-time $T_1=T_1\left(\boldsymbol{e}(0)\right)>0$.

The reduced order AS objectives are summarized as follows:

Objective 1. The reduced order AS scheme accomplishes if:

$$\lim_{t\to T_1}\left[e_1(t)\right]=\lim_{t\to T_1}\left[x_1(t)+x_2(t)\right]=[0],\ \lim_{t\to T_1}\left[e_2(t)\right]=\lim_{t\to T_1}\left[y_1(t)+y_2(t)\right]=[0],$$
$$\lim_{t\to T_1}\left[e_3(t)\right]=\lim_{t\to T_1}\left[z_1(t)+z_2(t)\right]=[0]$$

Objective 2. The reduced order AS error dynamical system (12) are globally stable in the finite-time T_1 given by Equation (16).

Assumption 1. Since chaotic trajectories are always bounded, there exist a positive constant U such that [25]:

$$U_x\geq|x_i|,\ U_y\geq|y_i|,\ U_z\geq|z_i|,\ U_w\geq|w_i|\quad i=1,\ 2$$

Assumption 2. It is assumed that the unknown model uncertainties and external disturbances are Norm-bounded [27]. That is:

$$\begin{array}{l}\left|D_1\left(x_1(t)\right)\right|\leq\psi_1,\ \left|D_2\left(y_1(t)\right)\right|\leq\psi_2,\ \left|D_3\left(z_1(t)\right)\right|\leq\psi_3,\ \left|D_4\left(w_1(t)\right)\right|\leq\psi_4,\\ \left|d_1\left(x_2(t)\right)\right|\leq\vartheta_1,\ \left|d_2\left(y_2(t)\right)\right|\leq\vartheta_2,\ \left|d_3\left(z_2(t)\right)\right|\leq\vartheta_3\end{array} \tag{13}$$

Accordingly, it is concluded that:

$$\left|d_1\left(x_2(t)\right)+D_1\left(x_1(t)\right)\right|\le\phi_1,\ \left|d_2\left(y_2(t)\right)+D_2\left(y_1(t)\right)\right|\le\phi_2,\ \left|d_3\left(z_2(t)\right)+D_3\left(z_1(t)\right)\right|\le\phi_3 \tag{14}$$

where ψ_i, ϑ_i and ϕ_i are unknown positive constants.

3.3. Controller Design

Theorem 1. For the arbitrary initial conditions, $\left(x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)\in R^4\neq x_2(0),\ y_2(0),\ z_2(0)\in R^3\right)$; of the master and slave systems respectively, the error system Equation (12) will converge to zero globally under the control law given by:

$$\left.\begin{aligned} u_1(t)&=(a-\alpha)\left(x_1(t)-y_1(t)\right)-\phi_1-l_1e_1(t)-e_1^{\rho}(t)\\ u_2(t)&=bx_2(t)+\gamma y_1(t)+w_1(t)-\phi_2-l_2e_2(t)-e_2^{\rho}(t)\\ u_3(t)&=(c-\beta)z_1(t)-w_1(t)-\phi_3-l_3e_3(t)-e_3^{\rho}(t)\end{aligned}\right\} \tag{15}$$

in the finite-time T_1 determined by the following:

$$T_1=\left(\sqrt{2}\right)^{1-\rho}\frac{1}{(1-\rho)}\left(\sqrt{\frac{1}{2}\left(e_1^2(0)+e_2^2(0)+e_3^2(0)\right)}\right)^{1-\rho},\quad \left\|\boldsymbol{e}(t)\right\|\equiv 0,\ \ t\ge T_1 \tag{16}$$

Then, objectives one and two are accomplished.

Proof of Theorem 1. Construct a Lyapunov function candidate as follows:

$$V\left(\boldsymbol{e}(t)\right)=\frac{1}{2}\left(\sum_{1}^{3}e_i^T(t)e_i(t)\right)\ge 0 \tag{17}$$

where $V\left(\boldsymbol{e}(t)\right)$ is a positive definite function. Now calculate the time derivatives of $V\left(\boldsymbol{e}(t)\right)$ along the trajectories of the error system Equation (12) that yields:

$$\dot V\left(e(t)\right)=\left(\begin{array}{l} e_1(t)\left[\begin{array}{l}\alpha\left(e_2(t)-e_1(t)\right)+(a-\alpha)\left(y_1(t)-x_1(t)\right)\\ +d_1\left(x_2(t)\right)+D_1\left(x_1(t)\right)+u_1(t)\end{array}\right]+\\ e_2(t)\left[\begin{array}{l}be_1(t)+\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)-\\ bx_2(t)-\gamma y_1(t)-w_1(t)+d_2\left(y_1(t)\right)+D_2\left(y_1(t)\right)+u_2(t)\end{array}\right]+\\ e_3(t)\left[\begin{array}{l}-\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+(\beta-c)z_1(t)\\ +w_1(t)+d_3\left(z_1(t)\right)+D_3\left(z_1(t)\right)+u_3(t)\end{array}\right]\end{array}\right) \tag{18}$$

