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Stability of a Monomial Functional Equation on a Restricted Domain

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Abstract: In this paper, we prove the stability of the following functional equation $\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(ix+y) - n!f(x) = 0$ on a restricted domain by employing the direct method in the sense of Hyers.

Keywords: stability; monomial functional equation; direct method

MSC: 39B82; 39B52

1. Introduction

Let V and W be real vector spaces, X a real normed space, Y a real Banach space, $n \in \mathbb{N}$ (the set of natural numbers), and $f : V \to W$ a given mapping. Consider the functional equation

$$\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(ix+y) - n!f(x) = 0 \tag{1}$$

for all $x, y \in V$, where ${}_nC_i := \frac{n!}{i!(n-i)!}$. The functional Equation (1) is called an n-monomial functional equation and every solution of the functional Equation (1) is said to be a monomial mapping of degree n. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := ax^n$ is a particular solution of the functional Equation (1). In particular, the functional Equation (1) is called an additive (quadratic, cubic, quartic, and quintic, respectively) functional equation for the case $n = 1$ ($n = 2$, $n = 3$, $n = 4$, and $n = 5$, respectively) and every solution of the functional Equation (1) is said to be an additive (quadratic, cubic, quartic, and quintic, respectively) mapping for the case $n = 1$ ($n = 2$, $n = 3$, $n = 4$, and $n = 5$, respectively).

A mapping $A : V \to W$ is said to be additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in V$. It is easy to see that $A(rx) = rA(x)$ for all $x \in V$ and all $r \in \mathbb{Q}$ (the set of rational numbers). A mapping $A_n : V^n \to W$ is called n-additive if it is additive in each of its variables. A mapping A_n is called symmetric if $A_n(x_1, x_2, \ldots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for every permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. If $A_n(x_1, x_2, \ldots, x_n)$ is an n-additive symmetric mapping, then $A^n(x)$ will denote the diagonal $A_n(x, x, \ldots, x)$ for $x \in V$ and note that $A^n(rx) = r^n A^n(x)$ whenever $x \in V$ and $r \in \mathbb{Q}$. Such a mapping $A^n(x)$ will be called a monomial mapping of degree n (assuming $A^n \not\equiv 0$). Furthermore, the resulting mapping after substitution $x_1 = x_2 = \ldots = x_l = x$ and $x_{l+1} = x_{l+2} = \ldots = x_n = y$ in $A_n(x_1, x_2, \ldots, x_n)$ will be denoted by $A^{l,n-l}(x, y)$. A mapping $p : V \to W$ is called a generalized polynomial (GP) mapping of degree $n \in N$ provided that there exist $A^0(x) = A^0 \in W$ and i-additive symmetric mappings $A^i : V^i \to W$ (for $1 \leq i \leq n$) such that $p(x) = \sum_{i=0}^{n} A^i(x)$, for all $x \in V$ and $A^n \not\equiv 0$. For $f : V \to W$, let Δ_h be the difference operator defined as follows:

$$\Delta_h f(x) = f(x+h) - f(x)$$

for $h \in V$. Furthermore, let $\Delta_h^0 f(x) = f(x)$, $\Delta_h^1 f(x) = \Delta_h f(x)$ and $\Delta_h \circ \Delta_h^n f(x) = \Delta_h^{n+1} f(x)$ for all $n \in N$ and all $h \in V$. For any given $n \in N$, the functional equation $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in V$ is well studied. In explicit form we can have

$$\Delta_h^n f(x) = \sum_{i=0}^{n} {}_nC_i (-1)^{n-i} f(x + ih).$$

The following theorem was proved by Mazur and Orlicz [1,2] and in greater generality by Djoković (see [3]).

Theorem 1. *Let V and W be real vector spaces, $n \in \mathbb{N}$ and $f : V \to W$, then the following are equivalent:*

(1) $\Delta_h^{n+1} f(x) = 0$ *for all* $x, h \in V$.
(2) $\Delta_{x_1} \circ \Delta_{x_2} \circ \ldots \circ \Delta_{x_{n+1}} f(x_0) = 0$ *for all* $x_0, x_1, x_2, \ldots, x_{n+1} \in V$.
(3) $f(x) = A^n(x) + A^{n-1}(x) + \cdots + A^2(x) + A^1(x) + A^0(x)$ *for all* $x \in V$, *where* $A^0(x) = A^0$ *is an arbitrary element of* W *and* $A^i(x)(i = 1, 2, \ldots, n)$ *is the diagonal of an i-additive symmetric mapping* $A^i : V^i \to W$.

