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Generalized Hyers-Ulam Stability of Trigonometric Functional Equations

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Abstract: In the present paper we study the generalized Hyers–Ulam stability of the generalized trigonometric functional equations

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2h(y),\ x, y \in S;$$

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(y)g(x) + 2h(x),\ x, y \in S,$$

where S is a semigroup, σ: $S \longrightarrow S$ is a involutive morphism, and μ: $S \longrightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. As an application, we establish the generalized Hyers–Ulam stability theorem on amenable monoids and when σ is an involutive automorphism of S.

Keywords: Hyers-Ulam stability; trigonometric functional equations; semigroup

1. Introduction

Let us consider S to be a semigroup (namely a set with an associative composition), μ: $S \longrightarrow \mathbb{C}$ to be a multiplicative function, and σ: $S \longrightarrow S$ to be an involutive morphism. That is, σ is an involutive automorphism:

$$\sigma(xy) = \sigma(x)\sigma(y) \text{ and } \sigma(\sigma(x)) = x \text{ for all } x, y \in S$$

or σ is an involutive anti-automorphism:

$$\sigma(xy) = \sigma(y)\sigma(x) \text{ and } \sigma(\sigma(x)) = x \text{ for all } x, y \in S.$$

From the functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2h(y),\ x, y \in S \tag{1}$$

we can obtain several other functional equations as a special case. For example, we can deduce: The Cauchy equation

$$f(xy) = f(x) + f(y),\ x, y \in S \tag{2}$$

($g = 1$, $\mu = 1$, $\sigma = I$), where I denotes the identity map.

The quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y),\ x, y \in S \tag{3}$$

$(g = 1, \mu = 1, f = h)$.

Wilson's functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y), \ x, y \in S \tag{4}$$

$(h = 0)$.

D'Alembert's functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)f(y), \ x, y \in S \tag{5}$$

$(g = f, h = 0)$.

D'Alembert's functional Equation (5) with $\mu = 1$ is also known as the cosine functional equation and has been studied extensively for a long period of time tracing back to d'Alembert [1]. This functional equation plays a crucial role in determining the sum of two vectors in various Euclidean and non-Euclidean geometries. The continuous solutions $f \colon \mathbb{R} \longrightarrow \mathbb{C}$ of d'Alembert's functional Equation (5) with $\mu = 1$ are known: A part from the trivial solution $f = 0$, the solutions of (5) are

$$f_\lambda(x) = \cos(\lambda x), \ \ x \in \mathbb{R}$$

where the parameter λ ranges over $\mathbb{C}$ (see for example [2]).

Several authors have succeeded to determine the general solution $f : S \longrightarrow \mathbb{C}$ of d'Alembert's functional Equation (5) in the abelian as well as non abelian case. Probably the very first result obtained for a non abelian group was presented by Kannappan [3]. Under the condition that f is abelian: $f(zxy) = f(zyx)$ for all $x, y, z \in S$, the solutions of the Equation (5) with $\mu = 1$ are of the form

$$f(x) = \frac{\gamma(x) + \gamma(\sigma(x))}{2}, \text{ where } \gamma : S \longrightarrow \mathbb{C}$$

is multiplicative.

In recent years, the theory of d'Alembert's functional Equation (5) with $\mu = 1$ has witnessed important development. For example, for the case of non abelian groups, as shown in works by Y. Dilian about compact groups [4–6], Stetkær [7] for step 2-nilpotent groups, Friis [8] for results on Lie groups and Davison [9,10] for general groups, even monoids.

In [11], Stetkær obtained the complex valued solutions of d'Alembert's functional Equation (5) for the case when μ is a character of the group S. The non-zero solutions of the Equation (5) are the normalized traces of certain representations of the group S on $\mathbb{C}^2$

Furthermore, in [12] Ebanks and Stetkær presented some new results on groups regarding the solutions of Wilson's functional Equation (4) with $\mu = 1$. We shall now also refer to Wilson's first generalization of d'Alembert's functional equation:

$$f(x+y) + f(x-y) = 2f(x)g(y), \ x, y \in \mathbb{R}.$$

The formulas constituting the solutions of this equation for the case of abelian groups are known, cf. Aczél [2], Sections 3.2.1 and 3.2.2.