Now introducing the control inputs, $u_i(t)$, $i=1,\ 2,\ 3$, Equation (15) to the right hand side of Equation (18) yields:

$$\dot{V}\left(e(t)\right) \le \begin{pmatrix} e_1(t)\begin{bmatrix} \alpha\left(e_2(t)-e_1(t)\right)+(a-\alpha)\left(y_1(t)-x_1(t)\right)+\left|d_1\left(x_2(t)\right)+D_1\left(x_1(t)\right)\right| \\ +(a-\alpha)\left(x_1(t)-y_1(t)\right)-\phi_1-l_1e_1(t)-e_1^{\rho}(t) \end{bmatrix}+ \\ e_2(t)\begin{bmatrix} be_1(t)+\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)-bx_2(t)-\gamma y_1(t)-w_1(t) \\ +\left|d_2\left(y_2(t)\right)+D_2\left(y_1(t)\right)\right|+bx_2(t)+\gamma y_1(t)+w_1(t)-\phi_2-l_2e_2(t)-e_2^{\rho}(t) \end{bmatrix}+ \\ e_3(t)\begin{bmatrix} -\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+w_1(t)+(\beta-c)z_1(t)+ \\ \left|d_3\left(z_2(t)\right)+D_3\left(z_1(t)\right)\right|-w_1(t)+(c-\beta)z_1(t)-\phi_3-l_3e_3(t)-e_3^{\rho}(t) \end{bmatrix} \end{pmatrix} \tag{19}$$

$$\Rightarrow\ \dot{V}\left(e(t)\right) \le \begin{bmatrix} -(l_1+\alpha)e_1^2(t)-(l_2-\gamma)e_2^2(t)-(l_3+\beta)e_3^2(t) \\ +\left(\alpha+b+z_1(t)\right)e_1(t)e_2(t)-y_1(t)e_1(t)e_3(t)-\sum_1^3 e_i^{\rho+1}(t) \end{bmatrix}$$

$$\Rightarrow\ \dot{V}\left(e(t)\right) \le \begin{bmatrix} -(l_1+\alpha)e_1^2(t)-(l_2-\gamma)e_2^2(t)-(l_3+\beta)e_3^2(t) \\ +\left(\alpha+b+U_z\right)\left|e_1(t)\right|\left|e_2(t)\right|+U_y(t)\left|e_1(t)\right|\left|e_3(t)\right|-\sum_1^3 e_i^{\rho+1}(t) \end{bmatrix}$$

Using Lemma 1, Equation (19) yields:

$$\dot{V}\left(e(t)\right) \le -\begin{bmatrix} \left(l_1+\alpha-\dfrac{\Omega_1\left(\alpha+b+U_z\right)^2}{2}-\dfrac{\Omega_2^{-1}U_y^{\ 2}}{2}\right)e_1^2(t) \\ +\left(l_2-\gamma-\dfrac{\Omega_1^{-1}\left(\alpha+b+U_z\right)^2}{2}\right)e_2^2(t) \\ +\left(l_3+\beta-\dfrac{\Omega_1 U_y^{\ 2}}{2}\right)e_3^2(t)-\sum_1^3 e_i^{\rho+1}(t) \end{bmatrix} \tag{20}$$

where Ω_1 and Ω_2 are two positive constants. Let us choose l_1, l_2 and l_3 as follows:

$$l_1 \ge \left(-\alpha+\frac{\Omega_1\left(\alpha+b+U_z\right)^2}{2}+\frac{\Omega_2^{-1}U_y^{\ 2}}{2}\right),\ l_2 \ge \gamma+\frac{\Omega_1^{-1}\left(\alpha+b+U_z\right)^2}{2}\ \text{ and }\ l_3 \ge -\beta+\frac{\Omega_2 U_y^{\ 2}}{2} \tag{21}$$

Using Equation (21), then, Equation (20) becomes as follows:

$$\begin{gathered} \dot{V}\left(e(t)\right) \le -\left(\left(e_1^{\rho+1}(t)+e_2^{\rho+1}(t)\right)+e_3^{\rho+1}(t)\right) \\ \Rightarrow\ \dot{V}\left(e(t)\right) \le -2^{\frac{\rho+1}{2}}\left(\left(\frac{1}{2}e_1^2(t)\right)^{\frac{\rho+1}{2}}+\left(\frac{1}{2}e_2^2(t)\right)^{\frac{\rho+1}{2}}+\left(\frac{1}{2}e_3^2(t)\right)^{\frac{\rho+1}{2}}\right) \end{gathered} \tag{22}$$

Using Lemma 2 to the above Equation (22), we obtain the following inequality:

$$\begin{gathered} \dot{V}\left(e(t)\right) \le -2^{\frac{\rho+1}{2}}\left(\frac{1}{2}\left(e_1^2(t)+e_2^2(t)+e_3^2(t)\right)\right)^{\frac{\rho+1}{2}} \\ \Rightarrow\ \dot{V}\left(e(t)\right) \le -2^{\frac{\rho+1}{2}}\left(V\left(e(t)\right)\right)^{\frac{\rho+1}{2}} \le 0 \end{gathered} \tag{23}$$

Thus, according to Lemma 3, the AS error system Equation (12) will converge to zero in the finite-time T_1. Accordingly, the slave chaotic Lu system will synchronize globally with the projection part of the hyperchaotic Li system in the finite-time T_1.

3.4. Simulation and Results

In this sub-section of the article, numerical simulation results using Mathematica 10*v* (Nizwa, Oman) are provided to verify the robustness and effectiveness of the proposed finite-time reduced order AS approach. Parameters of the hyperchaotic Li Equation (1) and the chaotic Lu Equation (2) systems which are set as: $a = 10,\ b = 35,\ c = 1.4,\ d = 5$ and $\alpha = 36,\ \beta = 3,\ \gamma = 20$, respectively. The initial states of the master and slave systems being taken as: $\left[x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)\right]^T = \left[1,\quad 2,\quad 3,\quad 4\right]^T$ and $\left[x_2(0),\ y_2(0),\ z_2(0)\right]^T = \left[2,\ 1,\ -1\right]^T$ respectively. According to the Theorem 1, the linear controller gains $l_1,\ l_2$ and l_3 are chosen as $l_1 = 30,\ l_2 = 35$ and $l_3 = 25$ and the constant number ρ being taken as $\rho = \dfrac{7}{10}$. In numerical simulation, the following unknown model uncertainties and external disturbances are applied to the master Equation (10) and slave Equation (11) systems respectively:

$$
\begin{aligned}
&D_1\left(x_1(t)\right) = -0.3\cos\left(5x_1(t)\right) + 0.2\sin(6t), && d_1\left(x_2(t)\right) = 0.25\cos\left(5x_2(t)\right) - 0.3\cos(3t) \\
&D_2\left(y_1(t)\right) = -0.25\sin\left(3y_1(t)\right) - 0.15\cos(5t), && d_2\left(y_2(t)\right) = 0.2\sin\left(4y_2(t)\right) + 0.15\sin(5t) \\
&D_3\left(z_1(t)\right) = -0.35\sin\left(4z_1(t)\right) + 0.3\sin(7t), && d_3\left(z_2(t)\right) = 0.3\sin\left(5z_2(t)\right) - 0.2\cos(4t)
\end{aligned}
$$

As a result, one can obtain $\phi_1 = 0.55,\ \phi_2 = 0.4$ and $\phi_3 = 0.6$. The corresponding numerical results are as follows.