In 2007, L. Cădariu and V. Radu [4] proved a stability of the monomial functional Equation (1) (see also [5–7]), in particular, the following result is given by the author in [6].

Theorem 2. *Let p be a non-negative real number with $p \neq n$, let $\theta > 0$, and let $f : X \to Y$ be a mapping such that*

$$\left\| \sum_{i=0}^{n} {}_nC_i (-1)^{n-i} f(ix + y) - n! f(x) \right\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2}$$

for all $x, y \in X$. Then there exist a positive real number K and a unique monomial function of degree n $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq K\|x\|^p \tag{3}$$

holds for all $x \in X$. The mapping $F : X \to Y$ is given by

$$F(x) := \lim_{s\to\infty} \frac{f(2^s x)}{2^{ns}}$$

for all $x \in X$.

The concept of stability for the functional Equation (1) arises when we replace the functional Equation (1) by an inequality (2), which is regarded as a perturbation of the equation. Thus, the stability question of functional Equation (1) is whether there is an exact solution of (1) near each solution of inequality (2). If the answer is affirmative with inequality (3), we would say that the Equation (1) is stable.

The direct method of Hyers means that, in Theorem 2, $F(x)$ satisfying inequality (3) is constructed by the limit of the sequence $\{\frac{f(2^s x)}{2^{ns}}\}_{s\in\mathbb{N}}$ as $s \to \infty$.

Historically, in 1940, Ulam [8] proposed the problem concerning the stability of group homomorphisms. In 1941, Hyers [9] gave an affirmative answer to this problem for additive mappings between Banach spaces, using the direct method. Subsequently, many mathematicians came to deal with this problem (cf. [10–17]).

In 1998, A. Gilányi dealt with the stability of monomial functional equation for the case $p = 0$ (see [18,19]) and he proved for the case when p is a real constant (see [20]). Thereafter, C.-K. Choi proved stability theorems for many kinds of restricted domains, but his theorems are mainly connected with the case of $p = 0$. If $p < 0$ in (2), then the inequality (2) cannot hold for all $x \in X$, so we have to restrict the domain by excluding 0 from X.

The main purpose of this paper is to generalize our previous result (Theorem 2) by replacing the real normed space X with a restricted domain S of a real vector space V and by replacing the control function $\theta(||x||^p + ||y||^p)$ with a more general function $\varphi : S^2 \to [0, \infty)$.

2. Stability of the Functional Equation (1) on a Restricted Domain

In this section, for a given mapping $f : V \to W$, we use the following abbreviation

$$D_n f(x, y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i} f(ix + y) - n!f(x)$$

for all $x, y \in V$.

Lemma 1. *The equalities*

$$ {}_nC_i = \sum_{\substack{j+k=2i \\ 0\le j,k\le n}} {}_nC_j \cdot {}_nC_k(-1)^{i+k} \quad (i = 0, \cdots, n) \tag{4}$$

and

$$ {}_nC_i = \sum_{\substack{j+k=2i-1 \\ 0\le j,k\le n}} {}_nC_j \cdot {}_nC_k(-1)^{n-k} \quad (i = 1, \cdots, n) \tag{5}$$

hold.

Proof. From the equalities

$$\begin{aligned}
&\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}x^{2i} = (x^2 - 1)^n, \\
&(x^2 - 1)^n = (x + 1)^n(x - 1)^n, \\
&(x + 1)^n(x - 1)^n = \left(\sum_{j=0}^{n} {}_nC_j x^j\right)\left(\sum_{k=0}^{n} {}_nC_k(-1)^{n-k}x^k\right) = \sum_{j=0}^{n}\sum_{k=0}^{n} {}_nC_j \cdot {}_nC_k(-1)^{n-k}x^{j+k},
\end{aligned}$$

we get the equality

$$\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}x^{2i} = \sum_{j=0}^{n}\sum_{k=0}^{n} {}_nC_j \cdot {}_nC_k(-1)^{n-k}x^{j+k} \tag{6}$$

for all $x \in \mathbb{R}$. Since the coefficient of the term x^{2i} of the left-hand side in (6) is ${}_nC_i(-1)^{n-i}$ and the coefficient of the term x^{2i} of the right-hand side in (6) is $\sum_{\substack{j+k=2i \\ 0\le j,k\le n}} {}_nC_j \cdot {}_nC_k(-1)^{n-k}$, we get the Equality (4). We easily know that the coefficient of the term x^{2i-1} of the left-hand side in (6) is 0 and the coefficient of the term x^{2i-1} of the right-hand side in (6) is $\sum_{\substack{j+k=2i-1 \\ 0\le j,k\le n}} {}_nC_j \cdot {}_nC_k(-1)^{n-k}$. So we get the Equality (5). □

We rewrite a refinement of the result given in [6].