In recent work, Stetkær ([13,14]) studied the solutions of Wilson's functional Equation (4) and in particular he proved that if f, g are solutions of (4) with $f \neq 0$ then g satisfies d'Alembert's functional Equation (5) [15]. Determining the solution formulas of f is still an open problem.

In 1940, Ulam posed the stability problem for group homomorphisms [16]. The first affirmative answer to Ulam's question was presented in 1941 by Hyers [17] on Banach spaces. In 1978 Rassias [18] generalized Hyers' theorem for linear mappings by considering an unbounded Cauchy difference for sum of powers of norms. Rassias' theorem has been generalized by Gavruta [19] who allowed the Cauchy difference to be bounded by a general control function. Since that period, the corresponding area has become a very vibrant domain of research and stability problems for several functional equations have been extensively investigated by a number of authors (cf. [20–39]).

The stability (superstability) of d'Alembert's functional equation was first obtained by Baker [40]. Another generalization of Baker's result was presented by Székelyhidi [41]. This involves an interesting generalization of the class of bounded functions on a group or semigroup. For a series of interesting stability and superstability results, one is also reffered to the following works [20,21,42–54] Bouikhalene and Elqorachi [55] for general groups.

The generalized Hyers-Ulam stability of the functional equations (1) and

$$f(xy)+\mu(y)f(x\sigma(y))=2f(y)g(x)+2h(x),\ x,y\in S \tag{6}$$

with $\mu=1$ was studied by Badora [42] and Akkouchi [56].

A variety of stability results regarding trigonometric functional equations and their generalizations are obtained (cf. [27,57]).

The main purpose of the present paper is to study the stability of the functional Equations (1) and (6). In the sequel, we obtain some properties of the stability of Equation (1) as well as Equation (6). As an application we prove the generalized Hyers-Ulam stability of Equations (1) and (6) on amenable monoids S and when σ is an involutive automorphism of S.

2. Generalized Hyers–Ulam Stability of Equation (1) on Non-Abelian Semigroups

In the present section, we obtain properties of the stability of Equation (1).

Theorem 1. *Let σ: $S\longrightarrow S$ be an involutive morphism of the semigroup S. Let μ: $S\longrightarrow\mathbb{C}$ be a multiplicative function such that $\mu(x\sigma(x))=1$ for all $x\in S$. Suppose that the functions f,g,h: $S\longrightarrow\mathbb{C}$ satisfy the functional inequality*

$$|f(xy)+\mu(y)f(x\sigma(y))-2f(x)g(y)-2h(y)|\leq\phi(y) \tag{7}$$

for all $x,y\in S$ and for some function ϕ: $S\longrightarrow\mathbb{R}^{+}$. Under these assumptions the following statements hold:
(1) If σ is an involutive anti-automorphism and f is unbounded, then g is a solution of the long d'Alembert functional equation

$$g(xy)+g(yx)+\mu(y)g(x\sigma(y))+\mu(y)g(\sigma(y)x)=4g(x)g(y) \tag{8}$$

for all $x,y\in S$.
(2) If σ is an involutive automorphism and f is unbounded, then g is a solution of the short d'Alembert functional Equation (5).

Proof. (1) Let f,g,h satisfy Inequality (7) with σ an involutive anti-automorphism. Then for all $x,y,z\in S$ we have

$$\begin{aligned}
&|2f(z)[g(xy)+\mu(y)g(x\sigma(y))+g(yx)+\mu(y)g(\sigma(y)x)-4g(x)g(y)]|\\
&=|2f(z)g(xy)+\mu(y)2f(z)g(x\sigma(y))+2f(z)g(yx)+\mu(y)2f(z)g(\sigma(y)x)-8f(z)g(x)g(y)|\\
&\leq|-f(zxy)-\mu(xy)f(z\sigma(y)\sigma(x))+2f(z)g(xy)+2h(xy)|\\
&+|\mu(y)[-f(zx\sigma(y))-\mu(x\sigma(y))f(zy\sigma(x))+2f(z)g(x\sigma(y))+2h(x\sigma(y))]|\\
&+|-f(zyx)-\mu(yx)f(z\sigma(x)\sigma(y))+2f(z)g(yx)+2h(yx)|\\
&+|\mu(y)[-f(z\sigma(y)x)-\mu(\sigma(y)x)f(z\sigma(x)y)+2f(z)g(\sigma(y)x)+2h(\sigma(y)x)]|\\
&+|f(zxy)+\mu(y)f(zx\sigma(y))-2f(zx)g(y)-2h(y)|
\end{aligned}$$

