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A Study of Controllability of Impulsive Neutral Evolution Integro-Differential Equations with State-Dependent Delay in Banach Spaces

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Abstract: In this paper, we study the problem of controllability of impulsive neutral evolution integro-differential equations with state-dependent delay in Banach spaces. The main results are completely new and are obtained by using Sadovskii's fixed point theorem, theory of resolvent operators, and an abstract phase space. An example is given to illustrate the theory.

Keywords: impulsive conditions; controllability; neutral evolution integro-differential equations; resolvent operators; state-dependent delay

AMS Subject Classification: 34A37; 46A50

1. Introduction

State-dependent delays are ubiquitous in applications, such as 3D printing and oil drilling. The formulation of the problem working with the control of nonlinear systems with state-dependent delays on the input can be studied by designing a "nonlinear predictor feedback" law that compensates the input delay. In [1], the authors introduced the concept of nonlinear predictor feedback starting from nonlinear systems with constant delays all the way through to predictor feedback for nonlinear systems with state-dependent delays. In [2], Bekiaris-Liberis considered nonlinear control systems with long, unknown input delays that depend on either time or the plant state and studied the robustness of nominal constant-delay predictor feedbacks. He showed that when the delay perturbation and its rate have sufficiently small magnitude, the local asymptotic stability of the closed-loop system, under the nominal predictor-based design, is preserved. For the special case of linear systems, and under only time-varying delay perturbations, he proved robustness of global exponential stability of the predictor feedback when the delay perturbation and its rate are small in any one of four different metrics. In addition, he presented two examples, one that is concerned with the control of a Direct Current (DC) motor through a network and another of a bilateral teleoperation between two robotic systems.

Very recently, Xuetao and Quanxin [3] studied a class of stochastic partial differential equations with Poisson jumps, which is more realistic for establishing mathematical models and has been widely applied in many fields. Under reasonable conditions, they not only established the existence and uniqueness of the mild solution for the investigated system but proved that the pth moment was exponentially stable by using fixed point theory. They also proved that the mild solution is almost surely the pth moment, and, therefore, exponentially stable using the well-known Borel-Cantelli lemma. In another publication, Xuetao, Quanxin, and Zhangsong [4] discussed the exponential stability problem of a class of nonlinear hybrid stochastic heat equations (with Markovian switching) in an infinite state space. Here, the fixed point theory was utilized to discuss the existence, uniqueness, and pth moment's exponential stability of the mild solution. Moreover, they acquired the Lyapunov exponents by combining fixed point theory and the Gronwall inequality. In [5], the authors investigated the stability problem for this class of new systems since Poisson jumps are considered to fill the mathematical gap. By using fixed point theory, they first studied the existence and uniqueness of the solution as well as the pth moment's exponential stability for the considered system. Then, based on the well-known Borel-Cantelli lemma, they proved that the solution was almost the pth moment and exponentially stable.

It is shown in [6], as another application, that the queuing delay involved in the congestion control algorithm is state-dependent and does not depend on the current time. Then, using an accurate formulation for buffers, networks with arbitrary topologies are built. At equilibrium, their model reduces to the widely used set-up. Using this model, the delay derivative is analyzed, and it is proven that the delay time derivative does not exceed one for the considered topologies. It is then shown that the considered congestion control algorithm globally stabilizes a delay-free single buffer network. Finally, using the specific linearization result for systems with state-dependent delays from Cooke and Huang [7], they showed the local stability of the single bottleneck network. Hartung, Herdman, and Turi [8] discussed the existence, uniqueness and numerical approximation for neutral equations with state-dependent delays.

There have been two main foci of mathematical control theory, which, at times, have seemed to work in entirely different directions. One of these is based on the idea that a good model of the object to be controlled is available, and one wants to minimize its behavior. For instance, physical principles and engineering specifications are used in order to calculate the optimal trajectory of a spacecraft that minimizes total travel time or fuel consumption. The techniques being used here are closely related to the classical calculus of variations and some areas of optimization; the end result is typically a preprogrammed flight plan. The other main focus is based on the constraints imposed by uncertainty about the model or about the environment in which the object operates. The central tool here is the use of feedback (state-dependent delay) in order to correct for deviations from the desired behavior. Thus, state-dependent delay systems are very important applicable systems.

The theory of semigroups of bounded linear operators is closely related to the solution of differential and integro-differential equations in Banach spaces. Using the method of semigroups, various types of solutions of semilinear evolution equations have been discussed by Pazy [9]. The theory of neutral differential equations in Banach spaces has been studied by several authors [10–15].

The notion of controllability is of great importance in mathematical control theory. It makes it possible to steer from any initial state of the system to any final state in some finite time using an admissible control. The concept of controllability plays a major role in finite-dimensional control theory; thus, it is natural to try to generalize it to infinite dimensions. The controllability of nonlinear systems, represented by ordinary differential equations in a finite dimensional space, is studied by means of fixed point principles [16]. This concept has been extended to infinite-dimensional spaces by applying semigroup theory [9]. Controllability of nonlinear systems, with different types of nonlinearity, has been studied with the help of fixed point principles [17]. Several authors have studied the problem of controllability of semilinear and nonlinear systems represented by differential and integro-differential equations in finite or infinite dimensional Banach spaces [18–21].

The impulsive differential systems can be used to model processes that are subjected to abrupt changes at certain moments. Examples include population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, etc. [22–24]. The study of dynamical systems with impulsive effects has been an object of investigations [25–28]. It has been extensively studied under various conditions on the operator A and the nonlinearity f by several authors [29–31]. Chalishajar and Acharya studied controllability of neutral impulsive differential inclusion with nonlocal conditions [32], (see also [33–36]), Anguraj and Karthikeyan [37] discussed the existence of solutions for impulsive neutral functional differential equations with nonlocal conditions. Ahmad, Malar, and Karthikeyan [38] studied nonlocal problems of impulsive integro-differential equations with a measure of noncompactness. Very recently, Klamka, Babiarz, and Niezabitowaski [39] did the survey based on Banach fixed point theorem in semilinear controllability problems and studied Schauder's fixed-point theorem in approximate controllability problems [40]. In addition, Balasubramaniam and Tamilalagan [41] studied approximate controllability of fractional neutral stochastic integro-differential inclusion with infinite delay by using Mainardi's function and Bohnenblust-Karlin's fixed point theorem. They also discussed approximate controllability of fractional stochastic equations driven by mixed fractional Brownian motion in a Hilber space with Hurst papramenter $H \in (\frac{1}{2}, 1)$ (see [42]). They proved the solvability and optimal controls for fractional stochastic integro differential equations (see [43]).

Motivated by the above-mentioned works, we show that a particular class of impulsive neutral evolution integro-differential systems with state-dependent delay in Banach spaces is controllable provided that some conditions are satisfied, using the theory of resolvent operators. The system considered here is untreated in the literature, which is a main motivation of the current work. Here, we have defined a new phase space for state-dependent infinite delay.

