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Solution of Excited Non-Linear Oscillators under Damping Effects Using the Modified Differential Transform Method

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Abstract: The modified differential transform method (MDTM), Laplace transform and Padé approximants are used to investigate a semi-analytic form of solutions of nonlinear oscillators in a large time domain. Forced Duffing and forced van der Pol oscillators under damping effect are studied to investigate semi-analytic forms of solutions. Moreover, solutions of the suggested nonlinear oscillators are obtained using the fourth-order Runge-Kutta numerical solution method. A comparison of the result by the numerical Runge-Kutta fourth-order accuracy method is compared with the result by the MDTM and plotted in a long time domain.

Keywords: forced duffing oscillator; forced van der Pol Oscillator; Padé approximant; laplace transform; semi-analytical solution

1. Introduction

Our concern in this work is to give semi-analytic solutions of excited Duffing and excited van der Pol oscillators under damping effect which are given in the forms

$$\text{Duffing equation} \qquad \frac{d^2x}{dt^2} + \eta\frac{dx}{dt} + \omega^2 x + \alpha x^3 = A\sin(\Omega\, t) \tag{1a}$$

$$\text{van der Pol equation} \qquad \frac{d^2x}{dt^2} - \epsilon(1-x^2)\frac{dx}{dt} + \omega^2 x = A\sin(\Omega\, t) \tag{1b}$$

where x is the position coordinate which is a function of the time t, ω is the system's natural frequency, η is a scalar parameter indicating the damping factor in Duffing equation, ϵ the nonlinearity and strength of the damping in van der Pol equation, respectively. α is a nonlinear parameter factor, A and Ω are the forcing amplitude and frequency, respectively.

Those two considered nonlinear oscillators have received remarkable attention in recent decades due to the variety of their engineering applications. For example, Duffing Equation (1a) used in studying the magneto-elastic mechanical systems [1], nonlinear vibrations of beams and plates [2,3] and vibrations induced by fluid flow [4] which are modeled by the nonlinear Duffing equation.

On the other hand, during the first half of the twentieth century, Balthazar van der Pol pioneered the fields of radio and telecommunications [5–10]. In an era when these areas were much less advanced than they are today, vacuum tubes were used to control the flow of electricity of transmitters and receivers. Simultaneously with Lorenz, Thompson, and Appleton, van der Pol experimented with oscillations in a vacuum tube triode circuit and concluded that all initial conditions converged to the same orbit of a finite amplitude. Since this behavior is different from the behavior of solutions of linear equations, van der Pol proposed a nonlinear differential Equation (1b) without excitation force, i.e., $A = 0$, commonly referred to as the (unforced) van der Pol equation [8], as a model for the behavior observed in the experiment. In studying the case $\eta << 1$, van der Pol discovered the importance of what has become known as relaxation oscillations [8].

The most common methods for constructing approximate analytical solutions to the nonlinear oscillator's equations are the perturbation methods [11]. These methods include the harmonic balance method, the elliptic Lindstedt-Poincaré method [11–13]. The Krylov Bogoliubov Mitropolsky method [14,15], the averaging [11–16] and multiple scales method [12] are widely used to obtain approximate solutions of nonlinear oscillators. A general common factor to all of these methods is that they solve weakly the nonlinear systems by using perturbation techniques to reduce the system into simpler equations which transform the physical problem into a purely mathematical one, for which a solution is readily available.

This work is the derivation to obtain approximate analytical oscillatory solutions for the nonlinear oscillator Equations (1a) and (1b) with initial conditions $x(0) = a$ and $\dot{x}(0) = b$ using the modified differential transform method. This is a powerful method for solving linear and nonlinear differential equations. This method was at first used as differential transform method in the engineering domain by Zhou [17] and in fluid flow problems [18–25]. The differential transform solution diverges by using finite number of terms. To solve this problem the modified differential transform method [26–30] was developed by combining the differential transform method (DTM) with the Laplace transform and Padé approximant [31] which can successfully predict the solution of differential equations with finite numbers of terms [32,33].

2. Differential Transform Method

A brief explanation of the differential transform method (DTM) is given in [27–31]; for an analytic function $x(t)$ in domain G, which can be represented by a power series around any arbitrary point in this domain. The differential transform of $x(t)$ is defined as follows:

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{2}$$

In Equation (2), $x(t)$ is the original function and $X(k)$ is the transformed corresponding function. The inverse transform of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k) t^k \tag{3}$$

Combining equations (2) and (3), we obtain the following equation

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{4}$$

In applications, $x(t)$ takes finite number of terms and Equation (4) can be written as

$$x(t) = \sum_{k=0}^{N} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{5}$$

Some basic transformation rules of the differential transform method which are used in this work are tabulated in Table 1.

Table 1. The fundamental operations of the differential transform method (DTM).