$$
\begin{aligned}
&+|\mu(x)(f(z\sigma(x)y)+\mu(y)f(z\sigma(x)\sigma(y))-2f(z\sigma(x))g(y)-2h(y))|\\
&+|f(zyx)+\mu(x)f(zy\sigma(x)-2f(zy)g(x)-2h(x)|\\
&+|\mu(y)[f(z\sigma(y)x)+\mu(x)f(z\sigma(y)\sigma(x))-2f(z\sigma(y))g(x)-2h(x)]|\\
&+|2g(y)[f(zx)+\mu(x)f(z\sigma(x))-2f(z)g(x)-2h(x)|\\
&+|2g(x)[f(zy)+\mu(x)f(z\sigma(y))-2f(z)g(y)-2h(y)|\\
&+|4g(x)h(y)+4g(y)h(x)+2\mu(y)h(x)-2h(xy)-2\mu(y)h(x\sigma(y))\\
&-2h(yx)-2\mu(y)h(\sigma(y)x)+2h(y)+2h(x)+2\mu(x)h(y)|\\
&\leq \phi(xy)+|\mu(y)|\phi(x\sigma(y))+\phi(yx)+|\mu(y)|\phi(\sigma(y)x)+\phi(y)\\
&+|\mu(x)|\phi(y)+\phi(x)+|\mu(y)|\phi(x)+2|g(y)|\phi(x)+2|g(x)|\phi(y)\\
&+|4g(x)h(y)+4g(y)h(x)+2\mu(y)h(x)-2h(xy)-2\mu(y)h(x\sigma(y))\\
&-2h(yx)-2\mu(y)h(\sigma(y)x)+2h(y)+2h(x)+2\mu(x)h(y)|.
\end{aligned}
$$

Since f is assumed to be unbounded, then g satisfies the functional Equation (8).
(2) If σ is an involutive automorphism, then, by using Inequality (7), $\mu(x\sigma(x))=1$ and the triangle inequality, we obtain

$$
\begin{aligned}
&|2f(z)[g(xy)+\mu(y)g(x\sigma(y))-2g(x)g(y)]|\\
&\leq |2f(z)g(xy)-f(zxy)-\mu(xy)f(z\sigma(x)\sigma(y))+2h(xy)|\\
&+|\mu(y)[2f(z)g(x\sigma(y))-f(zx\sigma(y))-\mu(x\sigma(y))f(z\sigma(x)y)+2h(x\sigma(y))]|\\
&+|f(zxy)+\mu(y)f(zx\sigma(y))-2f(zx)g(y)-2h(y)|\\
&+|\mu(x)[f(z\sigma(x)y)+\mu(y)f(z\sigma(x)\sigma(y))-2f(z\sigma(x))g(y)-2h(y)]|\\
&|2g(y)[f(zx)+\mu(x)f(z\sigma(x))-2f(z)g(x)-2h(x)]\\
&-2h(xy)-2\mu(y)h(x\sigma(y))+2h(y)+2\mu(x)h(y)+4g(y)h(x)|\\
&\leq \phi(xy)+|\mu(y)|\phi(x\sigma(y))+\phi(y)+|\mu(x)|\phi(y)+2|g(y)|\phi(x)\\
&+|2h(y)+2\mu(x)h(y)+4g(y)h(x)-2h(xy)-2\mu(y)h(x\sigma(y))|.
\end{aligned}
$$

The mapping f is assumed to be unbounded, so g is a solution of the short d'Alembert functional Equation (5). This completes the proof. □