2. Preliminaries

In this section, we recall some relevant definitions, notations, and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and $A(t)$ generates the evolution operator in X. In addition, $A(t), G(t,s), 0 \leq s \leq t \leq b$ are closed linear operators defined on a common domain $\mathcal{D} := D(A(t))$, which is dense in X. The notation $[D(A(t))]$ represents the domain of $A(t)$ endowed with the graph norm. Let $(Z, \|\cdot\|)$ and $(W, \|\cdot\|)$ be Banach spaces. The notation $\mathcal{L}(Z, W)$ represents the Banach space of bounded linear operators from Z onto W endowed with the uniform operator topology, and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. In this paper, we establish the controllability of impulsive neutral evolution integro-differential equations with state-dependent delay described by

$$\begin{aligned} \frac{d}{dt}\Big[x(t) - \int_{-\infty}^{t} C(t,s)x(s)ds\Big] &= A(t)x(t) + \int_{-\infty}^{t} G(t,s)x(s)ds + Bu(t) \\ &\quad + g\big(t, x_{\rho(t,x_t)}, \int_0^t k_1(t,s,x_{\rho(s,x_s)})ds\big),\ t \in I, t \neq t_k, \end{aligned} \tag{1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, ..., m, \tag{2}$$

$$x_0 = \phi \in \mathcal{B}_h, \tag{3}$$

where the unknown $x(\cdot)$ takes values in a Banach space X and the control function $u(\cdot) \in L^2(I, U)$, a Banach space of admissible control functions with U as a Banach space. Furthermore, B is a bounded linear operator from U to X. $C(t,s)$, for $0 \leq s \leq t \leq b$, is a bounded linear operator on X, I is an interval of the form $[0, b]$; $0 < t_1 < t_2 < ... < t_k < ... < b$ are prefixed numbers. The history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}_h$ defined axiomatically with $g : [0,b] \times \mathcal{B}_h \times X \to X$, $\rho : [0,b] \times \mathcal{B}_h \to (-\infty, b]$, $x(t_k^-) \in X$ and $I_k : X \to X$ are appropriate functions. The symbol $\Delta\xi(t)$ represents the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

To obtain our results, we assume that the abstract impulsive integro-differential system. System (1)–(3) has an associated resolvent operator of bounded linear operators $\mathcal{R}(t,s)$ on X.

Consider the space

$$\mathcal{PC} := \Big\{x : (-\infty, b] \to X \text{ such that } x(t_k) \text{ and } x(t_k^-) \text{ exist with } x(t_k) = x(t_k^-)$$

$$x(t) = \phi(t) \text{ for } t \in (-\infty, b], x_k \in C(I_k, X), k = 1, 2, ..., m\Big\}.$$

Definition 1. *A resolvent operator of the problems (1)–(3) is a bounded operator-valued function $\mathcal{R}(t,s)$, $0 \leq s \leq t \leq b$ on X, the space of bounded linear operators on X, having the following properties:*

(a) *$\mathcal{R}(t,t) = I_d$, for every $t \in [0,b]$ and $\mathcal{R}(t,.) \in \mathcal{PC}([0,t],X), \mathcal{R}(.,s) \in \mathcal{PC}([s,b],X)$, for every $x \in X, t \in [0,b]$ and $s \in [0,b]$. $\|\mathcal{R}(t,s)\| \leq Me^{\beta(t-s)}$, $t,s \in I$, for some constants M and β.*

(b) *For $x \in D(A), \mathcal{R}(t,s)x \in \mathcal{PC}([0,\infty),[D(A)]) \cap \mathcal{PC}'((0,\infty),X)$.*

(c) *For each $x \in X$, $\mathcal{R}(t,s)x$ is continuously differentiable in $s \in I$ and*

$$\frac{\partial \mathcal{R}}{\partial s}(t,s)x = -\mathcal{R}(t,s)A(s)x - \int_s^t \mathcal{R}(t,\tau)G(\tau,s)x d\tau.$$

(d) *For each $x \in X$ and $s \in I, \mathcal{R}(t,s)x$ is continuously differentiable in $t \in [s,b]$ and*

$$\frac{\partial \mathcal{R}}{\partial t}(t,s)x = A(t)\mathcal{R}(t,s)x + \int_s^t G(\tau,s)\mathcal{R}(t,\tau)x d\tau,$$

with $\frac{\partial \mathcal{R}}{\partial s}$ and $\frac{\partial \mathcal{R}}{\partial t}$ strongly continuous on $0 \leq s \leq t \leq b$.

Here, $\mathcal{R}(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. We need to use Sadovskii's fixed point theorem as stated below (see [44]).

Lemma 1 (Sadovskii's Fixed Point Theorem)**.** *Let N be the condensing operator on a Banach space X. If $N(S) \subset S$ for a convex, closed and bounded set S of X, then N has a fixed point in S.*

Remark 1. *Sadovskii's fixed point theorem is important for studying the stability of the given system. However, we have not discussed the stability, as it is a separate problem to study.*

In this paper, we present the abstract phase space $\mathcal{B}_h$. Assume that $h: (-\infty,0] \to (0,\infty)$ be a continuous function with $l = \int_{-\infty}^0 h(s)ds < +\infty$. Define,

$$\begin{aligned} \mathcal{B}_h \quad := \quad & \{\phi : (-\infty,0] \to X \text{ such that, for any } r > 0, \phi(\theta) \text{ is bounded and a measurable} \\ & \text{function on } [-r,0] \text{ and } \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0}\|\phi(\theta)\|ds < +\infty\}. \end{aligned}$$

Here, $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} \|\phi(\theta)\|ds, \ \ \forall \phi \in \mathcal{B}_h.$$

Then, it is easy to show that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 2. *Suppose $y \in \mathcal{B}_h$; then, for each $t > 0, y_t \in \mathcal{B}_h$. Moreover,*

$$l\|y(t)\| \leq \|y_t\|_{\mathcal{B}_h} \leq l \sup_{0\leq s\leq t} \|y(s)\| + \|y_0\|_{\mathcal{B}_h},$$

where $l := \int_{-\infty}^0 h(s)ds < +\infty$.

Proof. For any $t \in [0, b]$, it is easy to see that y_t is bounded and measurable on $[-a, 0]$ for $a > 0$, and

$$
\begin{aligned}
\|y_t\|_{\mathcal{B}_h} &= \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|y_t(\theta)\| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta \in [s,0]} \|y(t+\theta)\| ds + \int_{-t}^{0} h(s) \sup_{\theta \in [s,0]} \|y(t+\theta)\| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [t+s,t]} \|y(\theta_1)\| ds + \int_{-t}^{0} h(s) \sup_{\theta_1 \in [t+s,t]} \|y(\theta_1)\| ds \\
&\leq \int_{-\infty}^{-t} h(s) \left[\sup_{\theta_1 \in [t+s,0]} \|y(\theta_1)\| + \sup_{\theta_1 \in [0,t]} \|y(\theta_1)\| \right] ds + \int_{-t}^{0} h(s) \sup_{\theta_1 \in [0,t]} \|y(\theta_1)\| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [t+s,0]} \|y(\theta_1)\| ds + \int_{-\infty}^{0} h(s) ds. \sup_{s \in [0,t]} \|y(s)\| \\
&\leq \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [s,0]} \|y(\theta_1)\| ds + l \sup_{s \in [0,t]} \|y(s)\| \\
&\leq \int_{-\infty}^{0} h(s) \sup_{\theta_1 \in [s,0]} \|y(\theta_1)\| ds + l \sup_{s \in [0,t]} \|y(s)\| \\
&= \int_{-\infty}^{0} h(s) \sup_{\theta_1 \in [s,0]} \|y_0(\theta_1)\| ds + l \sup_{s \in [0,t]} \|y(s)\| \\
&= l \sup_{s \in [0,t]} \|y(s)\| + \|y_0\|_{\mathcal{B}_h}.
\end{aligned}
$$