Figure 3, depicts the time series of the convergence of the reduce order AS error signals to the zero state. As expected, one can observe that the AS error signals converged to the zero state quickly, which demonstrates the robustness and performance of the control action Equation (15) for the finite-time reduced-order AS scheme. Figure 4, illustrates the time series of the finite-time T_1 Equation (16). From the main Theorem 1, it can be checked that slave system Equation (11) is anti-synchronized with the projection part of the master system Equation (10) in the finite-time, when the control inputs are activated at $t = 0.3s$. Time series of the control inputs are depicted in Figures 5–7.

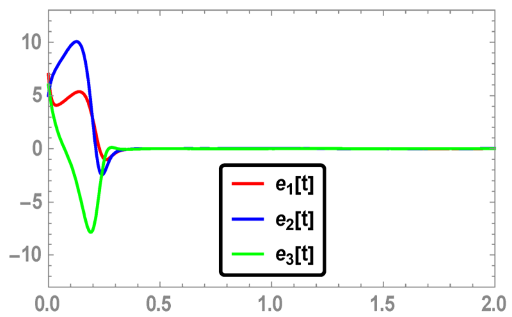


Figure 3. Times series of the AS error signals.

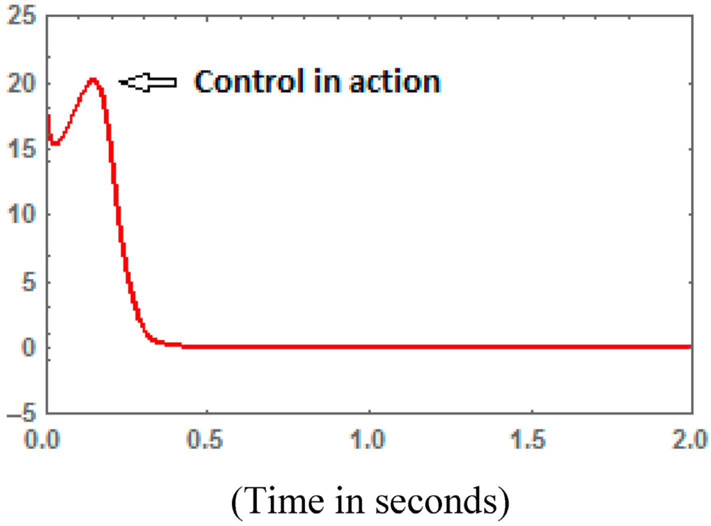


Figure 4. Time series of the estimated finite-time T_1.

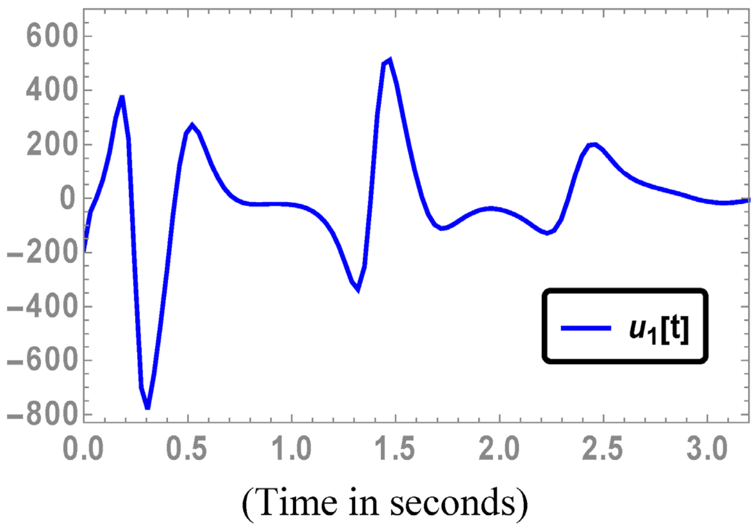


Figure 5. Times series of the control input $u_1(t)$.

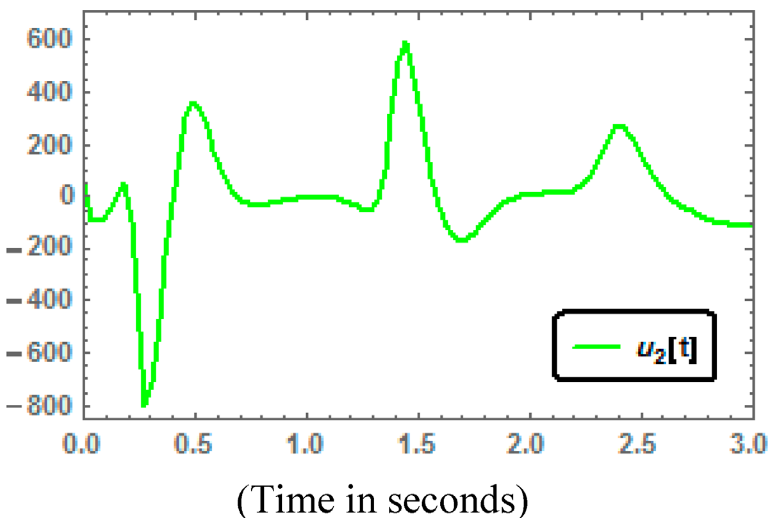
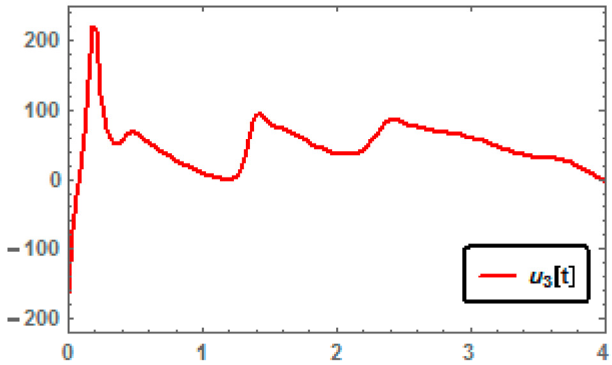


Figure 6. Time series of the control input $u_2(t)$.