Lemma 2. (Lemma 1 in [6]) *The equality*

$$\sum_{j=0}^{n} {}_nC_j D_n f(x, jx + y) - D_n f(2x, y) = n!(f(2x) - 2^n f(x)) \tag{7}$$

holds for all $x, y \in V$. In particular, if $D_n f(x,y) = 0$ for all $x, y \in V$, then

$$f(2x) = 2^n f(x) \tag{8}$$

Proof. By (4), (5), and the equality $\sum_{j=0}^{n} {}_nC_j = 2^n$, we get the equalities

$$\begin{aligned}
&\sum_{j=0}^{n} {}_nC_j D_n f(x, jx+y) - D_n f(2x, y) \\
&= \sum_{j=0}^{n} {}_nC_j \sum_{k=0}^{n} (-1)^{n-k} {}_nC_k f((j+k)x+y) - \sum_{j=0}^{n} {}_nC_j n! f(x) \\
&\quad - \sum_{i=0}^{n} {}_nC_i (-1)^{n-i} f(2ix+y) + n! f(2x) \\
&= \sum_{i=0}^{n} \sum_{\substack{j+k=2i \\ 0\le j,k\le n}} {}_nC_j (-1)^{n-k} {}_nC_k f(2ix+y) \\
&\quad + \sum_{i=1}^{n} \sum_{\substack{j+k=2i-1 \\ 0\le j,k\le n}} {}_nC_j (-1)^{n-k} {}_nC_k f((2i-1)x+y) \\
&\quad - \sum_{j=0}^{n} {}_nC_j n! f(x) - \sum_{i=0}^{n} {}_nC_i (-1)^{n-i} f(2ix+y) + n! f(2x) \\
&= - \sum_{j=0}^{n} {}_nC_j n! f(x) + n! f(2x)
\end{aligned}$$

for all $x, y \in V$. □

Lemma 3. *If f satisfies the functional equation $D_n f(x,y) = 0$ for all $x, y \in V\backslash\{0\}$ with $f(0) = 0$, then f satisfies the functional equation $D_n f(x,y) = 0$ for all $x, y \in V$.*

Proof. Since $D_n f(0,0) = 0$, $D_n f(0,y) = 0$ for all $y \in V\backslash\{0\}$, and

$$D_n f(x,0) = (-1)^n D_n f(-x, nx) - (-1)^n D_n f(-x, (n+1)x) + D_n f(x,x)$$

for all $x \in V\backslash\{0\}$, we conclude that f satisfies the functional equation $D_n f(x,y) = 0$ for all $x, y \in V$. □

We rewrite a refinement of the result given in [7].

Theorem 3. (Corollary 4 in [7]) *A mapping $f : V \to W$ is a solution of the functional Equation (1) if and only if f is of the form $f(x) = A^n(x)$ for all $x \in V$, where A^n is the diagonal of the n-additive symmetric mapping $A_n : V^n \to W$.*

Proof. Assume that f satisfies the functional Equation (1). We get the equation $\Delta_x^{n+1} f(y) = D_n f(x, x+y) - D_n f(x,y) = 0$ for all $x, y \in V$. By Theorem 1, f is a generalized polynomial mapping of degree at most n, that is, f is of the form $f(x) = A^n(x) + A^{n-1}(x) + \cdots + A^2(x) + A^1(x) + A^0(x)$ for all $x \in V$, where $A^0(x) = A^0$ is an arbitrary element of W and $A^i(x) (i = 1, 2, \ldots, n)$ is the diagonal of an i-additive symmetric mapping $A_i : V^i \to W$. On the other hand, $f(2x) = 2^n f(x)$ holds for all $x \in V$ by Lemma 2, and so $f(x) = A^n(x)$.