Since $\phi \in \mathcal{B}_h$, then $y_t \in \mathcal{B}_h$. Moreover,

$$
\|y_t\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|y_t(\theta)\| ds \geq \|y(t)\| \int_{-\infty}^{0} h(s) ds = l \|y(t)\|.
$$

The proof is complete. □

In this work, we employ an axiomatic definition for the phase space $\mathcal{B}_h$, which is similar to those introduced in [45]. More precisely, $\mathcal{B}_h$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with the seminorm $\|\cdot\|_{\mathcal{B}_h}$ and satisfying the following axioms:

(A_1) If $x : (-\infty, b] \to X, b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}_h$, then, for every $t \in [0, b]$, the following conditions hold:

- (a) x_t is in $\mathcal{B}_h$.
- (b) $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}_h}$.
- (c) $\|x_t\|_{\mathcal{B}_h} \leq K(t) \sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t) \|x_0\|_{\mathcal{B}_h}$, where $H > 0$ is constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bonded and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.

(A_2) The space $\mathcal{B}_h$ is complete.

Here, we consider some examples of phase spaces.

Example 1 (The phase space $\mathcal{PC}_\rho(X)$)**.** *A function $\psi : (-\infty, 0] \to X$ is said to be normalized piecewise continuous if ψ is left continuous and restriction of ψ to any interval $[-r, 0]$ is piecewise continuous. Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous nondecreasing function that satisfies the conditions $(g-1), (g-2)$ in the terminology of [45]. Next, we slightly modify the definition of phase spaces $\mathcal{C}_g$ and $\mathcal{C}_g^0$ in [45]. We denote by $\mathcal{PC}_g(X)$ the space formed by normalized piecewise continuous functions ψ such that ψ/g is bounded on $(-\infty, 0]$ and by $\mathcal{PC}_g^0(X)$, the subspace of $\mathcal{PC}_g(X)$ consisting of function ψ such that*

$[\psi(\theta)/g(\theta)] \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{B}_h = \mathcal{PC}_g(X)$ and $\mathcal{B}_h = \mathcal{PC}_g^0(X)$ endowed with the norm $\|y\|_{\mathcal{B}_h} := sup_{\theta\in(-\infty,0]}[\|\psi(\theta)\|/g(\theta)]$ are phase spaces in the sense defined above.

Example 2 (The phase space $\mathcal{PC}_r \times L^p(g, X)$)**.** *Let $r \geq 0$, $1 \leq p < \infty$ and let $g : (-\infty, -r] \to R$ be a nonneagative measurable function that satisfies the condition $(g-5), (g-6)$ in the terminology of [45]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $g(\Psi + \theta) \leq \gamma(\Psi)\rho(\theta)$, $\forall\ \Psi \leq 0$ and $\theta \in (-\infty, -r)\backslash N_\Psi$, where $N_\Psi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $\mathcal{B}_h = \mathcal{PC}_r \times L^p(g, X)$ consists of all classes of Lebesgue-measurable functions $\psi : (-\infty, 0] \to X$ such that $\psi|_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$ and $\rho\|\psi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in this space is defined by*

$$\|\psi\|_{\mathcal{B}_h} = \sup\Big\{\|\psi(\theta)\| : -r \leq \theta \leq 0\Big\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\psi(\theta)\|^p d\theta\Big)^{\frac{1}{p}}.$$

Proceeding as in the proof of [[45], Theorem 1.3.8], it follows that $\mathcal{B}_h$ is a space that satisfies axioms (A_1)–(A_3). Moreover, when $r = 0$, this space coincides with $C_0 \times L^p(g, X)$, and if, in addition, $p = 2$, we can take $H = 1$, $M(t)\gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \Big(\int_{-t}^{0} g(\theta)d\theta\Big)^{\frac{1}{2}}$ for $t \geq 0$.

Remark 2. *Let $\psi \in \mathcal{B}_h$ and $t \leq 0$. The notation ψ_t represents the function defined by $\psi_t(\theta) = \psi(t + \theta)$. Consequently, if the function $x(\cdot)$ in axiom (A_1) is such that $x_0 = \psi$, then $x_t = \psi_t$. We observe that ψ_t is well-defined for $t < 0$ since the domain of ψ is $(-\infty, 0]$. We also note that, in general, $\psi_t \notin \mathcal{B}_h$; consider, for example, functions of the type $x^\mu(t) = (t-\mu)^{-\alpha}\mathcal{K}_{(\mu,0]}$, $\mu > 0$, where $\mathcal{K}_{(\mu,0]}$ is the characteristic function of $(\mu, 0]$, $\mu < -r$ and $\alpha p \in (0, 1)$, in the space $\mathcal{PC}_r \times L^p(g, X)$.*

3. A Controllability Result for a Neutral System

In this section, we study the controllability results for the system (1)–(3). Throughout this section, M_1 is a positive constant such that $\|\mathcal{R}(t,s)\| \leq M_1$, for every $t \in I$. In the rest of this work, φ is a fixed function in $\mathcal{B}_h$ and $f_i : [0, b] \to X$, $i = 1, 2$ will be the functions defined by $f_1(t) = \int_{-\infty}^{0} C(t,s)\varphi(s)ds$ and $f_2(t) = \int_{-\infty}^{0} G(t,s)\varphi(s)ds$. We adopt the notation of mild solutions for (1)–(3) from the one given in [46].

Definition 2. *A function $x : (-\infty, b] \to X$ is called a mild solution of the impulsive neutral evolution integro-differential system (1)–(3) on [0,b] iff $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}_h$, f_1 is differentiable on [0,b], $f_1', f_2 \in L^1([0, b], X)$ and satisfies the following integral equation:*

$$\begin{aligned} x(t) = \mathcal{R}(t,0)\varphi(0) &+ \int_0^t \mathcal{R}(t,s)\big[f_1'(s) + f_2(s)\big]ds \\ &+ \int_0^t \mathcal{R}(t,s)\Big[Bu(s) + g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big)\Big]ds \\ &+ \sum_{0<t_k<t} \mathcal{R}(t,t_k)I_k(x(t_k^-)), \quad t \in [0,b]. \end{aligned} \tag{4}$$

Definition 3. *The system (1)–(3) is said to be controllable on the interval I iff, for every $x_0, x_b \in X$, there exists a control $u \in L^2(I, U)$ such that the mild solution $x(t)$ of (1)–(3) satisfies $x(0) = x_0$ and $x(b) = x_b$ [17].*

To establish our controllability result, we introduce the following assumptions:

(H1) The function $g : [0, b] \times \mathcal{B}_h \times \mathcal{B}_h \to X$ satisfies the following conditions:

(i) For each $t \in I$, the function $g(t, \cdot, \cdot) : \mathcal{B}_h \times \mathcal{B}_h \to X$ is continuous.
(ii) For every $\psi \in \mathcal{B}_h$, the function $g(\cdot, \psi_1, \psi_2) : I \to X$ is strongly measurable.
(iii) There exists $p_g \in \mathcal{PC}([0, b], [0, \infty))$ and a continuous non-decreasing function $\Omega_g : [0, \infty) \to (0, \infty)$ such that

$$\|g(t,\psi_1,\psi_2)\| \leq p_g(t)\Omega_g(\|\psi_1\|_{\mathcal{B}_h} + \|\psi_2\|_{\mathcal{B}_h}), \text{ for all } (t,\psi_1,\psi_2) \in [0,b] \times \mathcal{B}_h \times \mathcal{B}_h.$$

(H2) (i) The function g is continuous, and there exists a function $N_g \in L^1([0,b], R^+)$ such that

$$\|g(t,\psi_1,\eta_1) - g(t,\psi_2,\eta_2)\| \leq N_g(t)\big[\|\psi_1-\psi_2\| + \|\eta_1-\eta_2\|\big], \;\; t \in [0,b], \;\; \psi_1,\psi_2,\eta_1,\eta_2 \in \mathcal{B}_h.$$

(ii) The function $g : I \times \mathcal{B}_h \times X \to X$ is compact.

(H3) The linear operator $W : L^2(I,U) \to X$ defined by $Wu = \int_0^b \mathcal{R}(b,s)Bu(s)ds$ has an induced inverse operator W^{-1} that takes values in $L^2(I,U)/\ker W$, and there exists a constant K_1 such that $\|BW^{-1}\| \leq K_1$.

(**H4**) (i) $\rho : I \times \mathcal{B}_h \to X$ satisfies the Caratheodory condition, which is $\rho(t,x_t)$ is measurable with respect to t and continuous with respect to x_t.

(ii) The function $t \to \varphi_t$ is well-defined and continuous from the set $\mathcal{R}(\rho) = \{\rho(s,\psi) : (s,\psi) \in I \times \mathcal{B}_h, \rho(s,\psi) \leq 0\}$ into $\mathcal{B}_h$, and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho) \to (0,\infty)$ such that $\|\varphi_t\| \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}_h}$, for every $t \in \mathcal{R}(\subset)$.

(H5) For $I_k : \mathcal{B}_h \to X$, there exist constants $L_k > 0$ such that

$$\|I_k(\psi_1) - I_k(\psi_2)\| \leq L_k\|\psi_1 - \psi_2\|_{\mathcal{B}_h}, \;\; \psi_1,\psi_2 \in \mathcal{B}_h.$$

In addition, there exists $k > 0$ such that $\|I_k(x)\| \leq L$ for all $x \in X$ and $k = 1,2,...,m$.

(H6) For $k_1(t,s,x_{\rho(s,x_s)}) : I \times I \times \mathcal{B}_h \to X$, there exists a function $L_g \in L^1(I,R^+)$ such that

$$\|k_1(t,s,\phi_1) - k_1(t,s,\phi_2)\| \leq L_g(t)\|\phi_1 - \phi_2\|, \; \phi_1,\phi_2 \in \mathcal{B}_h.$$

(H7) There exists a positive constant Λ defined by

$$\Lambda = M_1\Big[(1+bK_1)\widetilde{K_b}\|N_g\|_{L^1([0,b])}(1+L_g b) + \textstyle\sum_{k=1}^m L_k\Big] < 1.$$

Remark 3. *In the remainder of this section, $\widetilde{M_b}$ and $\widetilde{K_b}$ are constants such that $\widetilde{M_b} = sup_{s\in[0,b]}\, M(s)$ and $\widetilde{K_b} = sup_{s\in[0,b]}\, K(s)$.*

Lemma 3. *Let $x : (-\infty,b] \to X$ be continuous on [0,b] and $x_0 = \phi$. If $(H4)$ holds, then $\|x_s\|_{\mathcal{B}_h} \leq (\widetilde{M_b} + J^\varphi)\|\varphi\|_{\mathcal{B}_h} + \widetilde{K_b}\, sup\, \{\|x(\theta)\| : \theta \in [0, max\{0,s\}]\}, s \in \mathcal{R}(\rho) \cup I$, where $J^\varphi = sup_{t\in\mathcal{R}(\rho)} J^\varphi(t)$ [47].*

Theorem 1. *If the assumptions $(H1)$–$(H4)$ are satisfied, $f_1 \in W^{1,1}([0,b],X)$ and $f_2 \in L^1([0,b],X)$, then the system (1)–(3) is controllable on I provided*

$$M_1\Big[\widetilde{K_b}(1+bK_1+bL_g)\lim_{\zeta\to\infty} inf\frac{\Omega_g(\zeta)}{\zeta}\int_0^b p_g(s)ds + L\Big] < 1.$$

Proof. Using assumption (H3) for an arbitrary function $x(\cdot)$, we define the control

$$\begin{aligned} u(t) = \;\; & W^{-1}\Big[x_b - \mathcal{R}(b,0)\varphi(0) - \textstyle\int_0^b \mathcal{R}(b,s)\big[f_1'(s) + f_2(s)\big]ds - \int_0^b \mathcal{R}(b,s) \\ & g(s,x_{\rho(s,x_s)}, \textstyle\int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau) - \sum_{k=1}^m \mathcal{R}(b,t_k)I_k(x(t_k^-))\Big](t). \end{aligned} \tag{5}$$

Consider the space $\mathcal{S}(b) = \{x \in \mathcal{PC}(I,X) : x(0) = \phi(0) \in \mathcal{B}_h\}$ endowed with the uniform convergence topology. For any $x \in S(b), \|x\|_b = \|x_0\|_{\mathcal{B}_h} + sup_{s\in[0,b]}\|x(s)\| = sup_{s\in[0,b]}\|x(s)\|$. Thus, $(S(b),\|.\|_b)$ is a Banach space. For each positive number r, set

$$B_r := \{x \in \mathcal{PC} : \|x\| \leq r\}.$$

It is clear that B_r is a bounded, closed, convex set in $S(b)$.

Consider the map $\Gamma : \mathcal{S}(b) \to \mathcal{S}(b)$ by $\Gamma x(\sigma) = \phi(\sigma)$, for $\sigma < 0$, and for all $t \in I$,

$$\begin{aligned}\Gamma x(t) &= \mathcal{R}(t,0)\varphi(0) - \int_0^t \mathcal{R}(t,s)\big[f_1'(s) + f_2(s)\big]ds + \int_0^t \mathcal{R}(t,s)BW^{-1}\Big[x_b - \mathcal{R}(b,0)\varphi(0)\\
&- \int_0^b \mathcal{R}(b,s)\big[f_1'(s) + f_2(s)\big]ds - \int_0^b \mathcal{R}(b,s)g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big)\\
&- \sum_{k=1}^{m} \mathcal{R}(b,t_k)I_k(\bar{x}(t_k^-))\Big](s)ds + \int_0^t \mathcal{R}(t,s)g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau ds\\
&+ \sum_{0<t_k<t} \mathcal{R}(t,t_k)I_k(\bar{x}(t_k^-)),\end{aligned}$$

where $\bar{x} : (-\infty, b] \to X$ is such that $\bar{x}_0 = \phi$ and $\bar{x} = x$ on I. From (A_1), the strong continuity of $(\mathcal{R}(t,s))_{t\geq s}$, and our assumption on ϕ; we infer that $\Gamma x(\cdot)$ is well-defined and continuous.