(Time in seconds)

Figure 7. Times series of the control input $u_3(t)$.

4. Finite-Time Increased Order Anti-Synchronization Scheme

4.1. Problem Statement

To achieve the finite-time increased order AS, the chaotic Lu system is considered as the master system and the hyperchaotic Li system as the slave system. Let us consider the following master-slave system increased order AS scheme between the chaotic Lu and the hyperchaotic Li systems as:

$$\left.\begin{aligned}
&(\text{Master system})\\
&\dot{x}_1(t)=\alpha\left(y_1(t)-x_1(t)\right)+d_1\left(x_1(t)\right)\\
&\dot{y}_1(t)=\gamma y_1(t)-x_1(t)z_1(t)+d_2\left(y_1(t)\right)\\
&\dot{z}_1(t)=-\beta z_1(t)+x_1(t)y_1(t)+d_3\left(z_1(t)\right)\\
&(\text{Slave system})\\
&\dot{x}_2(t)=a\left(y_2(t)-x_2(t)\right)+D_1\left(x_2(t)\right)+u_1(t)\\
&\dot{y}_2(t)=bx_2(t)-w_2(t)+x_2(t)z_2(t)+D_2\left(y_2(t)\right)+u_2(t)\\
&\dot{z}_2(t)=-cz_2(t)+w_2(t)-x_2(t)y_2(t)+D_3\left(z_2(t)\right)+u_3(t)\\
&\dot{w}_2(t)=dx_2(t)+y_2(t)+D_4\left(w_2(t)\right)+u_4(t)
\end{aligned}\right\} \tag{24}$$

where $\left[x_1(t),\ y_1(t),\ z_1(t)\right]^T\in R^3$ and $\left[x_2(t),\ y_2(t),\ z_2(t),\ w_2(t)\right]^T\in R^4$ are the state variables, $\alpha,\ \beta,\ \gamma$ and $a,\ b,\ c,?d$ are the control parameters of the master and slave systems alternatively. $d_j\left(j=1,\ 2,\ 3\right)$ and $D_i\ \left(i=1,\ 2,\ 3,\ 4\right)$ are the unknown model uncertainties and external disturbances present in the master and slave systems respectively and $\boldsymbol{u}(t)\in R^4$ is the control input. The error system for the increased order AS scheme Equation (24) is described as follows:

$$\left.\begin{aligned}&\dot{e}_1(t)=\alpha\left(e_2(t)-e_1(t)\right)+(a-\alpha)\left(y_1(t)-x_1(t)\right)+D_1\left(x_2(t)\right)+d_1\left(x_1(t)\right)+u_1(t)\\&\dot{e}_2(t)=\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)+bx_2(t)-\gamma y_2(t)-w_2(t)+D_2\left(y_2(t)\right)\\&\qquad+d_2\left(y_1(t)\right)+u_2(t)\\&\dot{e}_3(t)=-\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+(\beta-c)z_2(t)+w_2(t)+D_3\left(z_2(t)\right)+d_3\left(z_1(t)\right)+u_3(t)\\&\dot{e}_4(t)=de_1(t)-dx_1(t)+y_2(t)+D_4\left(w_2(t)\right)+u_4(t)\end{aligned}\right\}\tag{25}$$

Under these conditions, it is desired to design a controller that synthesis a smooth control input $\boldsymbol{u}(t)$. This control inputs accomplishes the increased order AS within finite-time $T_2=T_2\left(\boldsymbol{e}(0)\right)>0$. The increased order AS objectives are summarized as follows:

Objective 3. The increased order AS scheme is accomplished if:

$$\lim_{t\to T_2}\left[e_1(t)\right]=\lim_{t\to T_2}\left[x_1(t)+x_2(t)\right]=[0],\ \lim_{t\to T_2}\left[e_2(t)\right]=\lim_{t\to T_2}\left[y_1(t)+y_2(t)\right]=[0],$$
$$\lim_{t\to T_2}\left[e_3(t)\right]=\lim_{t\to T_2}\left[z_1(t)+z_2(t)\right]=[0],\ \lim_{t\to T_2}\left[e_4(t)\right]=\lim_{t\to T_2}\left[w_1(t)+w_2(t)\right]=[0]$$

Objective 4. The AS error dynamical system Equation (24) is globally stable in the finite time T_2 given by Equation (27).