Conversely, assume that $f(x) = A^n(x)$ for all $x \in V$, where $A^n(x)$ is the diagonal of the n-additive symmetric mapping $A_n : V^n \to W$. From $A^n(x+y) = A^n(x) + \sum_{i=1}^{n-1} {}_nC_i A^{n-i,i}(x,y)$,

$A^n(rx) = r^n A^n(x)$, $A^{n-i,i}(x, ry) = r^i A^{n-i,i}(x, y)$ $(x, y \in V, r \in \mathbb{Q})$, we see that f satisfies (1), which completes the proof of this theorem. □

Theorem 4. *Let S be a subset of a real vector space V and Y a real Banach space. Suppose that for each $x \in V \backslash \{0\}$ there exists a real number $r_x > 0$ such that $rx \in S$ for all $r \geq r_x$. Let $\varphi : S^2 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} 2^{-nk} \varphi(2^k x, 2^k y) < \infty \tag{9}$$

for all $x, y \in S$. If the mapping $f : S \to Y$ satisfies the inequality

$$\|D_n f(x, y)\| \leq \varphi(x, y) \tag{10}$$

for all $x, y \in S$, then there exists a unique monomial mapping of degree n $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \sum_{k=0}^{\infty} \frac{\Phi(2^k x)}{n! \cdot 2^{(k+1)n}} \tag{11}$$

for all $x \in S$, where

$$\Phi(x) := \sum_{j=0}^{n} {}_nC_j \varphi(x, jx + x) + \varphi(2x, x).$$

In particular, F is represented by

$$F(x) = \lim_{m \to \infty} \frac{f(2^m x)}{2^{mn}}$$

for all $x \in V$.

Proof. Let $x \in V \backslash \{0\}$ and m be an integer such that $2^m \geq r_x$. It follows from (7) in Lemma 2 and (10) that

$$\begin{aligned}
n! \left\| f(2^{m+1}x) - 2^n f(2^m x) \right\| &= \left\| \sum_{j=0}^{n} {}_nC_j D_n f(2^m x, 2^m(jx + x)) - D_n f(2^{m+1}x, 2^m x) \right\| \\
&\leq \sum_{j=0}^{n} \|{}_nC_j D_n f(2^m x, 2^m(j+1)x)\| + \|D_n f(2^{m+1}x, 2^m x)\| \\
&\leq \sum_{j=0}^{n} {}_nC_j \varphi(2^m x, 2^m(j+1)x) + \varphi(2^{m+1}x, 2^m x) \\
&= \Phi(2^m x).
\end{aligned}$$

From the above inequality, we get the following inequalities

$$\left\| \frac{f(2^m x)}{2^{mn}} - \frac{f(2^{m+1}x)}{2^{(m+1)n}} \right\| \leq \frac{\Phi(2^m x)}{n! \cdot 2^{n(m+1)}}$$

and

$$\left\| \frac{f(2^m x)}{2^{nm}} - \frac{f\left(2^{m+m'}x\right)}{2^{n(m+m')}} \right\| \leq \sum_{k=m}^{m+m'-1} \frac{\Phi(2^k x)}{n! \cdot 2^{n(k+1)}} \tag{12}$$

for all $m' \in \mathbb{N}$. So the sequence $\{\frac{f(2^m x)}{2^{mn}}\}_{m\in\mathbb{N}}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since $\lim_{m\to\infty} \frac{f(2^m 0)}{2^{mn}} = 0$ and Y is a real Banach space, we can define a mapping $F : V \to Y$ by

$$F(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^{nm}}$$

for all $x \in V$. By putting $m = 0$ and letting $m' \to \infty$ in the inequality (12), we obtain the inequality (11) if $x \in S$.

From the inequality (10), we get

$$\left\| \frac{D_n f(2^m x, 2^m y)}{2^{nm}} \right\| \leq \frac{\varphi(2^m x, 2^m y)}{2^{nm}}.$$

for all $x, y \in V\backslash\{0\}$, where $2^m \geq r_x, r_y$. Since the right-hand side in the above equality tends to zero as $m \to \infty$, we obtain that F satisfies the inequality (1) for all $x, y \in V\backslash\{0\}$. By Lemma 3 and $F(0) = 0$, F satisfies the Equality (1) for all $x, y \in V$. To prove the uniqueness of F, assume that F' is another monomial mapping of degree n satisfying the inequality (11) for all $x \in S$. The equality $F'(x) = \frac{F'(2^m x)}{2^{nm}}$ follows from the Equality (8) in Lemma 2 for all $x \in V\backslash\{0\}$ and $m \in \mathbb{N}$. Thus we can obtain the inequalities

$$\left\| F'(x) - \frac{f(2^m x)}{2^{nm}} \right\| = \left\| \frac{F'(2^m x)}{2^{nm}} - \frac{f(2^m x)}{2^{nm}} \right\| \leq \sum_{k=0}^{\infty} \frac{\Phi(2^{k+m}x)}{n! \cdot 2^{n(k+m+1)}} = \sum_{k=m}^{\infty} \frac{\Phi(2^k x)}{n! \cdot 2^{n(k+1)}},$$

for all $x \in V\backslash\{0\}$, where $2^m \geq r_x$. Since $\sum_{k=m}^{\infty} \frac{\Phi(2^k x)}{n! 2^{n(k+1)}} \to 0$ as $m \to \infty$ and $F'(0) = 0$, $F'(x) = \lim_{m\to\infty} 2^{-nm} f(2^m x)$ for all $x \in V$, i.e., $F(x) = F'(x)$ for all $x \in V$. This completes the proof of the theorem. □

We can give a generalization of Theorems 2 and 5 in [6] as the following corollary.