Original Function	Transformed Function
$\alpha u(t) \pm \beta v(t)$	$\alpha U(k) \pm \beta V(t)$
$u(t)\, v(t)$	$\sum_{t=0}^{k} U(l)\, V(k-l)$
$u(t)\, v(t)\, w(t)$	$\sum_{s=0}^{k} \sum_{m=0}^{k-s} U(s)\, V(m)\, W(k-s-m)$
$\dfrac{d^m u(t)}{dt^m}$	$\dfrac{(k+m)!}{k!} U(k+m)$
$\exp(t)$	$\dfrac{1}{k!}$
$\sin(\omega t + \alpha)$	$\dfrac{\omega^k}{k!} \sin(k\pi/2 + \alpha)$
$\cos(\omega t + \alpha)$	$\dfrac{\omega^k}{k!} \cos(k\pi/2 + \alpha)$

Now, we will apply the MDTM to obtain semi-analytic solutions for forced nonlinear Duffing and van der Pol differential equations under damping effect.

3. Forced Duffing Oscillator under Damping Effect

Consider a nonlinear differential Equation (1a) which describes the forced Duffing oscillator with damping effect and initial conditions $x(0) = a$ and $\dot{x}(0) = b$. The differential transform of this equation gives the recurrence relation

$$\begin{aligned} &(k+2)(k+1)X(k+2) + \omega^2 X(k) + \eta(k+1)X(k+1) \\ &\quad + \alpha \sum_{n=0}^{k} \sum_{m=0}^{n} X(m)X(n-m)X(k-n) - \frac{A\,\Omega^k}{k!} \sin k\pi/2 = 0 \end{aligned} \tag{6}$$

$$X(0) = a, \qquad X(1) = b \tag{7}$$

The recursive equations deduced from Equations (6) and (7) for different values of k is obtained as follows:

$$k = 0: \qquad 2X(2) + \eta\, b + \alpha\, a^3 + \omega^2 a = 0 \tag{8}$$

$$k = 1: \qquad 6X(3) + 2\,\eta\, X(2) + (\omega^2 + 3\,\alpha\, a^2)\, b - A\,\Omega = 0 \tag{9}$$

$$k = 2: \qquad 12X(4) + \omega^2 X(2) + 3\,\eta\, X(3) + 3\,\alpha\, a(b^2 + aX(2)) = 0 \tag{10}$$

and so on.

The recursive relation in Equations (8)–(10) can be solved successively and then by taking the inverse differential transform $x(t)$ is obtained.

3.1. *Example 1: Free Duffing Oscillator under Damping Effect*

Let

$$a = 1.0, \quad b = 0.0, \quad \omega = 1, \quad \eta = 0.05, \quad \alpha = 0.15, \quad A = 0 \tag{11}$$

The analytic expansion $x(t)$ of Equation (1a) for the given values in (11) is given as follows:

$$\begin{aligned} x(t) = 1.0 - 0.575\, t^2 + 0.00958333\, t^3 + 0.0693594\, t^4 \\ -0.00138839\, t^5 - 0.00830017\, t^6 \\ +0.0002253\, t^7 + 0.00136295\, t^8 + \cdots \end{aligned} \tag{12}$$

Figure 1 shows the comparison between the results obtained using the DTM, Equation (12), and the numerical results obtained by Runge-Kutta fourth-order accuracy method. It is clear that the results using the DTM have a reasonable agreement with the results obtained using only the fourth-order Runge-Kutta numerical method in a small range of the solution domain.

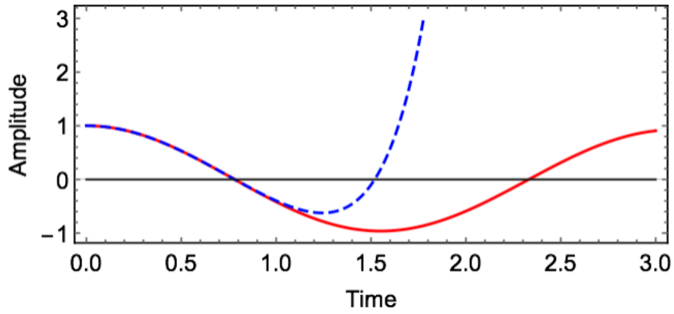

Figure 1. The red curve is the numerical solution and the dashed blue curve is the solution by the DTM.

Now we are improving the accuracy of the differential transform solution using the MDTM [22]. We first apply Laplace transform to the series solution given by Equation (12) to obtain

$$\begin{aligned} L[x(t)] = &\frac{54.954}{s^9} + \frac{1.1355}{s^8} - \frac{5.9761}{s^7} - \frac{0.1666}{s^6} \\ &+ \frac{1.6646}{s^5} + \frac{0.0575}{s^4} - \frac{1.15}{s^3} + \frac{1}{s} \end{aligned} \tag{13}$$

As the first step of the procedure of the MDTM [22] replacing s by $1/t$, calculating the Padé approximant of $[4/4]$ and letting $t = 1/s$ gives the following:

$$\left[\frac{4}{4}\right] = \frac{0.5525 + 10.6558\, s + 0.16667\, s^2 + s^3}{11.9025 + 0.6867\, s + 11.8058\, s^2 + 0.16667\, s^3 + s^4} \tag{14}$$

Taking the inverse Laplace transform to the Padé approximant $[4/4]$, Equation (14), to obtain the solution by the MDTM as follows:

$$x(t) = 0.9962\, e^{-0.0262\, t} \cos(1.0551\, t) + 0.0038\, e^{-0.0572\, t} \cos(3.2685\, t) \tag{15}$$

Figure 2 depicts the comparison between the MDTM results obtained by the Padé approximant of $[4/4]$ and the results obtained using the fourth-order Runge-Kutta numerical method. It is clear that the MDTM result obtained by the real part of Padé approximate gives an excellent agreement with the result obtained using the fourth-order Runge-Kutta numerical method.