4.2. Controller Design

Theorem 2. For the arbitrary initial conditions: $\left(x_1(0),\, y_1(0),\, z_1(0)\in R^3\neq x_2(0),\, y_2(0),\, z_2(0),\, w_2(0)\in R^4\right)$; of the master and slave systems respectively, the error system Equation (25) will converge to zero globally under the control law given by:

$$\left.\begin{aligned}&u_1(t)=(a-\alpha)\left(x_1(t)-y_1(t)\right)-\phi_1-l_1e_1(t)-e_1^{\rho}(t)\\&u_2(t)=-bx_2(t)+\gamma y_2(t)+w_2(t)-\phi_2-l_2e_2(t)-e_2^{\rho}(t)\\&u_3(t)=(c-\beta)z_2(t)-w_2(t)-\phi_3-l_3e_3(t)-e_3^{\rho}(t)\\&u_4(t)=dx_1(t)-y_2(t)-\psi_4-l_4e_4(t)-e_4^{\rho}(t)\end{aligned}\right\}\tag{26}$$

in the finite-time T_2, determined by:

$$T_2=\left(\sqrt{2}\right)^{1-\rho}\frac{1}{(1-\rho)}\left(\sqrt{\frac{1}{2}\left(e_1^2(0)+e_2^2(0)+e_3^2(0)+e_4^2(0)\right)}\right)^{1-\rho},\ \ \left\|\boldsymbol{e}(t)\right\|\equiv 0,\ \ t\geq T_2\tag{27}$$

Then, the objectives 3 and 4 are accomplished.

Proof of Theorem 2. Construct a Lyapunov function candidate as follows:

$$V\left(\boldsymbol{e}(t)\right)=\frac{1}{2}\left(\sum_{1}^{4}e_i^T(t)e_i(t)\right)\geq 0\tag{28}$$

Now calculate the time derivatives of $V\left(\boldsymbol{e}(t)\right)$ along the trajectories of the error system Equation (25) that yields:

$$\dot{V}(e(t))=\begin{pmatrix} e_1(t)\left[\alpha(e_2(t)-e_1(t))+(a-\alpha)(y_1(t)-x_1(t))+d_1(x_1(t))+D_1(x_1(t))+u_1(t)\right] \\ +e_2(t)\begin{bmatrix}\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)+bx_2(t)-\gamma y_2(t) \\ -w_2(t)+d_2(y_1(t))+D_2(y_2(t))+u_2(t)\end{bmatrix} \\ +e_3(t)\begin{bmatrix}-\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+(\beta-c)z_2(t)+w_2(t) \\ +d_3(z_1(t))+D_3(z_2(t))+u_3(t)\end{bmatrix} \\ +e_4(t)\left[de_1(t)-dx_1(t)+y_2(t)+D_4(w_2(t))+u_4(t)\right]\end{pmatrix} \tag{29}$$

Now introducing the control inputs $u_i(t)$, $i=1, 2, 3, 4$, Equation (26) to the right hand side of Equation (29) that yields:

$$\begin{aligned}\Rightarrow \dot{V}(e(t)) &\le \begin{pmatrix} e_1(t)\begin{bmatrix}\alpha(e_2(t)-e_1(t))+(a-\alpha)(y_1(t)-x_1(t))+\left|d_1(x_2(t))+D_1(x_1(t))\right| \\ +(a-\alpha)(x_1(t)-y_1(t))-\phi_1-l_1e_1(t)-e_1^{\rho}(t)\end{bmatrix} \\ +e_2(t)\begin{bmatrix}\gamma e_2(t)+z_1(t)e_1(t)-x_2(t)e_3(t)+bx_2(t)-\gamma y_2(t)-w_2(t) \\ +\left|d_2(y_2(t))+D_2(y_1(t))\right|-w_2(t)-bx_2(t)+\gamma y_2(t)+w_2(t)-\phi_2- \\ l_2e_2(t)-e_2^{\rho}(t)\end{bmatrix} \\ +e_3(t)\begin{bmatrix}-\beta e_3(t)-y_1(t)e_1(t)+x_2(t)e_2(t)+(\beta-c)z_2(t)+w_2(t) \\ +\left|d_3(z_2(t))+D_3(z_1(t))\right|-(\beta-c)z_2(t)-w_2(t)-\phi_3-l_3e_3(t)-e_3^{\rho}(t)\end{bmatrix} \\ +e_4(t)\left[de_1(t)-dx_1(t)+y_2(t)+\left|d_4(w_2(t))\right|+dx_1(t)-y_2(t)-\psi_4-l_4e_4(t)-e_4^{\rho}(t)\right]\end{pmatrix} \\ \Rightarrow \dot{V}(e(t)) &\le \begin{pmatrix}-(\alpha+l_1)e_1^2(t)-(l_2-\gamma)e_2^2(t)-(\beta+l_3)e_3^2(t)-l_4e_4^2(t) \\ +(\alpha+z_1(t))e_1(t)e_2(t)-y_1(t)e_1(t)e_3(t)+de_1(t)e_4(t)-\sum_1^4 e_i^{\rho+1}(t)\end{pmatrix} \\ \Rightarrow \dot{V}(e(t)) &\le \begin{pmatrix}-(\alpha+l_1)e_1^2(t)-(l_2-\gamma)e_2^2(t)-(\beta+l_3)e_3^2(t)-l_4e_4^2(t) \\ +(\alpha+U_z)\left|e_1(t)\right|\left|e_2(t)\right|+U_y\left|e_1(t)\right|\left|e_3(t)\right|+d\left|e_1(t)\right|\left|e_4(t)\right|-\sum_1^4 e_i^{\rho+1}(t)\end{pmatrix}\end{aligned} \tag{30}$$

Using Lemma 1, then, Equation (30) yields as follows:

$$\dot{V}(e(t))\le-\begin{bmatrix}\left(l_1+\alpha-\frac{\Omega_1(\alpha+U_z)^2}{2}-\frac{\Omega_2^{-2}U_y^2}{2}-\frac{d^2}{2}\right)e_1^2(t)+\left(l_2-\gamma-\frac{\Omega_2^{-2}(\alpha+U_z)^2}{2}\right)e_2^2(t) \\ +\left(l_3+\beta-\frac{\Omega_1U_y^2}{2}\right)e_3^2(t)+\left(l_4-\frac{d^2}{2}\right)e_4^2-\sum_1^4 e_i^{\rho+1}(t)\end{bmatrix} \tag{31}$$

where Ω_1 and Ω_2 are two positive constants. Let us choose l_1, l_2, l_3 and l_4 such that:

$$\begin{aligned}&l_1\ge-\alpha+\frac{\Omega_1(\alpha+U_z)^2}{2}+\frac{\Omega_2^{-2}U_y^2}{2}+\frac{d^2}{2},\ l_2\ge\gamma+\frac{\Omega_2^{-2}(\alpha+b-U_z)^2}{2} \\ &l_3\ge-\beta+\frac{\Omega_1U_y^2}{2}\ \text{and}\ l_4\ge\frac{d^2}{2}\end{aligned} \tag{32}$$