It is easy to see that $\Gamma\mathcal{S}(b) \subset \mathcal{S}(b)$. We prove that there exists $r > 0$ such that $\Gamma(B_r(\varphi|_I, \mathcal{S}(b)) \subseteq B_r(\varphi|_I, \mathcal{S}(b))$. If this property fails, then for every $r > 0$, $\exists\ x^r \in B_r(\varphi|_I, \mathcal{S}(b))$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - \varphi(0)\|$. Then, from Lemma 3, we find that

$$\begin{aligned}r &< \|\Gamma x^r(t^r) - \varphi(0)\| \leq \|\mathcal{R}(t,0)\varphi(0) - \varphi(0)\| + \int_0^{t^r} \|\mathcal{R}(t,s)\|\|f_1'(s) + f_2(s)\|ds\\
&+ \int_0^{t^r} \|\mathcal{R}(t,s)\|\|BW^{-1}\|\Big[\|x_b\| + \|\mathcal{R}(b,0)\varphi(0)\| + \int_0^b \|\mathcal{R}(b,s)\|\|f_1'(s) + f_2(s)\|ds\\
&+ \int_0^{t^r} \|\mathcal{R}(b,s)\|\|g\big(s, \bar{x^r}_{\rho(s,\bar{x^r}_s)}, \int_0^s k_1(s,\tau,\bar{x^r}_{\rho(\tau,\bar{x^r}_\tau)})d\tau\big)\|\\
&+ \sum_{k=1}^{m} \|\mathcal{R}(b,t_k)\|\|I_k(x(t_k^-))\Big](s)ds\\
&+ \int_0^{t^r} \|\mathcal{R}(t,s)\|\|g\big(s, \bar{x^r}_{\rho(s,\bar{x^r}_s)}, \int_0^s k_1(s,\tau,\bar{x^r}_{\rho(\tau,\bar{x^r}_\tau)})d\tau\big)\|\Big]ds\\
&+ \sum_{0<t_k<t} \|\mathcal{R}(t,t_k)\|\|I_k(x(t_k^-))\|\\
&\leq [M_1 + 1]H\|\varphi\|_{\mathcal{B}_h} + M_1\|f_1'(s) + f_2(s)\|_{L^1([0,b],X)}\\
&+ bM_1K_1\Big[\|x_b\| + [M_1 + 1]\|H\varphi\|_{\mathcal{B}_h} + M_1\|f_1'(s) + f_2(s)\|_{L^1([0,b],X)}\\
&+ M_1\Omega_g[1 + bL_g]\big[(\widetilde{M_b} + J^\varphi)\|\varphi\|_{\mathcal{B}_h} + \widetilde{K_b}(r + \|\varphi(0)\|)\big]\int_0^b p_g(s)ds\\
&+ M_1L\Big] + M_1\Omega_g[1 + bL_g]\big[(\widetilde{M_b} + J^\varphi)\|\varphi\|_{\mathcal{B}_h} + \widetilde{K_b}(r + \|\varphi(0)\|)\big]\int_0^b p_g(s)ds\\
&+ M_1L.\end{aligned}$$

Therefore,

$$1 \leq M_1\Big[\widetilde{K_b}(1 + bK_1 + bL_g)\lim_{\zeta\to\infty}\inf\frac{\Omega_g(\zeta)}{\zeta}\int_0^b p_g(s)ds + L\Big],$$

which contradicts our assumption.

Let $r > 0$ be such that $\Gamma(B_r(\phi|_I, \mathcal{S}(b))) \subset B_r(\varphi|_I, \mathcal{S}(b)), r^*$ is the number defined by $r^* = [1 + bL_g](\widetilde{M_b} + J^\varphi)\|\varphi\|_{\mathcal{B}_h} + \widetilde{K_b}(r + \|\varphi(0)\|)$ and $r^{**} = \Omega_g(r^*)\int_0^b p_g(s)ds$.

To prove that Γ is a condensing operator, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 x(t) = \mathcal{R}(t,0)\varphi(0) + \int_0^t \mathcal{R}(t,s)\big[f_1'(s) + f_2(s)\big]ds$$
$$+ \int_0^t \mathcal{R}(t,s)BW^{-1}\Big[x_b - \mathcal{R}(b,0)\varphi(0) - \int_0^b \mathcal{R}(b,s)\big[f_1'(s) + f_2(s)\big]ds$$
$$+ \int_0^b \mathcal{R}(b,s)g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big)ds - \sum_{k=1}^m \mathcal{R}(b,t_k)I_k(\bar{x}(t_k^-))\Big](s)ds$$
$$\Gamma_2 x(t) = \int_0^t \Big[\mathcal{R}(t,s)g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big)\Big]ds + \sum_{0<t_k<t} \mathcal{R}(t,t_k)I_k(\bar{x}(t_k^-)), \quad t \in I.$$

On the other hand, for $x, y \in B_r(\phi|_I, \mathcal{S}(b))$ and $t \in [0,b]$, we see that

$$\begin{aligned}
\|\Gamma_1 x(t) - \Gamma_1 y(t)\| &\le M_1 \int_0^t N_g(s)\Big[\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h} + L_g b\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h}\Big]ds \\
&+ M_1K_1 \int_0^t \mathcal{K}_g(s)\Big[\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h} \\
&+ L_g b\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h}\Big]ds + \sum_{k=1}^m M_1\| I_k(\bar{x}(t_k^-)) - I_k(\bar{y}(t_k^-))\|_{\mathcal{B}_h} \\
&\le M_1 \int_0^t N_g(s)\big[1 + L_g b\big]\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h} ds \\
&+ M_1K_1 \int_0^t \mathcal{K}_g(s)\big[1 + L_g b\big]\|\bar{x^r}_{\rho(s,\bar{x^r}_s)} - \bar{y^r}_{\rho(s,\bar{y^r}_s)}\|_{\mathcal{B}_h} ds \\
&+ \sum_{k=1}^m M_1\| I_k(\bar{x}(t_k^-)) - I_k(\bar{y}(t_k^-))\|_{\mathcal{B}_h} \\
&\le M_1\big[1 + bK_1\big]\widetilde{K_b} \int_0^t N_g(s)\big[1 + L_g(s)b\big]\sup_{0\le\zeta\le s}\|x(\zeta) - y(\zeta)\|ds \\
&+ M_1 \sum_{k=1}^m L_k \sup_{0\le\zeta\le s}\|x(\zeta) - y(\zeta)\| \\
&\le \Big[M_1\big[1 + bK_1\big]\widetilde{K_b} \int_0^t N_g(s)\big[1 + L_g(s)b\big]ds \\
&+ M_1 \sum_{k=1}^m L_k\Big]\sup_{0\le\zeta\le s}\|x(\zeta) - y(\zeta)\| \\
&\le \Lambda\|x(\zeta) - y(\zeta)\|,
\end{aligned}$$

where $\Lambda = M_1\Big[(1 + bK_1)\widetilde{K_b}\|N_g\|_{L^1([0,b])}(1 + L_g b) + \sum_{k=1}^m L_k\Big] < 1$, which implies that $\Gamma_1(\cdot)$ is a contraction to $B_r(\phi|_I, \mathcal{S}(b))$.