Corollary 1. *Let p and r be real numbers with $p < n$ and $r > 0$, let X be a normed space, $\varepsilon > 0$, and $f : X \to Y$ be a mapping such that*

$$\|D_n f(x, y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{13}$$

for all $x, y \in X$ with $\|x\|, \|y\| > r$. Then there exists a unique monomial mapping of degree n $F : X \to Y$ satisfying

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\varepsilon(2^p+2^n+1+\sum_{j=0}^{n} {}_nC_j(j+1)^p)}{n!(2^n-2^p)}\|x\|^p & (\text{for } \|x\| > r), \\ \frac{((k+1)^p+k^p)\varepsilon\|x\|^p}{n} + (k^p + \sum_{i=2}^{n} {}_nC_i(ik+i-k)^p + n!(k+1)^p) \\ \times \frac{\varepsilon(2^p+2^n+1+\sum_{j=0}^{n} {}_nC_j(j+1)^p)}{n!n(2^n-2^p)}\|x\|^p & (\text{for } \|kx\| > r). \end{cases}$$

In particular, if $p < 0$, then f is a monomial mapping of degree n itself.

Proof. If we set $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^p)$ and $S = \{x \in X | \|x\| > r\}$, then there exists a unique monomial mapping of degree n $F : X \to Y$ satisfying

$$\|f(x) - F(x)\| \leq \frac{\varepsilon(2^p + 2^n + 1 + \sum_{j=0}^{n} {}_nC_j(j+1)^p)}{n!(2^n - 2^p)}\|x\|^p \tag{14}$$

for all $x \in X$ with $\|x\| > r$ by Theorem 4. Notice that if F is a monomial mapping of degree n, then $D_n F((k+1)x, -kx) = 0$ for all $x \in X$ and $k \in \mathbb{R}$. Hence the equality

$$\begin{aligned}(-1)^n \cdot n(f(x) - F(x)) =& D_n f((k+1)x, -kx) + (-1)^n (F - f)(-kx) \\ &+ \sum_{i=2}^{n} {}_nC_i (-1)^{n-i} (F - f)(i(k+1)x - kx) \\ &- n!(F - f)((k+1)x)\end{aligned}$$

holds for all $x \in X$ and $k \in \mathbb{R}$. So if $F : X \to Y$ is the monomial mapping of degree n satisfying (14), then $F : X \to Y$ satisfies the inequality with a real number k

$$\begin{aligned}\|f(x) - F(x)\| \le& \frac{((k+1)^p + k^p)\varepsilon\|x\|^p}{n} + (k^p + \sum_{i=2}^{n} {}_nC_i (ik + i - k)^p + n!(k+1)^p) \\ &\times \frac{\varepsilon(2^p + 2^n + 1 + \sum_{j=0}^{n} {}_nC_j (j+1)^p)}{n!n(2^n - 2^p)} \|x\|^p \end{aligned} \tag{15}$$

for all $\|kx\| > r$.

Moreover, if $p < 0$, then $\lim_{k\to\infty}(k^p + \sum_{i=2}^{n} C_i^n (ik + i - k)^p + n!(k+1)^p) = 0$ and $\lim_{k\to\infty}(k^p + (k+1)^p) = 0$. Hence we get

$$f(x) = F(x)$$

for all $x \in X\backslash\{0\}$ by (15). Since $\lim_{k\to\infty} k^p = 0$ and the inequality

$$\begin{aligned}\|f(0) - F(0)\| \le& \frac{1}{n}\|D_n(f - F)(kx, -kx) + (-1)^n (F - f)(-kx) \\ &+ \sum_{i=2}^{n} {}_nC_i (-1)^{n-i} (F - f)((i-1)kx) - n!(F - f)(kx)\| \\ \le& \frac{1}{n}\left[2 + \frac{2^p + 2^n}{n!(2^n - 2^p)}(1 + \sum_{i=1}^{n-1} {}_nC_{i+1} i^p + n!)\right] k^p \varepsilon \|x\|^p\end{aligned}$$