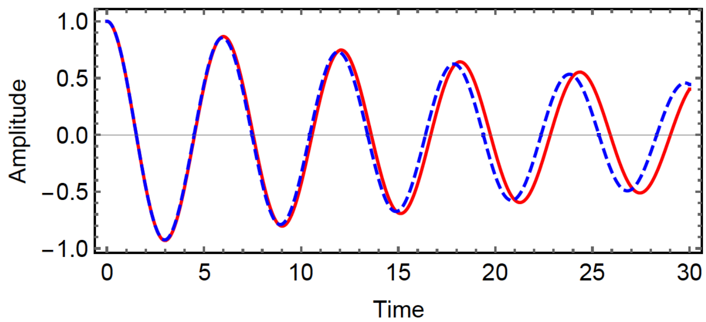

Figure 2. The red curve is the numerical solution and the dashed blue curve is the solution by the modified differential to transform method (MDTM) in Equation (15).

3.2. *Example 2: Forced Nonlinear Duffing Oscillator with Damping Effect*

In Equation (1a) let

$$a = 1.0,\ b = 0.0,\ \eta = 0.03,\ \alpha = 0.03,\ A = 0.15,\ \omega = 1.0,\ \text{and}\ \Omega = 0.8 \tag{16}$$

In this case the analytic expansion takes the form:

$$\begin{aligned} x(t) = 1.0 - 0.515\,t^2 + 0.02515\,t^3 + 0.0440155\,t^4 - 0.00219932\,t^5 \\ -0.0015002\,t^6 + 0.0000701175\,t^7 + 0.0000273301\,t^8 + \cdots \end{aligned} \tag{17}$$

Figure 3 illustrates the comparison of results obtained by the differential transform method which given by Equation (17) and the fourth-order Runge-Kutta numerical method. Clearly, the weakness of accuracy of the DTM result even for short time domain.

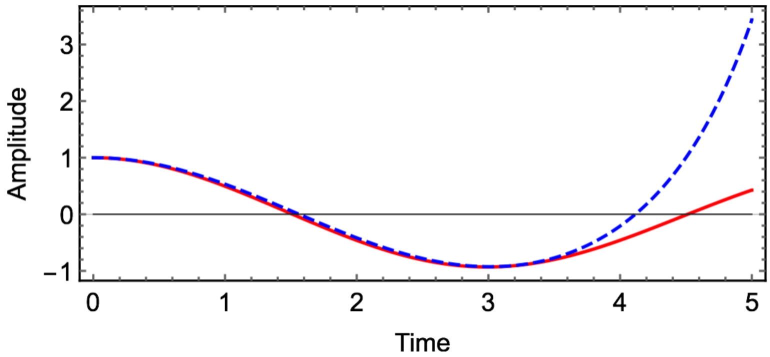

Figure 3. The red curve is the numerical solution and the dashed blue curve is the solution by the DTM in Equation (17).

To apply the modified differential transform method, we first get the Laplace transform to the time series solution which is given by Equation (17), yields

$$\begin{aligned} L[x(t)] = \frac{1}{s} - \frac{1.03}{s^3} + \frac{0.1509}{s^4} + \frac{1.0564}{s^5} - \frac{0.2639}{s^6} \\ -\frac{1.08015}{s^7} + \frac{0.3534}{s^8} - \frac{1.10195}{s^9} \end{aligned} \tag{18}$$

Using the concept of the MDTM [22]. Replace s by $1/t$ in Equation (18), calculate the Padé approximant of [4/4] and after that let $t = 1/s$ which gives the following:

$$\left[\frac{4}{4}\right] = \frac{0.1392 + 0.64\,s + 0.03\,s^2 + s^3}{0.6592 + 0.0192\,s + 1.67\,s^2 + 0.03s^3 + s^4} \tag{19}$$

Applying the inverse Laplace transform to the Padé approximant of [4/4] in Equation (19) to obtain the solution by the MDTM in the form:

$$\begin{aligned} x(t) = -0.0236\cos(0.8\,t) + 0.3832\sin(0.8\,t) \\ +e^{-0.015\,t}\left(1.0236\cos(1.0148\,t) - 0.287\sin(1.0148\,t)\right) \end{aligned} \tag{20}$$

Figure 4 shows the comparison between the MDTM result obtained by the Padé approximant of [4/4] and the result obtained using the fourth-order Runge-Kutta numerical method. The MDTM results obtained by the real part of the Padé approximant of [4/4] are clearly in excellent agreement with the result obtained using Runge-Kutta fourth-order accuracy numerical method.