Now, we prove that $\Gamma_2(\cdot)$ is completely continuous from $B_r(\phi|_I, \mathcal{S}(b))$ into $B_r(\phi|_I, \mathcal{S}(b))$.

First, we prove that the set $\Gamma_2(B_r(\phi|_I, \mathcal{S}(b)))$ is relatively compact on X, for every $t \in [0,b]$.

The case $t = 0$ is trivial. Let $0 < \epsilon < t < b$. From the assumptions, we can fix the numbers $0 = t_0 < t_1 < \ldots t_n = t - \epsilon$ such that $\|\mathcal{R}(t,s) - \mathcal{R}(t,s')\| \le \epsilon$ if $s, s' \in [t_i, t_{i+1}]$, for some $i = 0, 1, 2, \ldots, n-1$. Let $x \in B_r(\phi|_I, \mathcal{S}(b))$. From the mean value theorem for the Bochner integral (see [48]), we see that

$$
\begin{aligned}
\Gamma_2(t) = & \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathcal{R}(t,t_i) g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big) ds \\
& + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \big(\mathcal{R}(t,s) - \mathcal{R}(t,t_i)\big) g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big) ds \\
& + \int_{t_n}^{t} \mathcal{R}(t,s) g\big(s, \bar{x}_{\rho(s,\bar{x}_s)}, \int_0^s k_1(s,\tau,\bar{x}_{\rho(\tau,\bar{x}_\tau)})d\tau\big) ds + \sum_{0<t_k<t} \mathcal{R}(t,t_k) I_k(\bar{x}(t_k^-)), \\
& \in \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{co(\{\mathcal{R}(t,t_i) g(s,\psi_1,\psi_2) : \psi_1,\psi_2 \in B_r(0,\mathcal{B}_h), s \in [0,b]\})} \\
& + \epsilon r^{**} + M_1 \Omega_g(r^*) \int_{t-\epsilon}^{t} p_g(s) ds + \sum_{0<t_k<t} \overline{co(\{\mathcal{R}(t,t_k)\psi_k(x) : \psi \in B_r(0,\mathcal{B}_h)} \\
& \in \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{co(\{\mathcal{R}(t,t_i) g(s,\psi_1,\psi_2) : \psi_1,\psi_2 \in B_r(0,\mathcal{B}_h), s \in [0,b]\})} \\
& + \epsilon B_{r^{**}}(0,X) + C_\epsilon + \sum_{0<t_k<t} \overline{co(\{\mathcal{R}(t,t_k)\psi_k(x) : \psi \in B_r(0,\mathcal{B}_h)},
\end{aligned}
$$

where dia $(C_\epsilon) \to 0$ when $\epsilon \to 0$. This proves that $\Gamma_2(B_r(\phi|_I, \mathcal{S}(b)))(t)$ is totally bounded and hence relatively compact in X, for every $t \in [0,b]$.

Second, we prove that the set $\Gamma_2(B_r(\phi|_I, \mathcal{S}(b)))$ is equicontinuous on $[0,b]$.

Let $0 < \epsilon < t < b$ and $0 < \delta < \epsilon$ be such that $\|\mathcal{R}(t,s) - \mathcal{R}(t,s')\| \le \epsilon$, for every $s, s' \in [\epsilon, b]$ with $|s - s'| \le \delta$. Under these conditions, $x \in B_r(\phi|_I, \mathcal{S}(b))$ and $0 < h \le \delta$ with $t + h \in [0,b]$. We get

$$
\begin{aligned}
\|\Gamma_2 x(t+h) - \Gamma_2 x(t)\| = & \Big\| \int_0^{t+h} \big[\mathcal{R}(t+h,s) g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big] ds \\
& + \sum_{0<t_k<t+h} \mathcal{R}(t+h,t_k) I_k(x(t_k^-)) \\
& - \Big[\int_0^{t} \big[\mathcal{R}(t,s) g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big] ds \\
& + \sum_{0<t_k<t} \mathcal{R}(t,t_k) I_k(x(t_k^-)) \Big] \Big\| \\
\le & \Big\| \int_0^{t+h} \big[\mathcal{R}(t+h,s) g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big] ds \\
& - \int_0^{t} \big[\mathcal{R}(t,s) g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big] ds \Big\| \\
& + \Big\| \sum_{0<t_k<t+h} \mathcal{R}(t+h,t_k) I_k(x(t_k^-)) - \sum_{0<t_k<t} \mathcal{R}(t,t_k) I_k(x(t_k^-)) \Big] \Big\| \\
\le & I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 \le & \int_0^{t-\epsilon} \Big\| \big[\mathcal{R}(t+h,s) - \mathcal{R}(t,s)\big] g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)}d\tau\big) \Big\| ds \\
& + \int_{t-\epsilon}^{t} \Big\| \big[\mathcal{R}(t+h,s) - \mathcal{R}(t,s)\big] g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big) \Big\| ds \\
& + \int_t^{t+h} \Big\| \mathcal{R}(t+h,s) g\big(s, x_{\rho(s,x_s)}, \int_0^s k_1(s,\tau,x_{\rho(\tau,x_\tau)})d\tau\big) \Big\| ds \\
& \le \epsilon r^{**} + 2M_1 \Omega_g(r^*) \int_{t-\epsilon}^{t} p_g(s) ds + M_1 \Omega_g(r^*) \int_t^{t+h} p_g(s) ds,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leq \Big\| \sum_{0<t_k<t+h} \mathcal{R}(t+h,t_k)I_k(x(t_k^-)) - \sum_{0<t_k<t} \mathcal{R}(t+h,t_k)I_k(x(t_k^-)\Big\| \\
&+ \Big\| \sum_{0<t_k<t} \mathcal{R}(t+h,t_k)I_k(x(t_k^-) - \sum_{0<t_k<t} \mathcal{R}(t,t_k)I_k(x(t_k^-))\Big]\Big\| \\
&\leq \Big\| \sum_{t<t_k<t+h} \mathcal{R}(t+h,t_k)I_k(x(t_k^-))\Big\| + \sum_{0<t_k<t} \Big\|\mathcal{R}(t+h,t_k)I_k(x(t_k^-) - \mathcal{R}(t,t_k)I_k(x(t_k^-))\Big\|,
\end{aligned}
$$

which shows that the set of functions $\Gamma_2(B_r(\phi|_I,\mathcal{S}(b)))$ is right-equicontinuous at $t \in (0,b]$. A similar procedure proves the right-equicontinuity at zero and left-equicontinuity at $t \in (0,b]$. Thus, $\Gamma_2(B_r(\phi|_I,\mathcal{S}(b)))$ is equicontinuous on I.

Finally, we show that the map $\Gamma_2(\cdot)$ is continuous on $B_r(\phi|_I,\mathcal{S}(b))$.

Let $(x^n)_{n\in N}$ be a sequence in $B_r(\phi|_I,\mathcal{S}(b))$ and $x \in B_r(\phi|_I,\mathcal{S}(b))$ such that $x^n \to x$ in $\mathcal{S}(b)$. At first, we study the convergence of sequences $(\bar{x^n}_{\rho(s,\bar{x_s^n})})_{n\in\mathcal{N}}, s \in I$.