Using Equation (32), then, Equation (31) $\Rightarrow$

$$\dot{V}(e(t)) \le -\left(e_1^{\rho+1}(t)+e_2^{\rho+1}(t)+e_3^{\rho+1}(t)+e_4^{\rho+1}(t)\right)$$

$$\Rightarrow \dot{V}(e(t)) \le -2^{\frac{\rho+1}{2}}\left(\left(\frac{1}{2}e_1^2(t)\right)^{\frac{\rho+1}{2}}+\left(\frac{1}{2}e_2^2(t)\right)^{\frac{\rho+1}{2}}+\left(\frac{1}{2}e_3^2(t)\right)^{\frac{\rho+1}{2}}+\left(\frac{1}{2}e_4^2(t)\right)^{\frac{\rho+1}{2}}\right) \tag{33}$$

Using Lemma 2, Equation (33) yields:

$$\Rightarrow \dot{V}(e(t)) \le -2^{\frac{\rho+1}{2}}\left(\frac{1}{2}\left(e_1^2(t)+e_2^2(t)+e_3^2(t)+e_4^2(t)\right)\right)^{\frac{\rho+1}{2}} \tag{34}$$

Using Lemma 3, yields the following inequality:

$$\dot{V}(e(t)) \le -2^{\frac{\rho+1}{2}}\left(V(e(t))\right)^{\frac{\rho+1}{2}} \le 0 \tag{35}$$

Hence, the closed-loop system Equation (25) is globally stable in the finite-time T_2. This completes the proof.

4.3. Simulation and Results

The parameters and initial conditions for the hyperchaotic Li and chaotic Lu systems are selected in the same way as in Sub-Section 3.3. The linear controller gains l_1, l_2 and l_3 are chosen as $l_1 = 35$, l_2=30, l_3=20 and $l_4 = 17$ and the constant number ρ being taken as $\rho = \frac{7}{10}$. In simulation, the following model uncertainties and external disturbances are applied to the master and slave systems respectively.

$$\begin{aligned}
&d_1(x_1(t)) = 0.35\sin(7x_1(t)) - 0.25\cos(10t), && D_1(x_2(t)) = -0.2\sin(5x_2(t)) + 0.25\sin(3t)\\
&d_2(y_1(t)) = 0.2\cos(5y_1(t)) + 0.15\sin(5t), && D_2(y_2(t)) = -0.25\sin(3y_2(t)) - 0.15\cos(5t)\\
&d_3(z_1(t)) = 0.3\sin(4z_1(t)) - 0.25\cos(5t), && D_3(z_2(t)) = -3\cos(4z_2(t)) + 0.2\sin(5t)\\
& && D_4(w_1(t)) = 0.2\sin(2w_2(t)) - 0.25\sin(3t)
\end{aligned}$$

As a result, one can see that $\phi_1 = 0.5$, $\phi_2 = 0.4$, $\phi_3 = 0.6$ and $\psi_4 = 0.45$. The corresponding numerical results are as follows:

Time series of the increased order AS error signals is depicted in Figure 8. As expected, one can observe that the AS error signals converged to zero state quickly with small amplitude of the oscillations.

Figure 9, shows the time series of the finite-time T_2 Equation (27). From the main Theorem 2, it can be checked that the slave hyperchaotic Li system is anti-synchronized with the master chaotic Lu system under the control action Equation (26) in the finite-time, when the control inputs are activated at $t \approx 0.3\ s$. Times series of the control inputs are depicted in Figures 10–13.

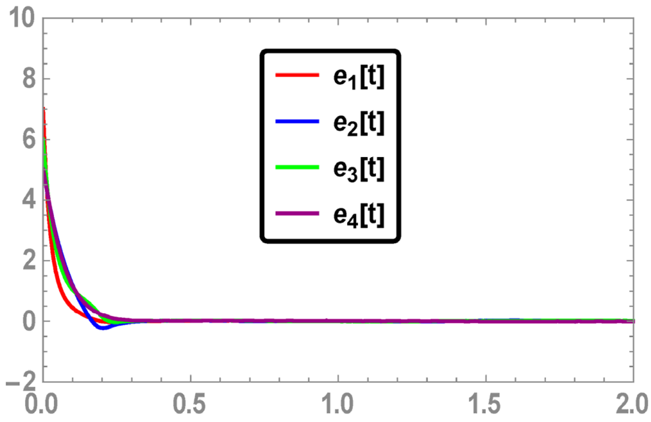


Figure 8. Times series of the AS error signals.

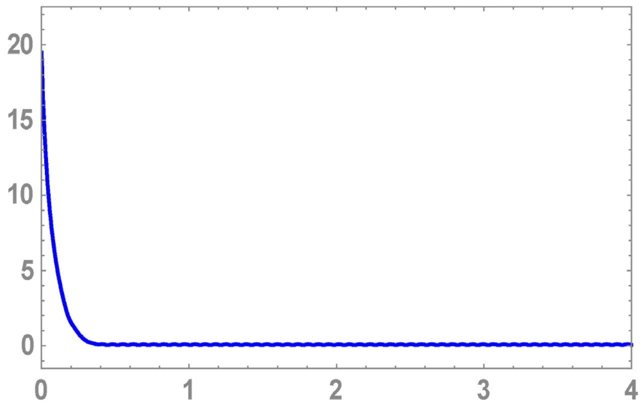


Figure 9. Time series of the estimated finite time T_2.