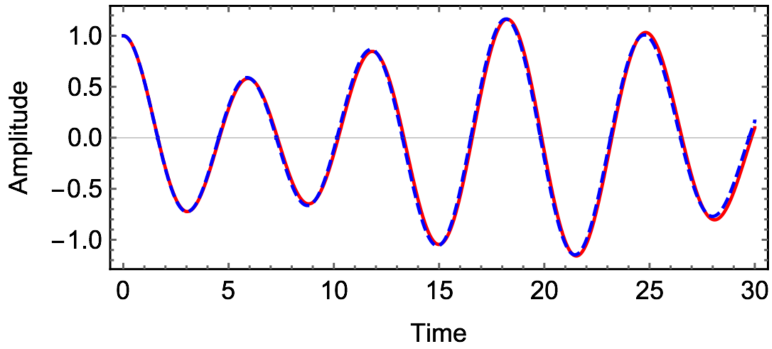

Figure 4. The red curve is the numerical result and the dashed blue curve is the solution by the DTM given in Equation (20).

4. Forced van der Pol Oscillator under Damping Effect

Nayfeh and Mook from [12] derived the ***Rayleigh equation*** in the form

$$y'' + \epsilon\left[\frac{1}{3}(y')^2 - 1\right]y' + y = 0, \quad \text{with} \quad y' = dy/d\tau \tag{21}$$

Setting $x = y'$ we find the ***van der Pol equation***

$$x'' + \epsilon\left[x^2 - 1\right]x + x = 0 \quad \text{with} \quad x = dx/d\tau \tag{22}$$

which arises in the study of circuits containing vacuum tubes and given by Equation (1b) with $A = 0$.

Electrical circuit involving a triode results a forced van der Pol oscillator Equation (1b) where $A \neq 0$, see Figure 5. The circuit contains: a triode, a resistor R, a coupled inductor L and mutual inductance M. In the serial RCL circuit there is a current i, and towards the triode anode (plate) a current i_a, while there is a voltage u_g on the triode control grid, see Figure 5. The van der Pol oscillator is forced by an AC voltage source E_s.

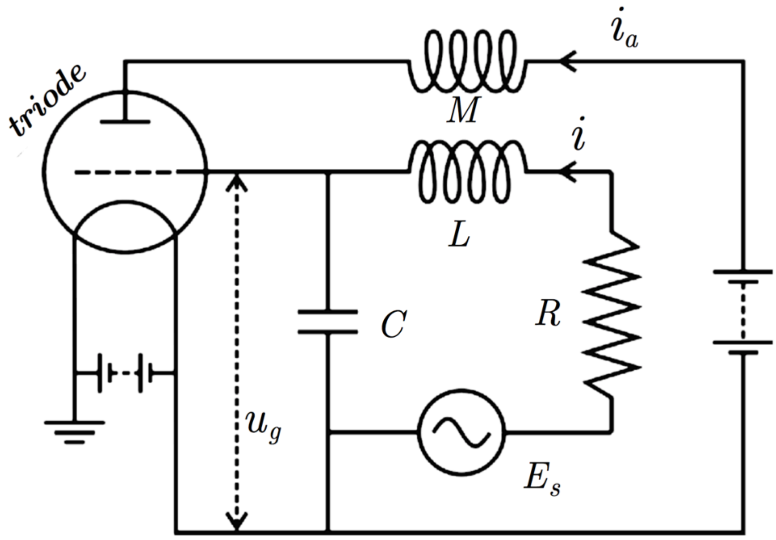

Figure 5. Electrical circuit involving a triode, resulting in a forced van der Pol oscillator.

To investigate a semi-analytic solution of forced van der Pol oscillator under damping effect we apply the MDTM to Equation (1b) with initial conditions $x(0) = a$ and $\dot{x} = b$. The differential transform of this equation has the recurrence relation

$$(k+2)(k+1)X(k+2)+\omega^2 X(k)+\epsilon\Big[(k+1)X(k+1)$$
$$-\sum_{n=0}^{k}\sum_{m=0}^{n}(k-n+1)X(m)X(n-m)X(k-n+1)\Big]$$
$$-\frac{A\,\Omega^k}{k!}\sin\left(\frac{k\pi}{2}\right)=0 \tag{23}$$
$$X(0)=a,\quad X(1)=b \tag{24}$$

The recursive equations deduced from Equations (23) and (24) for $k = 0, 1, 2$ are obtained as follows:

$$k=0: \quad 2X(2)+\omega^2 a+\epsilon\,(a^2-1)b=0 \tag{25}$$
$$k=1: \quad 6X(3)+\omega^2 b+2\,\epsilon\,X(2)(a^2-1)+2\,\epsilon\,a\,b^2-A\,\Omega=0 \tag{26}$$
$$k=2: \quad 12X(4)+\omega^2 X(2)+\epsilon[3(a^2-1)X(3)+6\,a\,b\,X(2)+b^3]=0 \tag{27}$$

and so on.