If $s \in I$ is such that $\rho(s,\bar{x}_s) > 0$, we can fix $N \in \mathcal{N}$such that $\rho(s,\bar{x^n}_s) > 0$, for every $n > N$ (by assumption $(H4)(i)$). In this case, for $n > N$, we see that

$$
\begin{aligned}
&\Big\|\overline{x}^n{}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)}\Big\|_{\mathcal{B}_h} + \Big\| \int_0^s k_1(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)}d\tau - \int_0^s k_1(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)}d\tau\Big\|_{\mathcal{B}_h} \\
&= I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_3 &\leq \Big\|\overline{x}^n{}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s^n)}\Big\|_{\mathcal{B}_h} + \Big\|\overline{x}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)})\Big\|_{\mathcal{B}_h} \\
&\leq \widetilde{K_b}\Big\|x^n(\theta) - x(\theta)\Big\|_{\rho(s,\overline{x}_s^n)} + \Big\|\overline{x}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)})\Big\|_{\mathcal{B}_h} \\
&\leq \widetilde{K_b}\Big\|x^n - x\Big\|_b + \Big\|\overline{x}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)})\Big\|_{\mathcal{B}_h},
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\leq \int_0^s \Big\|k(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)} - k(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)}\Big\|_{\mathcal{B}_h} d\tau \\
&\leq bL_g(s)\Big\|\overline{x}^n{}_{\rho(s,\overline{x}^n{}_s)} - \overline{x}_{\rho(s,\overline{x}_s)}\Big\|_{\mathcal{B}_h} \\
&\leq bL_g(s)\Big[\widetilde{K_b}\Big\|x^n - x\Big\|_b + \Big\|\overline{x}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)})\Big\|_{\mathcal{B}_h}\Big],
\end{aligned}
$$

which proves that $(\overline{x}^n{}_{\rho(s,\overline{x}_s^n)}) \to (\overline{x}_{\rho(s,\overline{x}_s)})$, and $\int_0^s k_1(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)})d\tau \to \int_0^s k_1(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)})d\tau$ in $\mathcal{B}_h$ as $n \to \infty$, for every $s \in I$ such that $\rho(s,\bar{x}_s) > 0$. Similarly, if $\rho(s,\bar{x}_s) < 0$ and $N \in \mathcal{N}$ such that $\rho(s,\bar{x}_s^n) < 0$, for every $n > N$, we get

$$
\Big\|\overline{x}^n{}_{\rho(s,\overline{x}_s^n)} - \overline{x}_{\rho(s,\overline{x}_s)}\Big\|_{\mathcal{B}_h} = \Big\|\varphi_{\rho(s,\overline{x}_s^n)} - \varphi_{\rho(s,\overline{x}_s)}\Big\|_{\mathcal{B}_h},
$$

and

$$
\Big\| \int_0^s k_1(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)}d\tau - \int_0^s k_1(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)}d\tau\Big\|_{\mathcal{B}_h} = bL_g(s)\Big\|\varphi_{\rho(s,\overline{x}_s^n)} - \varphi_{\rho(s,\overline{x}_s)}\Big\|_{\mathcal{B}_h},
$$

which also shows that $(\overline{x}^n{}_{\rho(s,\overline{x}_s^n)}) \to (\overline{x}_{\rho(s,\overline{x}_s)})$ and $\int_0^s k_1(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)})d\tau \to \int_0^s k_1(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)})d\tau$ in $\mathcal{B}_h$ as $n \to \infty$, for every $s \in I$ such that $\rho(s,\bar{x}_s) < 0$. Combining the previous arguments, we can prove that $(\overline{x}^n{}_{\rho(s,\overline{x}_s^n)}) \to \varphi$, for every $s \in I$ such that $\rho(s,\overline{x}_s) = 0$.

From previous remarks,

$$
g\big(s,\overline{x}^n{}_{\rho(s,\overline{x}_s^n)}, \int_0^s k_1(s,\tau,\overline{x}^n{}_{\rho(\tau,\overline{x}^n{}_\tau)})d\tau\big) \to g\big(s,\overline{x}_{\rho(s,\overline{x}_s)}, \int_0^s k_1(s,\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)})d\tau\big), \forall\, s \in [0,b].
$$

Now, assumption (H1) and the Lebesgue Dominated Convergence Theorem permit us to assert that $\Gamma x^n \to \Gamma x$ in $\mathcal{S}(b)$. Thus, $\Gamma(\cdot)$ is continuous, which completes the proof that $\Gamma_2(\cdot)$ is completely continuous. Those arguments enable us to conclude that $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing map on $\mathcal{S}(b)$. By Lemma 1, there exists a fixed point $x(\cdot)$ for Γ on $\mathcal{S}(b)$. Obviously, $x(\cdot)$ is a mild solution of the system (1)–(3) satisfying $x(b) = x_b$. $\square$

4. Example

Consider the partial integro-differential equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big[z(t,\xi) + \textstyle\int_{-\infty}^{t}(t-s)^{\beta}e^{-\eta(t-s)}z(s,\xi)ds\Big] \\
&= \frac{\partial^2 z(t,\xi)}{\partial \xi^2} + a_0(t,\xi)z(t,\xi) + \textstyle\int_{-\infty}^{t} e^{-\gamma(t-s)}\frac{\partial^2 z(s,\xi)}{\partial \xi^2}ds + \mu(t,\xi) \\
&+ \textstyle\int_{-\infty}^{t} a_1(s-t)z\Big(s - \rho_1(t)\rho_2\big(\int_0^{\pi} a_2(\theta)|z(s,\theta)|^2 d\theta, \xi\big)\Big)ds, \\
&+ \textstyle\int_{-\infty}^{t}\int_0^{s} k_1(s,\xi,\tau-s)z\Big(\tau - \rho_1(s)\rho_2\big(\int_0^{\pi} k_2(\theta)|z(\tau,\theta)|^2 d\theta, \xi\big)\Big)d\tau ds,
\end{aligned} \tag{6}$$

$$z(t,0) = z(t,\pi) = 0, \tag{7}$$

$$z(\tau,\xi) = \varphi(\tau,\pi), \quad \tau \le 0, \quad 0 \le \xi \le \pi, \tag{8}$$

$$\Delta|_{t=t_k} = I_k(z(\xi)) = (\zeta_k|z(\xi)| + t_k)^{-1},\ z \in X,\ 1 \le k \le m, \tag{9}$$

where $a_0(t,\xi)$ is continuous on $0 \le \xi \le \pi,\ 0 \le t \le b$ and the constant ζ_i is small, $0 < t_1 < t_2 < \dots < t_m$. In this system, $\beta \in (0,1), \eta, \gamma$ are positive numbers, $a_i : \mathbb{R} \to \mathbb{R}, \rho_i : [0,\infty) \to [0,\infty), i = 1,2$ are continuous, and the function $a_2(\cdot)$ is positive. Moreover, we have identified $\varphi(\theta)(\xi) = \varphi(\theta,\xi)$. Put $x(t) = z(t,\xi)$ and $u(t) = \mu(t,\xi)$, where $\mu(t,\xi) : I \times [0,\pi] \to [0,\pi]$ is continuous and

$$\begin{aligned}
g(t, x_{\rho(t,x_t)}, H(x_{\rho(t,x_t)})) = &\int_{-\infty}^{t} a_1(s-t)z\Big(s - \rho_1(t)\rho_2\big(\int_0^{\pi} a_2(\theta)|z(s,\theta)|^2 d\theta, \xi\big)\Big)ds \\
&+ H(x_{\rho(t,x_t)}),
\end{aligned}$$

where

$$H(x_{\rho(t,x_t)}) = \int_{-\infty}^{t}\int_0^{s} k_1(s,\xi,\tau-s)z\Big(\tau - \rho_1(s)\rho_2\big(\int_0^{\pi} k_2(\theta)|z(\tau,\theta)|^2 d\theta, \xi\big)\Big)d\tau ds.$$