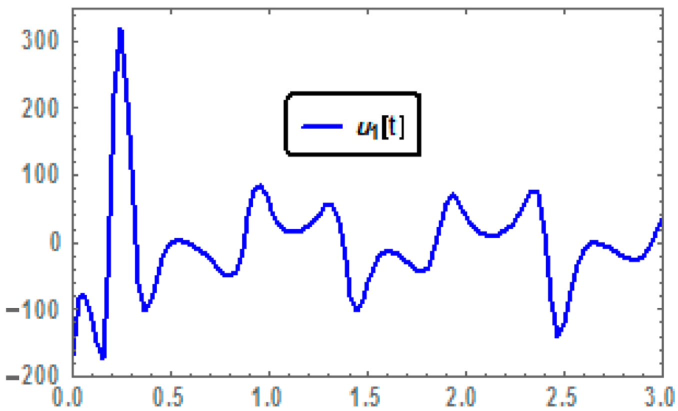


Figure 10. Times series of the control input $u_1(t)$.

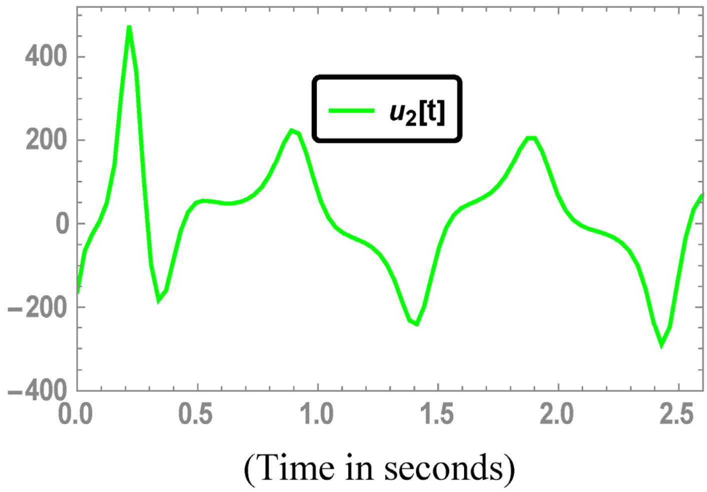


Figure 11. Times series of the control input $u_2(t)$.

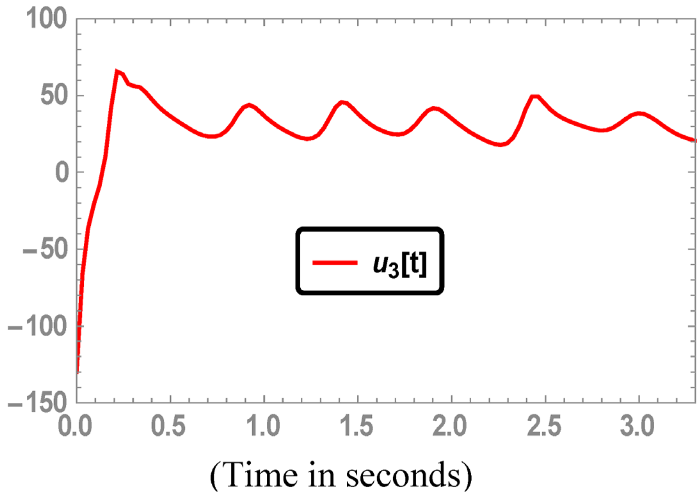


Figure 12. Times series of the control input $u_3(t)$.

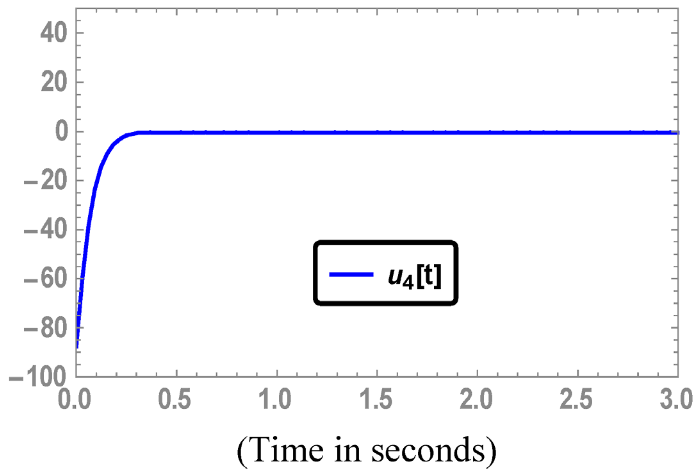


Figure 13. Times series of the control input $u_4(t)$.

5. Conclusions

In this article, it has been established that the finite-time reduced order and increased order AS of chaotic systems can be accomplished. Sufficient conditions for the controller parameter design are derived. The closed-loop systems are then simulated and the anti-synchronization behavior is analyzed. The simulation results fully verify the analytical findings. The obtained results show that the proposed finite-time reduced order and increased order AS schemes are comparable with the published notable results in terms of AS speed and quality under the effect of both unknown model uncertainties and external disturbances. It has been confirmed that the reduced order and increased order AS error signals converged to the equilibrium point in the finite time with smaller amplitude of the oscillations.

In practical applications, it is difficult to exactly fix the values of the systems parameters in advance, which is a limitation of the proposed finite-time AS approach and will be addressed in our next research article or elsewhere.

Acknowledgments

The authors would like to thank the honorable reviewers who suggested many worthy changes to improve the quality of this paper.