holds for any fixed $x \in X\backslash\{0\}$ with $\|x\| > r$ and all natural numbers k, we get

$$f(0) = F(0).$$

□

Theorem 5. *Let S be a subset of a real vector space V and Y a real Banach space. Suppose that for each $x \in V$ there exists a real number $r_x > 0$ such that $rx \in S$ for all $r \le r_x$. Let $\varphi : V^2 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} 2^{nk} \varphi(2^{-k}x, 2^{-k}y) < \infty \tag{16}$$

for all $x, y \in S$. Suppose that a mapping $f : V \to Y$ satisfies the inequality (10) for all $x, y \in S$, where $ix + y \in S$ for all $i = 0, 1, \ldots, n$. Then there exists a unique monomial mapping of degree n $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \sum_{k=0}^{\infty} \frac{2^{nk}}{n!} \Phi\left(\frac{x}{2^{k+1}}\right) \tag{17}$$

for all x with $nx \in S$, where $\Phi(x)$ is defined as in Theorem 4. In particular, F is represented by

$$F(x) = \lim_{m\to\infty} 2^{nm} f(2^{-m}x)$$

for all $x \in V$.

Proof. Let $x \in V$ and m be an integer such that $2^{-m}(n+1) \leq r_x$. It follows from (7) in Lemma 2 and (10) that

$$\begin{aligned} n! \left\| f\left(\frac{x}{2^m}\right) - 2^n f\left(\frac{x}{2^{m+1}}\right) \right\| &= \left\| \sum_{j=0}^{n} {}_nC_j D_n f\left(\frac{x}{2^{m+1}}, \frac{(j+1)x}{2^{m+1}}\right) - D_n f\left(\frac{x}{2^m}, \frac{x}{2^{m+1}}\right) \right\| \\ &\leq \sum_{j=0}^{n} {}_nC_j \left\| D_n f\left(\frac{x}{2^{m+1}}, \frac{(j+1)x}{2^{m+1}}\right) \right\| + \left\| D_n f\left(\frac{x}{2^m}, \frac{x}{2^{m+1}}\right) \right\| \\ &\leq \sum_{j=0}^{n} {}_nC_j \varphi\left(\frac{x}{2^{m+1}}, \frac{(j+1)x}{2^{m+1}}\right) + \varphi\left(\frac{x}{2^m}, \frac{x}{2^{m+1}}\right) \\ &= \Phi\left(\frac{x}{2^{m+1}}\right) \end{aligned}$$

for all $x \in V$. From the above inequality, we get the inequality

$$\left\| 2^{nm} f\left(\frac{x}{2^m}\right) - 2^{n(m+m')} f\left(\frac{x}{2^{m+m'}}\right) \right\| \leq \sum_{k=m}^{m+m'-1} \frac{2^{nk}}{n!} \Phi\left(\frac{x}{2^{k+1}}\right) \tag{18}$$

for all $x \in V$ and $m' \in \mathbb{N}$. So the sequence $\{2^{nm} f\left(\frac{x}{2^m}\right)\}_{m \in \mathbb{N}}$ is a Cauchy sequence by the inequality (16). From the completeness of Y, we can define a mapping $F : V \to Y$ by

$$F(x) = \lim_{m \to \infty} 2^{nm} f\left(\frac{x}{2^m}\right)$$

for all $x \in V$. Moreover, by putting $m = 0$ and letting $m' \to \infty$ in (18), we get the inequality (17) for all $x \in S$ with $(n+1)x \in S$. From the inequality (10), if m is a positive integer such that $\frac{ix+y}{2^m} \in S$ for all $i = 0, 1, \ldots, n$, then we get

$$\left\| 2^{nm} D_n f\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\| \leq 2^{nm} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right),$$

for all $x, y \in V$. Since the right-hand side in this inequality tends to zero as $m \to \infty$, we obtain that F is a monomial mapping of degree n. To prove the uniqueness of F assume that F' is another monomial mapping of degree n satisfying the inequality (17) for all $x \in S$ with $(n+1)x \in S$. So the equality $F'(x) = 2^{nm} F'\left(\frac{x}{2^m}\right)$ holds for all $x \in V$ by (8) in Lemma 2. Thus, we can infer that

$$\begin{aligned} \left\| F'(x) - 2^{nm} f\left(\frac{x}{2^m}\right) \right\| &= \left\| 2^{nm} F'\left(\frac{x}{2^m}\right) - 2^{nm} f\left(\frac{x}{2^m}\right) \right\| \\ &\leq \sum_{k=0}^{\infty} 2^{n(m+k)} \Phi\left(\frac{x}{2^{m+k+1}}\right) \\ &\leq \sum_{k=m}^{\infty} 2^{nk} \Phi\left(\frac{x}{2^{k+1}}\right) \end{aligned}$$

for all positive integers m, where $\frac{(n+1)x}{2^m} \leq r_x$. Since $\sum_{k=m}^{\infty} 2^{nk} \Phi\left(\frac{x}{2^{k+1}}\right) \to 0$ as $m \to \infty$, we know that $F'(x) = \lim_{m \to \infty} 2^{nm} f\left(\frac{x}{2^m}\right)$ for all $x \in V$. This completes the proof of the theorem. $\square$