Theorem 2. *Let σ: $S \longrightarrow S$ be an involutive automorphism of the amenable semigroup S. Let μ: $S \longrightarrow \mathbb{C}$ be a multiplicative function such that $\mu(x\sigma(x))=1$ for all $x \in S$. Suppose that the functions f, g, h: $S \longrightarrow \mathbb{C}$ satisfy the functional inequality*

$$|f(xy)+\mu(y)f(x\sigma(y))-2f(x)g(y)-2h(y)| \leq \phi(y) \tag{9}$$

for all $x, y \in S$ and for some function ϕ: $S \longrightarrow \mathbb{R}^+$. Under the additionally assumption that f is unbounded, there is a mapping $H : S \longrightarrow \mathbb{C}$ such that

$$H(xy)+\mu(y)H(x\sigma(y))=2H(x)g(y)+(1+\mu(x))H(y) \tag{10}$$

and

$$|h(x)-H(x)| \leq \frac{1}{2}\phi(x) \tag{11}$$

for all $x, y \in S$.

Proof. For each y fixed in S, the function

$$x \longrightarrow f(xy) + \mu(y)f(x\sigma(y)) - f(x)g(y)$$

is bounded. Since S is an amenable semigroup, then, from [58], there is an invariant mean on $B(S,\mathbb{C})$—the space of the complex-valued bounded functions on S, which we denote by m. We can now define the following mapping $H : S \longrightarrow \mathbb{C}$ by

$$H(x) = m[\frac{f_x + \mu(x)f_{\sigma(x)}}{2} - g(x)f], \ \ x \in S,$$

where $f_x(y) = f(yx)$, $x, y \in S$. For all $x, y \in S$, we have

$$H(xy) + \mu(y)H(x\sigma(y)) = m[\frac{f_{xy} + \mu(xy)f_{\sigma(x)\sigma(y)}}{2} - g(xy)f]$$

$$+\mu(y)m[\frac{f_{x\sigma(y)} + \mu(x\sigma(y))f_{\sigma(x)y}}{2} - g(x\sigma(y))f].$$

From Theorem 2 (2), g is a solution of the short d'Alembert functional Equation (5), so we obtain

$$H(xy) + \mu(y)H(x\sigma(y)) = m[\frac{f_{xy} + \mu(xy)f_{\sigma(x)\sigma(y)}}{2} + \frac{\mu(y)f_{x\sigma(y)} + \mu(x)f_{\sigma(x)y}}{2} - 2g(x)g(y)f]$$

$$= m[\frac{(f_y + \mu(y)f_{\sigma(y)})_x}{2} - g(y)f_x] + \mu(x)[\frac{(f_y + \mu(y)f_{\sigma(y)})_{\sigma(x)}}{2} - g(y)f_{\sigma(x)})]$$

$$+2g(y)[\frac{f_x + \mu(x)f_{\sigma(x)}}{2} - g(x)f]]$$

$$= m[\frac{f_y + \mu(y)f_{\sigma(y)}}{2} - g(y)f]_x + \mu(x)m[\frac{f_y + \mu(y)f_{\sigma(y)}}{2} - g(y)f]_{\sigma(x)}$$

$$+2g(y)m[\frac{f_x + \mu(x)f_{\sigma(x)}}{2} - g(x)f]$$

$$= H(y) + \mu(x)H(y) + 2g(y)H(x) = (1 + \mu(x))H(y) + 2g(y)H(x).$$

Now, by using the definition of H, Inequality (9) and the definition of m, we obtain

$$|h(y) - H(y)| = \frac{1}{2}|m[f_y + \mu(y)f_\sigma(y) - 2g(y)f] - 2h(y)|$$

$$\leq \frac{1}{2}sup_{x\in S}|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y) - 2h(y)| \leq \frac{1}{2}\phi(y)\,.$$

for all $y \in S$. This completes the proof. □

Theorem 3. *Let M be a monoid (a semigroup with identity element e). Let σ: $M \longrightarrow S$ be an involutive automorphism of the amenable monoid M. Let μ: $S \longrightarrow \mathbb{C}$ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in M$. Suppose that the functions f, g, h: $S \longrightarrow \mathbb{C}$ satisfy the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y) - 2h(y)| \leq \phi(y) \tag{12}$$

for all $x, y \in M$ and for some function ϕ: $S \longrightarrow \mathbb{R}^+$. Under the additionally assumption that f is unbounded, there are mappings $F, H : M \longrightarrow \mathbb{C}$ such that