4.1. *Example 3:*

Let

$$a=1.0,\;\; b=0.0,\;\; \omega=1.0,\;\; \epsilon=0.04,\;\; A=0.04,\;\; \text{and}\;\; \Omega=1.4 \tag{28}$$

For the values given in Equation (28) the analytic expansion of the nonlinear van der Pol Equation (1b) is given by:

$$x(t)=1.0-0.5\,t^2+0.009333\,t^3+0.041667\,t^4t^4-0.003381\,t^5$$
$$-0.001327\,t^6+0.000599\,t^7-.000009\,t^8+\cdots \tag{29}$$

To use the modified differential transform method we take the Laplace transform of the series solution in Equation (29), yields:

$$L[x(t)]=\frac{1}{s}-\frac{1}{s^3}+\frac{0.056}{s^4}+\frac{1}{s^5}-\frac{0.40576}{s^6}-\frac{0.9552}{s^7}+\frac{3.01838}{s^8}$$
$$-\frac{0.353677}{s^9}-\frac{25.0076}{s^{10}}+\frac{31.4072}{s^{11}} \tag{30}$$

Forthwith as in [22] we replace s by $1/t$, calculating the Padé approximant of [3/3] and [4/4] and letting $t = 1/s$ gives us the following:

$$\left[\frac{3}{3}\right]=\frac{5.24571+93.6735\,s+s^2}{93.6175+6.24571\,s+93.6735\,s^2+s^3} \tag{31}$$

$$\left[\frac{4}{4}\right]=\frac{0.13895+7.72697\,s+0.168318\,s^2+s^3}{7.71754+0.251268\,s+8.72697\,s^2+0.168318\,s^3+s^4} \tag{32}$$

Taking the inverse Laplace transform of the Padé approximant of [3/3] and [4/4] in Equations (31) and (32) we obtain the semi-analytic solutions respectively, as follows:

$$x(t) \approx e^{-0.02802\,t}(\cos t + 0.02803\ \sin t), \tag{33}$$

$$\begin{aligned} x(t) \approx\ & e^{-0.0062\,t}(\cos t + 0.01264\ \sin t) \\ & + e^{-0.07798\,t}(0.0003\cos(2.7785\,t) - 0.0023\sin(2.7785\,t)) \end{aligned} \tag{34}$$

Figure 6a illustrated to show a comparison of the solution by the differential transform method, Equation (29), and the numerical solution by the fourth-order Runge-Kutta method. It is clear that, the result obtained by DTM have not reasonable agreement with the numerical result by Runge-Kutta for a long time domain.

Figure 6b shows the comparison between the MDTM results obtained by the real part of Padé approximant of orders [3/3] and [4/4] whose given by Equations (33) and (34), respectively, and the result obtained by the fourth-order Runge-Kutta numerical method. The MDTM result obtained by the real part of Padé approximant [3/3] in Equation (33) shows some discrepancies in comparison to the result obtained using the fourth-order Runge-Kuta numerical method. However, it is clear that the result of the result of the MDTM by the real part of Padé approximant [4/4] in Equation (34) has excellent agreement and seems to coincide with the numerical result.

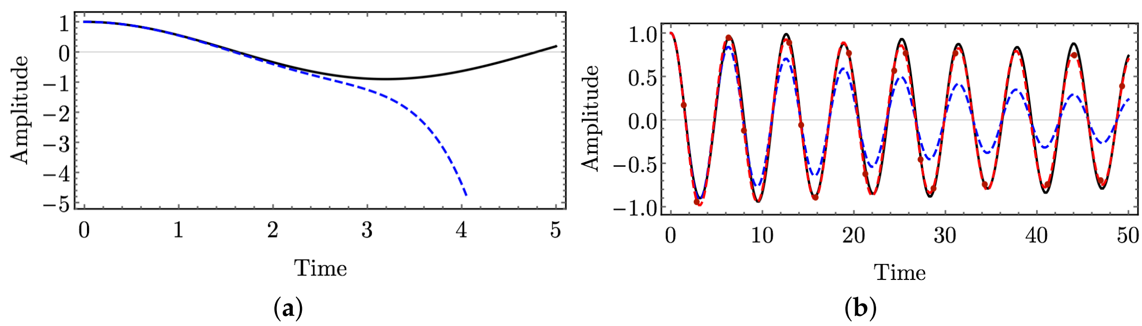

Figure 6. Example 3. (**a**) The black curve is the numerical solution and the dashed blue curve is the solution by the DTM given by Equation (29); (**b**) The dashed blue curve is the solution by the MDTM and Padé [3/3] approximant solution. The dashed red curve is the solution by the MDTM and Padé [4/4] approximant. The black curve is the numerical solution.