We can give a generalization of Theorem 3 in [6] as the following corollary.

Corollary 2. *Let p and r be real numbers with $p > n$ and $r > 0$, and X a normed space. Let $f : X \to Y$ be a mapping satisfying the inequality (13) for all x, y with $\|x\|, \|y\| < r$. Then there exists a unique monomial mapping of degree n $F : X \to Y$ satisfying*

$$\|f(x) - F(x)\| \le \frac{\varepsilon(2^p + 2^n + 1 + \sum_{j=0}^{n} {}_nC_j (j+1)^p)}{n!(2^n - 2^p)} \|x\|^p$$

for all $x \in X$ with $\|x\| < \frac{r}{n+1}$.

The following example shows that the assumption $p \neq n$ cannot be omitted in Corollarys 1 and 2. This example is an extension of the example of Gajda [21] for the monomial functional inequality (13) (see also [22]).

Example 1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\psi(x) = \begin{cases} x^n, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \ge 1. \end{cases} \tag{19}$$

Consider that the function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \sum_{m=0}^{\infty} (n+1)^{-nm} \psi((n+1)^m x) \tag{20}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$\left| \sum_{i=0}^{n} {}_nC_i (-1)^i f(ix + y) - n! f(x) \right| \le 4 \cdot (n+1)!(n+1)^{2n}(|x|^n + |y|^n). \tag{21}$$

for all $x, y \in \mathbb{R}$, but there do not exist a monomial mapping of degree n $F : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that $|f(x) - F(x)| \le d|x|^n$ for all $x \in \mathbb{R}$.

Proof. It is clear that f is bounded by 2 on $\mathbb{R}$. If $|x|^n + |y|^n = 0$, then f satisfies (21). And if $|x|^n + |y|^n \ge \frac{1}{(n+1)^n}$, then

$$|D_n f(x, y)| \le 2 \cdot 2 \cdot (n+1)! \le 4 \cdot (n+1)!(n+1)^n(|x|^n + |y|^n),$$

which means that f satisfies (21). Now suppose that $0 < |x|^n + |y|^n < \frac{1}{(n+1)^n}$. Then there exists a nonnegative integer k such that

$$\frac{1}{(n+1)^{n(k+2)}} \le |x|^n + |y|^n < \frac{1}{(n+1)^{n(k+1)}}. \tag{22}$$

Hence $(n+1)^{nk}|x|^n < \frac{1}{(n+1)^n}$, $(n+1)^{nk}|y|^n < \frac{1}{(n+1)^n}$, $|(n+1)^m(x + iy)| < 1$, and $|(n+1)^m y| < 1$ for all $m = 0, 1, \dots, k-1$. Hence, for $m = 0, 1, \dots, k-1$,

$$\sum_{i=0}^{n} {}_nC_i (-1)^i \psi((n+1)^m(ix + y)) - n!\psi((n+1)^m x) = 0. \tag{23}$$

From the definition of f, the inequality (22), and the inequality (23), we obtain that

$$\begin{aligned}|D_n f(x,y)| = &\left|\sum_{m=0}^{\infty}(n+1)^{-nm}\Big(\sum_{i=0}^{n}{}_nC_i(-1)^i\psi((n+1)^m(ix+y)) - n!\psi((n+1)^m x)\Big)\right| \\ \leq &\sum_{m=0}^{\infty}(n+1)^{-nm}\left|\sum_{i=0}^{n}{}_nC_i(-1)^i\psi((n+1)^m(ix+y)) - n!\psi((n+1)^m x)\right| \\ \leq &\sum_{m=k}^{\infty}(n+1)^{-nm}\left|\sum_{i=0}^{n}{}_nC_i(-1)^i\psi((n+1)^m(ix+y)) - n!\psi((n+1)^m x)\right| \\ \leq &\sum_{m=k}^{\infty}(n+1)^{-nm}2\cdot(n+1)! \leq 4\cdot(n+1)^{-nk}(n+1)! \\ \leq &4\cdot(n+1)^{2n}(n+1)!(|x|^n + |y|^n). \end{aligned} \tag{24}$$