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(x)g(y) + (1 + \mu(x))H(y) \tag{13}$$

$$F(xy) + \mu(y)F(x\sigma(y)) = 2F(x)g(y) + (1 + \mu(x))H(y) \tag{14}$$

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) \tag{15}$$

$$|h(x) - H(x)| \leq \frac{\phi(x)}{2} \tag{16}$$

and

$$|f(x) + \mu(x)f(\sigma(x)) - 2F(x)| \leq 2\phi(x) \tag{17}$$

for all $x, y \in M$.

Proof. From Theorem 2, there is an H: $M \longrightarrow \mathbb{C}$ such that

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(x)g(y) + (1 + \mu(x))H(y)$$

and

$$|h(x) - H(x)| \leq \frac{\phi(x)}{2}$$

for all $x, y \in M$. By replacing x by e in Inequality (12), we obtain

$$|f(y) + \mu(y)f(\sigma(y)) - 2f(e)g(y) - 2h(y)| \leq \phi(y)$$

for all $y \in M$. If we set $F = f(e)g + H$, we obtain

$$|f(x) + \mu(x)f(\sigma(x)) - 2F(x)| \leq |f(x) + \mu(x)f(\sigma(x)) - 2f(e)g(x) - 2h(x)| + |2h(x) - 2H(x)|$$

$$\leq \phi(x) + \phi(x) = 2\phi(x)$$

for all $x \in M$.

On the other hand, we have

$$F(xy) + \mu(y)F(x\sigma(y)) = c\sigma[g(xy) + \mu(y)g(x\sigma(y))] + [H(xy) + \mu(y)H(x\sigma(y))$$

$$= 2f(e)g(x)g(y) + 2H(x)g(y) + (1 + \mu(x))H(y) = 2(F(x)g(y) + (1 + \mu(x)H(y)$$

for all $x, y \in M$. This completes the proof. □

3. Generalized Hyers–Ulam Stability of Equation (6) on Non-Abelian Semigroups

In this section, we obtain the stability of Equation (6) on an amenable monoid.

Theorem 4. *Let σ: $S \longrightarrow S$ be an involutive automorphism of the semigroup S. Let μ: $S \longrightarrow \mathbb{C}$ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Suppose that the functions f, g, h: $S \longrightarrow \mathbb{C}$ satisfy the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(y)g(x) - 2h(x)| \leq \phi(x) \tag{18}$$

for all $x, y \in S$ and for some function ϕ: $S \longrightarrow \mathbb{R}^+$. Under the additionally assumption that f is unbounded, g is a solution of the short d'Alembert functional Equation (5).

Proof. By using Inequality (18), $\mu(x\sigma(x)) = 1$, and σ an involutive automorphism, we obtain

$$|2f(z)[g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)]|$$

$$\leq |2f(z)g(xy) - f(xyz) - \mu(z)f(xy\sigma(z)) + 2h(xy)|$$

$$+ |\mu(y)[2f(z)g(x\sigma(y)) - f(x\sigma(y)z) - \mu(z)f(x\sigma(y)\sigma(z)) + 2h(x\sigma(y))]|$$

$$+ |f(xyz) + \mu(yz)f(x\sigma(yz)) - 2f(yz)g(x) - 2h(x)$$
$$+ \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z) - 2f(y\sigma(z))g(x) - 2h(x)]$$
$$+ |2g(x)[f(yz) + \mu(z)f(y\sigma(z)) - 2f(z)g(y) - 2h(y)]$$
$$- 2\mu(y)h(x\sigma(y)) + 2h(x) + 2\mu(z)h(x) + 4g(x)h(y) - 2h(xy)|$$
$$\leq \phi(xy) + \phi(x) + |\mu(z)|\phi(x) + |\mu(y)|\phi(x\sigma(y)) + 2|g(x)|\phi(y)$$
$$+ | - 2h(xy) - 2\mu(y)h(x\sigma(y)) + 2h(x) + 2\mu(z)h(x) + 4g(x)h(y)|$$

for all $x, y, z \in S$. The mapping f is assumed to be unbounded and μ is bounded, so g is a solution of the short d'Alembert functional Equation (5). This completes the proof. □