The system (6)–(9) is the abstract forms of (1)–(3). We choose the space $X = L^2([0,\pi])$ and $\mathcal{B}_h = \mathcal{C}_0 \times L^2(g,X)$ (see Example 1 for details). We also consider the operators $A, G(t,s) : D(A) \subset X \to X,\ 0 \le s \le t \le b$, given by $Ax = x'', G(t,s)x = e^{-\gamma(t-s)}Ax$, for $x \in D(A) := \{x \in X : x'', x(0) = x(\pi) = 0\}$ and $C(t,s) = t^{\beta}e^{-\eta(t-s)}x$, for $x \in X$. It is well-known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on X. Furthermore, A has a discrete spectrum with eigenvalues $-n^2, n \in N$, and corresponding normalized eigenfunctions are given by $z_n(y) = \sqrt{\frac{2}{\pi}}\sin ny$.

In addition, $\{z_n : n \in N\}$ is an orthonormal basis of X, and

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z,z_n)z_n,$$

for $z \in X$ and $t \ge 0$. In addition, for $\alpha \in (0,1)$, the fractional power $(-A)^{\alpha} : D((-A)^{\alpha}) \subset X \to X$ of A is given by

$$(-A)^{\alpha}z = \sum_{n=1}^{\infty} n^{2\alpha}(z,z_n)z_n,$$

where $D((-A)^{\alpha}) := \{z \in X : (-A)^{\alpha}z \in X\}$.

Now, we define operator $A(t)z = Az(\xi) + a_0(t,\xi)z$, $z \in D(A(t))$, $t \geq 0, y \in [0,\pi]$, where $D(A(t)) = D(A)$, $t \geq 0$. By assuming that $\xi \to a_0(t,\xi)$ is continuous in t, and there exists $\omega > 0$ such that $a_0(t,\xi) \leq -\omega$ for all $t \in I$, $\xi \in [0,\pi]$, it follows that the system

$$\begin{aligned} z'(t) &= A(t)z(t), \quad t \geq s, \\ z(s) &= x \in X \end{aligned}$$

has an (associated) evolution family $(U(t,s))_{t\geq s}$ with $U(t,s)y = T(t-s)e^{\int_s^t a(\tau,x)d\tau}y$, for $y \in X$ and $\|U(t,s)\| \leq e^{-(1+\rho)(t-s)}$, for every $t \geq s$.

Under the above conditions, we can represent the system

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[z(t,\xi) + \int_{-\infty}^{t}(t-s)^{\beta}e^{-\eta(t-s)}z(s,\xi)ds\Big] \\ &\quad = \frac{\partial^2 z(t,\xi)}{\partial \xi^2} + a_0(t,\xi)z(t,\xi) + \int_{-\infty}^{t} e^{-\gamma(t-s)}\frac{\partial^2 z(s,\xi)}{\partial \xi^2}ds + \mu(t,\xi), \end{aligned} \tag{10}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \geq 0, \tag{11}$$

$$\Delta|_{t=t_k} = I_k(z(\xi)) = (\zeta_k|z(\xi)| + t_k)^{-1}, \tag{12}$$

in the abstract form (1)–(3).

Lemma 4 below is a consequence of [27].

Lemma 4. *There exists an operator resolvent for the system (6)–(9). Consider the problem of controllability for the system (6)–(9). For this, the following conditions are assumed:*

(i) *The function $a_1(\cdot)$ is continuous, and $\mathcal{L}_g(\int_{-\infty}^{0}\frac{(a_1(s))^2}{g(s)}ds)^{\frac{1}{2}} < \infty$.*
(ii) *The functions $\rho_i : [0,\infty) \to [0,\infty)$, i = 1,2 are continuous.*
(iii) *The functions φ, $A\varphi$ belong to $\mathcal{B}_h$, and the expressions $sup_{t\in[0,b]}[\int_{-\infty}^{0}\frac{(t-\tau)^{2\alpha}}{g(\tau)} \times e^{2\eta\tau}d\tau]$ and $(\int_{-\infty}^{0}\frac{e^{2\gamma\tau}}{g(\tau)}d\tau)^{\frac{1}{2}}$ are finite.*

Define the operators $g : I \times \mathcal{B}_h \to X$, $f_i : I \to X$, $i = 1,2$; and $I_i : \mathcal{B}_h \to X$, given by

$$\begin{aligned} g(t,\psi_1,\psi_2)(\xi) &= \int_{-\infty}^{0} a_1(s)\Big(\psi_1(s,\xi) + \psi_2(s,\xi)\Big)ds, \\ f_1(t)(\xi) &= \int_{-\infty}^{0}(t-s)^{\alpha}e^{\omega(t-s)}\psi(s,\xi)ds, \\ f_2(t)(\xi) &= \int_{-\infty}^{0} e^{\gamma(t-s)}A\psi(s,\xi)ds, \\ I_i(z(\xi)) &= (\zeta_i|z(\xi)| + t_i)^{-1}, \\ \rho(s\cdot\psi) &= s - \rho_1(s)\rho_2\Big(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)\Big|^2\Big)d\theta, \end{aligned}$$

which are well-defined, then the system (6)–(9) are represented in the abstract forms (1)–(3). Moreover, g is a bounded linear operator $\|g(\cdot)\| \leq \mathcal{K}_g$. With these choices of g, f_1, f_2, ρ and $B = I$, the identity operator, assume that the linear operator W from $L^2(J,U)/\ker W$ into X, defined by

$$Wu = \int_0^b T(b-s)e^{\int_s^b a(\tau,x)d\tau}\mu(s,\cdot)ds,$$

has an invertible operator and satisfies the condition (H3).

Furthermore, all of the conditions stated in Theorem 1 are satisfied, and it is possible to choose a_1, a_2 and check (5). Hence, by Theorem 1, the system (6)–(9) controllable on I.

5. Conclusions

This paper contains some controllability results for the impulsive neutral evolution integro-differential equations with state-dependent delay in Banach spaces using the theory of resolvent operators and Sadovskii's fixed point theorem. It is proved that, under some constraints, the first-order impulsive neutral evolution integro-differential system is exactly controllable. The result shows that Sadovskii's fixed point theorem can effectively be used in control problems to obtain sufficient conditions.

We have considered the deterministic model without considering the noise disturbance, but one can extend the same problem to stochastic state delay differential equations/inclusions with Markovian switching or with the Gaussian process. Stability and robust stability of state delay systems are the future aspects of the current work.

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