Therefore, f satisfies (21) for all $x, y \in \mathbb{R}$. Now, we claim that the functional Equation (1) is not stable for $p = n$ in Corollarys 1 and 2. Suppose on the contrary that there exists a monomial mapping of degree n $F : R \to R$ and constant $d > 0$ such that $|f(x) - F(x)| \leq d|x|^n$ for all $x \in R$. Notice that $F(x) = x^n F(1)$ for all rational numbers x. So we obtain that

$$|f(x)| \leq (d + |F(1)|)|x|^n \tag{25}$$

for all $x \in Q$. Let $k \in N$ with $k + 1 > d + |F(1)|$. If x is a rational number in $(0, (n+1)^{-k})$, then $(n+1)^m x \in (0, 1)$ for all $m = 0, 1, \ldots, k$, and for this x we get

$$\begin{aligned}f(x) = &\sum_{m=0}^{\infty}(n+1)^{-nm}\psi((n+1)^m x) \geq \sum_{m=0}^{k}(n+1)^{-nm}((n+1)^m x)^n \\ = &(k+1)x^n > (d + |F(1)|)x^n\end{aligned}$$

which contradicts (25). □

3. Conclusions

The advantage of this paper is that we do not need to prove the stability of additive quadratic, cubic, and quartic functional equations separately. Instead we can apply our main theorem to prove the stability of those functional equations simultaneously.

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References

1. Mazur, S.; Orlicz, W. Grundlegende eigenschaften der polynomischen operationen. Erste mitteilung. *Stud. Math.* **1934**, *5*, 50–68.
2. Mazur, S.; Orlicz, W. Grundlegende eigenschaften der polynomischen operationen. Zweite mitteilung. *Stud. Math.* **1934**, *5*, 179–189.
3. Djoković, D.Z. A representation theorem for $(X_1 - 1)(X_2 - 1)\cdots(X_n - 1)$ and its applications. *Ann. Polon. Math.* **1969** *22*, 189–198.
4. Cădariu, L.; Radu, V. The fixed points method for the stability of some functional equations. *Caepathian J. Math.* **2007**, *23*, 63–72.
5. Choi, C.-K. Stability of a Monomial Functional Equation on Restricted Domains of Lebesgue Measure Zero. *Results Math.* **2017**, 1–11, doi:10.1007/s00025-017-0742-0.
6. Lee, Y.-H. On the stability of the monomial functional equation. *Bull. Korean Math. Soc.* **2008**, *45*, 397–403.

7. Mckiernan, M.A. On vanishing n-th odered difference and Hamel bases, *Ann. Polon. Math.* **1967**, *19*, 331–336.
8. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience Publishers: New York, NY, USA, 1960.
9. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224.
10. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66.
11. Cholewa, P.W. Remarks on the stability of functional equations. *Aequ. Math.* **1984**, *27*, 76–86.
12. Czerwik, S. *Functional Equations and Inequalities in Several Variables*; World Scientific: Hackensacks, NJ, USA, 2002.
13. Găvruta, P. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436.
14. Jung, S.-M. On the Hyers-Ulam stability of the functional equations that have the quadratic property. *J. Math. Anal. Appl.* **1998**, *222*, 126–137.
15. Lee, Y.H.; Jun, K.W. On the stability of approximately additive mappings. *Proc. Am. Math. Soc.* **2000**, *128*, 1361–1369.
16. Skof, F. Local properties and approximations of operators. *Rend. Sem. Mat. Fis. Milano* **1983**, *53*, 113–129.
17. Rassias, T.M.On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300.
18. Gilányi, A. On Hyers-Ulam stability of monomial functional equations. *Abh. Math. Sem. Univ. Hamburg* **1998**, *68*, 321–328.
19. Gilányi, A. Hyers-Ulam stability of monomial functional equations on a general domain. *Proc Natl. Acad. Sci. USA* **1999**, 96, 10588–10590.
20. Gilányi, A. On the stability of monomial functional equations. *Publ. Math. Debrecen* **2000**, *56*, 201–212.
21. Gajda, Z. On stability of additive mappings. *Int. J. Math. Math. Sci.* **1991**, *14*, 431–434.
22. Xu, T.Z.; Rassias, J.M.; Rassias, M.J.; Xu, W.X. A fixed point approach to the stability of quintic and sextic functional equations in quasi—Normed spaces. *J. Inequal. Appl.* **2010**, *2010*, 423231.