Author Contributions

This work was carried out in collaboration among all authors. Azizan Bin Saaban proposed the main idea, performed the literature review and suggested the problem. Israr Ahmad and Adyda Binti Ibrahin designed the study and performed the mathematical analysis. Mohammad Shahzad performed the numerical simulations and revised the manuscript. All the authors read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Wu, X.; Zhu, C.; Kan, H. An improved secure communication scheme based passive synchronization of hyperchaotic complex nonlinear system. *Appl. Math. Comput.* **2015**, *252*, 201–214.
2. Hammami, S. State feedback-based secure image cryptosystem hyperchaotic synchronization. *ISA Trans.* **2014**, doi:10.1016/j.isatra.2014.05.027.
3. Pisarchik, A.N.; Arecchi, F.T.; Meucci, R.; Garbo, A.D. Synchronization of Shilnikov chaos in CO_2 laser with feedback. *Laser Phys.* **2014**, *11*, 1235–1239.
4. Olusola, O.I.; Vincent, U.E.; Njah, A.N.; Idow, B.A. Global stability and synchronization criterion of linearly coupled gyroscope. *Nonlinear Dyn.* **2013**, *18*, 258–269.
5. Ren, F.; Cao, J. Anti-synchronization of stochastic perturbed delayed chaotic neural networks. *Neural Comput. Appl.* **2009**, *18*, 515–521.

6. Mobayen, S. Design of LMI-based global sliding mode controller for uncertain nonlinear systems with application to Genesio's chaotic system. *Complexity* **2014**, doi:10.1002/cplx.21545.
7. Ahmad, I.; Saaban, A.B.; Ibrahim, A.B.; Shahzad, M. Global chaos synchronization of new chaotic system using linear active control. *Complexity* **2015**, *21*, 379–389.
8. Al-Sawalha, M.; Noorani, M. Anti-synchronization of two hyperchaotic systems via nonlinear control. Commun. *Nonlinear Sci. Numer. Simul*. **2009**, *14*, 3402–3411.
9. Xia, W.; Cao, J. Pinning synchronization of delayed dynamical networks via periodically intermittent control. *Chaos* **2009**, *19*, 013120, doi:10.1063/1.3071933.
10. Li, X.-F.; Leung, A.C.-S.; Liu, X.-J.; Han, X.-P.; Chu, Y.-D. Adaptive synchronization of identical chaotic and hyper-chaotic systems with uncertain parameters. *Nonlinear Anal. RWA* **2010**, *11*, 2215–2223.
11. Cao, J.; Sivasamy, R.; Rakkiyappan, R. Sampled-data synchronization of chaotic Lur'e systems with time delay. *Circuits Syst. Signal Process.* **2015**, doi:10.1007/s00034-015-0105-6.
12. Cao, J.; Ho, D.W.C.; Yang, Y. Projective synchronization of a class of delayed chaotic systems via impulsive control. *Phys. Lett. A* **2009**, *373*, 3128–3133.
13. Gutiérrez, R.; Sevilla-Escoboza, R.; Piedrahita, P.; Finke, C.; Feudel, U.; Buldú, M.; Huerta-Cuellar, G.; Jaimes-Reátegui, R.; Moreno, Y.; Boccaletti, S. Generalized synchronization in relay systems with instantaneous coupling. *Phys. Rev. E* **2013**, *88*, doi:10.1103/PhysRevE.88.052908.
14. Song, Q.; Cao, J. Synchronization and anti-synchronization for chaotic systems. *Chaos Solitons Fract.* **2007**, *33*, 929–939.
15. Shahzad, M.; Ahmad, I. Experimental study of synchronization & Anti-synchronization for spin orbit problem of Enceladus. *Int. J. Control Sci. Eng*. **2013**, *3*, 41–47.
16. Srivastava, M.; Ansari, S.P.; Agrawal, S.K.; Das, S.; Leung, A. Anti-synchronization between identical and non-identical fractional-order chaotic systems using active control method. *Nonlinear Dyn*. **2014**, *76*, 905–914.
17. Vincent, U.E.; Laoye, J.A. Synchronization, anti-synchronization and current transport in non-identical chaotic ratchets. *Phys. A* **2007**, *384*, 230–240.
18. Bhalekar, S.; Daftardar-Gejji, V. Antisynchronization of nonidentical fractional-order chaotic systems using active control. *Int. J. Differ. Equ*. **2011**, doi:10.1155/2011/250763.
19. Ouannas, A. Synchronization and inverse synchronization of some different dimensional discrete-time chaotic dynamical systems via scaling matrices. *Int. J. Chaos Control Model. Simul.* **2014**, *3*, 1–17.
20. Al-Sawalha, M.; Noorani, M. Adaptive reduced-order anti-synchronization of chaotic systems with fully unknown parameters. *Commun. Nonlinear Sci. Numer. Simul*. **2010**, *15*, 3022–3034.
21. Al-Sawalha, M.; Noorani, M. Adaptive increasing-order synchronization and anti-synchronization of chaotic systems with uncertain parameters. *Chin. Phys. Lett*. **2011**, *25*, doi:10.1088/0256-307X/28/11/110507.
22. Wang, W.; Li, L.; Peng, H.; Kurths, J.; Xiao, J.; Yang, Y. Finite-time anti-synchronization control of memristive neural networks with stochastic perturbations. *Neural Process. Lett*. **2014**, doi:10.1007/s11063-014-9401-6.
23. Liu, L.; Liu, C.; Zhang, Y. Analysis of a novel four-dimensional hyperchaotic system. *Chin. J. Phys*. **2008**, *46*, 386–393.

24. Lu, J.; Chen, G. A new chaotic attractor coined. *Int. J. Bifurc. Chaos* **2002**, *12*, 659–662.
25. Yang, C. One input control of exponential synchronization for a four-dimensional chaotic system. *Appl. Math. Comput*. **2013**, *219*, 5152–5161.
26. Wang, H.; Han, Z.; Xie, Q.; Zhang, W. Finite-time chaos control via nonsingular terminal sliding mode control. *Commun. Nonlinear Sci. Numer. Simul*. **2009**, *14*, 2728–2733.
27. Jawaada, W.; Noorani, M.; Al-sawalha, M. Robust active sliding mode anti-synchronization of hyperchaotic systems with uncertainties and external disturbances. *Nonlinear Anal. Real World Appl*. **2012**, *13*, 2403–2413.