Theorem 5. *Let σ: $S \longrightarrow S$ be an involutive automorphism of the amenable semigroup S. Let μ: $S \longrightarrow \mathbb{C}$ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Suppose that the functions $f, g, h : S \longrightarrow \mathbb{C}$ satisfy the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y) - 2h(y)| \leq \phi(y) \tag{19}$$

for all $x, y \in S$ and for some function ϕ: $S \longrightarrow \mathbb{R}^+$. Under the additionally assumption that f is unbounded, there is a mapping $H : S \longrightarrow \mathbb{C}$ such that

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(y)g(x) + 2H(x) \tag{20}$$

and

$$|h(x) - H(x)| \leq \frac{1}{2}\phi(x) \tag{21}$$

for all $x, y \in S$.

Proof. For a mapping l: $S \longrightarrow \mathbb{C}$, we define the new functions ${}_xl$ and l^μ by ${}_xl(y) = l(xy)$ and $l^\mu(x) = \mu(x)l(\sigma(x))$ for all $x, y \in S$.

From Inequality (19) for each fixed x in S, the function

$$\frac{{}_xf + ({}_xf)^\mu}{2} - fg(x)$$

is bounded. Since, S is amenable semigroup, then there is an invariant mean m on $B(S, \mathbb{C})$. By replacing m by M with $M(l) = m(l^\mu)$, we can choose m such that $m(l^\mu) = m(l)$ for all $l \in B(S, \mathbb{C})$.

The following mapping

$$H(x) = m[\frac{{}_xf + ({}_xf)^\mu}{2} - g(x)f], \quad x \in S$$

is well defined on S.

On the other hand, we obtain

$$\mu(y)({}_{x\sigma(y)}f)^\mu(z) = \mu(y)\mu(z)_{x\sigma(y)}f(\sigma(z))$$
$$= \mu(y)\mu(z)f(x\sigma(y)\sigma(z)) = \mu(yz)f(x\sigma(yz)) = \mu(yz)_xf(\sigma(yz))$$
$$= ({}_xf)^\mu(yz) = ({}_y({}_xf)^\mu)(z)\,,$$

which implies that

$$\mu(y)({}_{x\sigma(y)}f)^\mu =_y ({}_xf)^\mu$$

for all $x, y \in S$.

$$({}_{xy})^{\mu}(z) = \mu(z)_{xy}f(\sigma(z)) = \mu(z)f(xy\sigma(z)) = \mu(z)_{y}({}_{x}f)(\sigma(z)) .$$

Therefore, we have $({}_{xy}f)^{\mu} = ({}_{y}({}_{x}f))^{\mu}$ for all $x, y \in S$.

$$\mu(y)({}_{x\sigma(y)}f)^{\mu}(z) = \mu(y)\mu(z)f(x\sigma(y)\sigma(z))$$

$$= \mu(yz)f(x\sigma(yz)) = \mu(yz)_{x}f(\sigma(yz)) = ({}_{x}f)^{\mu}(z) .$$

Therefore, we have $\mu(y)({}_{x\sigma(y)}f)^{\mu} =_{y} ({}_{x}f)^{\mu}$ for all $x, y \in S$.

By using the definition of H, we obtain

$$H(xy) + \mu(y)H(x\sigma(y)) = m[\frac{{}_{xy}f + ({}_{xy}f)^{\mu}}{2} - g(xy)f]$$

$$+\mu(y)m[\frac{{}_{x\sigma(y)}f + ({}_{x\sigma(y)}f)^{\mu}}{2} - g(x\sigma(y))f].$$