4.2. *Example 4:*

Let

$$a = 1.0,\ \ b = 0.0,\ \ \epsilon = 0.004,\ \ A = 0.9,\ \ \omega = 2.0\ \text{ and }\ \Omega = 1.4 \tag{35}$$

The nonlinear differential Equation (1b) in this case has the analytic solution in the form:

$$\begin{aligned} x(t) = 1 - 2\,t^2 + 0.21\,t^3 + 0.666667\,t^4 - 0.06578\,t^5 - 0.088329'\,t^6 \\ +0.010248\,t^7 + 0.005618\,t^8 - 0.002084\,t^9 + 0.000145\,t^{10} \\ +0.000545\,t^{11} - 0.000143\,t^{12} + \cdots \end{aligned} \tag{36}$$

To apply the MDTM we first take Laplace transform of the DTM solution given in Equation (36), yields:

$$\begin{aligned} L[x(t)] = & \frac{1}{s} - \frac{4}{s^2} + \frac{1.26}{s^4} + \frac{16}{s^5} - \frac{7.8936}{s^6} - \frac{63.5968}{s^7} + \frac{51.647808}{s^8} \\ & + \frac{226.503501}{s^9} - \frac{756.140505}{s^{10}} + \frac{524.935235}{s^{11}} \\ & + \frac{21755.582199}{s^{12}} - \frac{68682.662616}{s^{13}} \end{aligned} \tag{37}$$

Replace s by $1/t$ in Equation (37) and calculate the Padé approximants [4/4] and [6/6] and after that let $t = 1/s$ to obtain

$$\left[\frac{4}{4}\right] = \frac{0.554007 + 3.493991\, s - 0.860981\, s^2 - s}{15.0608 - 1.629917\, s - 0.506009\, s^2 - 0.860981\, s^3 - s^4} \tag{38}$$

$$\left[\frac{6}{6}\right] = \frac{45.6864 + 71.9781\, s + 3.2114\, s^2 + 38.5489\, s^3 + 0.7508\, s^4 + s^5}{283.4087 + 7.7739s + 225.2278s^2 + 4.9547s^3 + 42.5489s^4 + 0.7508s^5 + s^6} \tag{39}$$

The inverse Laplace transform of (38) and (39), respectively, gives the semi-analytical solutions as follows:

$$\begin{aligned} x(t) = & -0.0058\, e^{-2.2603\, t} + 0.01583\, e^{1.6398\, t} \\ & + e^{-0.12024\, t}(0.9899 \cos(2.0122\, t) + 0.0394 \sin(2.0122\, t)) \end{aligned} \tag{40}$$

$$\begin{aligned} x(t) = & \; e^{-0.00698\, t}(0.0091\ \cos(1.3895\, t) + 0.4314\ \sin(1.3895\, t)) \\ & + e^{-0.00015\, t}(\cos(2.00267\, t) - 0.29905\ \sin 92.00267\, t)) \end{aligned} \tag{41}$$

Figure 7 illustrates the comparison of the fourth-order Runge-Kutta numerical solution and the solution of the MDTM by the real part of the Padé approximant of order [4/4]. In Figure 8, it is clear that the solution by the MDTM is unstable. For this reason we have another attempt to obtain a more accurate and stable semi-analytical solution. To resolve this problem we use again the MDTM but with the real part of the Padé approximant of order [6/6] given by Equation (41).

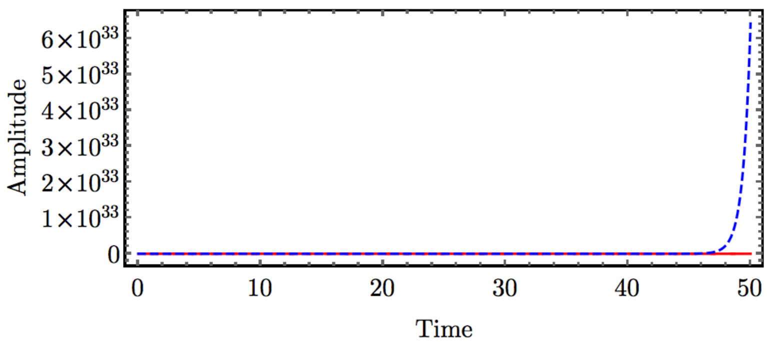

Figure 7. The red curve is the numerical solution and the dashed blue curve is the DTM solution in Equation (36).

Example 4 shows that the solution by the MDTM with the real part of the Padé approximant [6/6] not only matches perfectly with the numerical solution for a long time domain, as in Figure 8, but also shows that the phase plane given by the two methods seems to be identical, show Figure 9.

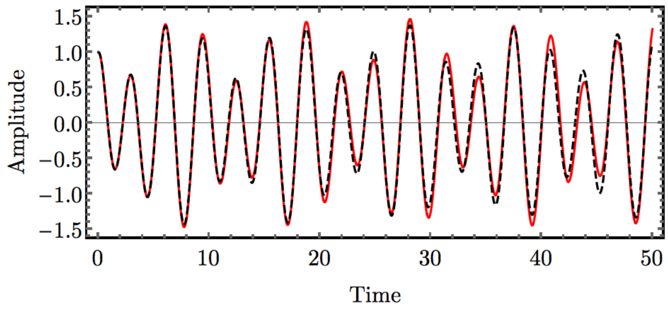

Figure 8. The red curve is the numerical solution and the dashed blue curve is the MDTM solution by Padé approximant [6/6] in Equation (41).