From Theorem 4, g is a solution of the short d'Alembert functional Equation (5). Since m is additive, by using the above relations, we obtain

$$H(xy) + \mu(y)H(x\sigma(y)) = m[\frac{{}_{xy}f + ({}_{xy}f)^{\mu}}{2} - 2g(x)g(y)f]$$

$$+ [\mu(y)\frac{{}_{x\sigma(y)}f + \mu(y)({}_{x\sigma(y)}f)^{\mu}}{2}] = m[[\frac{{}_{y}({}_{x}f) +_{y} (({}_{x}f))^{\mu}}{2} -_{y} fg(x)]$$

$$+ [\frac{\mu(y)_{x\sigma(y)f} + ({}_{y}({}_{x}f))^{\mu}}{2} - ({}_{y}f)^{\mu}g(x)] + 2g(x)[\frac{{}_{y}f + ({}_{y}f)^{\mu}}{2} - fg(y)]$$

$$= m[[\frac{{}_{y}({}_{x}f) +_{y} (({}_{x}f))^{\mu}}{2} -_{y} fg(x)] + m[\frac{\mu(y)_{x\sigma(y)}f + ({}_{y}({}_{x}f))^{\mu}}{2} - ({}_{y}f)^{\mu}g(x)]$$

$$+ 2g(x)m[\frac{{}_{y}f + ({}_{y}f)^{\mu}}{2} - fg(y)].$$

Since m is invariant and $m(l^{\mu}) = m(l)$ for all bounded functions l on S, then we obtain

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(y)g(x) + 2H(x)$$

for all $x, y \in S$.

Finally, from Inequality (19) and the definition of H, we have

$$|h(y) - H(y)| = \frac{1}{2}|m[{}_{y}f + ({}_{y}f)^{\mu} - 2g(y)f] - 2h(y)|$$

$$\leq \frac{1}{2}sup_{x\in S}|f(yx) + \mu(x)f(y\sigma(x)) - 2g(y)f(x) - 2h(y)| \leq \frac{1}{2}\phi(y) .$$

for all $y \in S$. This completes the proof. $\square$

Theorem 6. *Let M be a monoid. Let σ: $M \longrightarrow S$ be an involutive automorphism of the amenable monoid M. Let μ: $S \longrightarrow \mathbb{C}$ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in M$. Suppose that the functions f, g, h: $S \longrightarrow \mathbb{C}$ satisfy the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(y)g(x) - 2h(x)| \leq \phi(x) \tag{22}$$

for all $x, y \in M$ and for some function ϕ: $S \longrightarrow \mathbb{R}^+$. Under the additional assumption that f is unbounded, there are mappings $F, H : M \longrightarrow \mathbb{C}$ such that

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(y)g(x) + 2H(x) \tag{23}$$

$$F(xy) + \mu(y)F(x\sigma(y)) = 2F(y)g(x) + 2H(x) \tag{24}$$

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) \tag{25}$$

$$|h(x) - H(x)| \leq \frac{\phi(x)}{2} \tag{26}$$

and

$$|f(x) - F(x)| \leq \phi(x) \tag{27}$$

for all $x, y \in M$.

Proof. From Theorem 4, there is a mapping H: $M \longrightarrow \mathbb{C}$ such that

$$H(xy) + \mu(y)H(x\sigma(y)) = 2H(y)g(x) + 2H(x)$$

and

$$|h(x) - H(x)| \leq \frac{\phi(x)}{2}$$

for all $x, y \in M$. By setting $y = e$ in Inequality (22), we obtain

$$|2f(x) - 2f(e)g(x) - 2h(x)| \leq \phi(x)\,.$$

Let $F = f(e)g + H$. For all $x \in M$, we have

$$|f(x) - F(x)| \leq |f(x) - f(e)g(x) - h(x)| + |h(x) - H(x)| \leq \frac{\phi(x)}{2} + \frac{\phi(x)}{2} = \phi(x)\,.$$

On the other hand, we have

$$F(xy) + \mu(y)F(x\sigma(y)) = f(e)\sigma[g(xy) + \mu(y)g(x\sigma(y))] + [H(xy) + \mu(y)H(x\sigma(y))$$
$$= 2f(e)g(x)g(y) + 2H(y)g(x) + 2H(x) = 2F(y)g(x) + 2H(x)$$

for all $x, y \in M$. This completes the proof. □

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