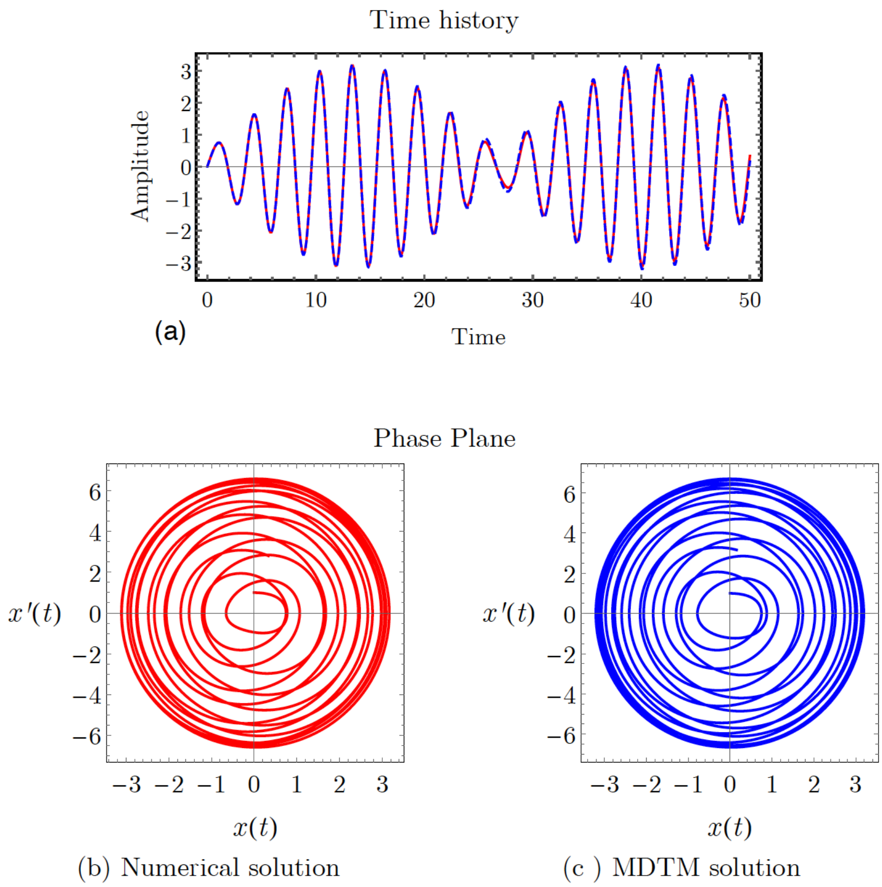

Figure 9. Red plot is the numerical solution and the blue plot is the Padé approximant solution of order [6/6] given by Equation (41).

5. Conclusions

The main goal of researchers who are interested in solving nonlinear differential equations is to obtain analytical solutions along with numerical solutions. These researchers have relied on some methods such as the multiple time scales method and the harmonic balance method and others. Other researchers have taken another turn and used the modified differential to transform method (MDTM) to obtain semi-analytic solutions of free non-linear oscillation by adding Laplace transform and Padé approximant [4/4]. Here, we extend their studies and provided semi-analytical solutions of forced oscillations of Duffing and van der Pol under damping effects.

Through some applications, we have provided records to indicate the success of the semi-analytical solution by the MDTM of nonlinear differential equations. The study focused on van der Pol and Duffing nonlinear oscillators under damping effects, due to their importance in science and engineering.

The numerical results demonstrate the validity and applicability of analytic solutions that we obtained by using the MDTM with appropriate values of parameters. This assures us the extent of success of using the modified differential to transform method to obtain analytical solutions of non-linear differential equations.

To get the analytical solution using the MDTM method one can obtain the result of the real part of Padé approximant of any order like [3/3], [4/4], [5/5] and higher where at least one of these approximant gives a satisfactory accurate analytic solution.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Guckenheimer, J.; Holmes, P. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*; Springer-Verlag: New York, NY, USA, 1983.
2. Ahmadian, M.T.; Mojahedi, M.; Moeenfard, H. Free vibration analysis of a nonlinear beam using homotopy and modified LindstedtPoincaré methods. *J. Solid Mech.* **2009**, *1*, 29–36.
3. Bakhtiari-Nejad, F.; Nazari, M. Nonlinear vibration analysis of isotropic cantilever plate with viscoelastic laminate. *Nonlinear Dyn.* **2009**, *56*, 325–356.
4. Srinil, N.; Zanganeh, H. Modeling of coupled cross-flow/in-line vortex-induced vibrations using double Duffing and van der Pol oscillators. *Ocean Eng.* **2012**, *53*, 83–97.
5. Cartwright, M. L. Balthazar van der Pol. *J. Lond. Math. Soc.* **1960**, *35*, 367–376.
6. Cartwright, M. L. *Van der Pols Equation for Relaxation Oscillations, in Contributions to the Theory of Nonlinear Oscillations II*; Princeton Annals of Mathematics Studies 2; Princeton University Press: Princeton, NJ, USA, 1952; p. 318.
7. Stumpers, F.L.H.M. Balth. van der Pols work on nonlinear circuits. *IRE Trans. Circuit Theory* **1960**, *7*, 366–367.
8. Van der Pol, B. A theory of the amplitude of free and forced triode vibrations. *Radio Rev.* **1920**, *1*, 701–710.
9. Van der Pol, B. Relaxation Oscillations I. *Phil. Mag.* **1926**, *2*, 978–992.
10. Van der Pol, B. The nonlinear theory of electric oscillations. *Proc. IRE* **1934**, *22*, 1051–1086.
11. Nayfeh, A.H. *Perturbation Methods*; John Wiley: New York, NY, USA, 1973.
12. Nayfeh, A.H.; Mook, D.T. *Nonlinear Oscillations*; John Wiley: New York, NY, USA, 1979.
13. Nayfeh, A.H. *Introduction to Perturbation Methods*; Jhon Wiley: New York, NY, USA, 1981.
14. Krylov, N.; Bogolioubov, N. *Introduction to Nonlinear Mechanics*; Princeton University Press: Princeton, NJ, USA, 1943.
15. Bogolioubov, N.N.; Mitropolsky, Y.A. *Asymptotic Methods in the Theory of Nonlinear Oscillations*; Gordon and Breach: New York, NY, USA, 1961.
16. Sanders, J.A.; Verhulst, F. *Averaging Methods in Nonlinear Dynamical Systems*; Springer-Verlag: New York, NY, USA, 1985.
17. Zhou, J.K. *Differential Transformation and Its Application for Electrical Circuits*; Huazhong University Press: Wuhan, China, 1986.
18. Mao, Q. Design of shaped piezoelectric modal sensors for cantilever beams with intermediate support by using differential transformation method. *Appl. Acoust.* **2012**, *73*, 144–149.
19. Rashidi, M.M.; Erfani, E. Analytical method for solving steady MHD convective and slip flow due to a rotating disk with viscous dissipation and Ohmic heating. *Eng. Comput.* **2012**, *29*, 562–579.
20. Rashidi, M.M.; Domairry, G. New analytical solution of the three dimensional Navier Stokes equations. *Mod. Phys. Lett. B* **2009**, *23*, 3147–3155.
21. Su, X.H.; Zheng, L.C. Approximate solutions to MHD Falkner Skan flow over permeable wall. *Appl. Math. Mech. (Engl. Ed.)* **2011**, *32*, 401–408.
22. Momani, S.; Odibat, Z.; Erturk, V.S. Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation. *Phys. Lett. A* **2007**, *370*, 379–387.

23. Kangalgil, F.; Ayaz, F. Solitary wave solutions for the KdV and mKdV equations by differential transform method. *Chaos Solitons Fractals* **2009**, *41*, 464–472.
24. Biazar, J.; Eslami, M. Analytic solution for Telegraph equation by differential transform method. *Phys. Lett. A* **2010**, *374*, 2904–2906.
25. Hesam, S.; Nazemi, A.R.; Haghbin, A. Analytical solution for the Fokker Planck equation by differential transform method. *Sci. Iran.* **2012**, *19*, 1140–1145.
26. El-Shahed, M. Application of differential transform method to non-linear oscillatory systems. *Commun. Nonlinear Sci. Numer. Simul.* **2008**, *13*, 1714–1720.
27. Momani, S.; Erturk, V.S. Solutions of non-linear oscillators by the modified differential transform method. *Comput. Math. Appl.* **2008**, *55*, 833–842.
28. Yildirim, A.; Gokdogan, A.; Merdan, M. Chaotic systems via multistep differential transformation method. *Can. J. Phys.* **2012**, *90*, 391–406.
29. Mirzabeigy, A.; Yildirim, A. Approximate periodic solution for nonlinear jerk equation as a third-order nonlinear equation via modified differential transform method. *Eng. Comput.* **2014**, *31*, 622–633.
30. Erturk, V.S.; Yildirim, A.; Momani, S.; Khan, Y. The differential transform method and Padé approximants for a fractional population growth model. *Int. J. Numer. Methods Heat Fluid Flow* **2012**, *22*, 791–802.
31. Baker, G.A. *Essentials of Padé Approximants*; Academic Press: London, UK, 1975.
32. Nourazar, S.; Mirzabeigy, A. Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method. *Sci. Iran. B* **2013**, *20*, 364–368.
33. Momani, S.; Ertürk, V.S. Solutions of non-linear oscillators by the modified differential transform method. *Comput. Math. Appl.* **2008**, *55*, 833–842.