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Article

Some Properties for Multiple Twisted (p,q)-L-Function and Carlitz's Type Higher-Order Twisted (p,q)-Euler Polynomials

Kyung-Won Hwang [1] and Cheon Seoung Ryoo [2,*]

1 Department of Mathematics, Dong-A University, Busan 49315, Korea; khwang@dau.ac.kr
2 Department of Mathematics, Hannam University, Daejeon 34430, Korea
* Correspondence: ryoocs@hnu.kr

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Abstract: The main goal of this paper is to study some interesting identities for the multiple twisted (p,q)-L-function in a complex field. First, we construct new generating functions of the new Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials. By applying the Mellin transformation to these generating functions, we obtain integral representations of the multiple twisted (p,q)-Euler zeta function and multiple twisted (p,q)-L-function, which interpolate the Carlitz-type higher order twisted (p,q)-Euler numbers and Carlitz-type higher order twisted (p,q)-Euler polynomials at non-positive integers, respectively. Second, we get some explicit formulas and properties, which are related to Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials. Third, we give some new symmetric identities for the multiple twisted (p,q)-L-function. Furthermore, we also obtain symmetric identities for Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials by using the symmetric property for the multiple twisted (p,q)-L-function.

Keywords: higher order twisted (p,q)-Euler numbers and polynomials; q-L-function; multiple twisted (p,q)-L-function; symmetric identities

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1. Introduction

Many researchers have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, zeta function, and Hurwitz zeta function. Recently, some generalizations of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, zeta function, and Hurwitz zeta function were introduced (see [1–11]). Luo and Zhou [6] introduced the l-function and q-L-function. Ryoo [7] discussed generalized Barnes-type multiple q-Euler polynomials twisted by use of the roots of unity. Kim constructed the Barnes-type multiple q-zeta function and q-Euler polynomials (see [9]). In [10], Simsek defined the twisted (h,q)-Bernoulli numbers and polynomials of the twisted (h,q)-zeta function and L-function. Many (p,q)-extensions of some special numbers, polynomials, and functions have been studied (see [1–5]). In this paper, we introduce the multiple twisted (p,q)-L-function in the complex field and Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials. We obtain some new symmetric identities for the multiple twisted (p,q)-L-function. We also give symmetric identities for Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials by using the symmetric property for the multiple twisted (p,q)-L-function.

Throughout this paper, we use the following: $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers; $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ is the set of nonpositive integers; $\mathbb{Z}$ is the set of integers; $\mathbb{R}$ is the set of real numbers; $\mathbb{C}$ is the set of complex numbers; and:

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} = \sum_{m_1,\cdots,m_r=0}^{\infty} .$$

The binomial formulae are known as:

$$(1-a)^m = \sum_{i=0}^{m} \binom{m}{i} (-a)^i, \text{ where } \binom{m}{i} = \frac{m(m-1)\ldots(m-i+1)}{i!},$$

and:

$$\frac{1}{(1-a)^m} = (1-a)^{-m} = \sum_{i=0}^{\infty} \binom{-m}{i} (-a)^i = \sum_{i=0}^{\infty} \binom{m+i-1}{i} a^i.$$

The q-number is defined by:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-3} + q^{n-2} + q^{n-1}, \quad q \neq 1.$$

By using the q-number, Luo and Zhou defined the q-L-function $L_q(s,a)$ and q-l-function $l_q(s)$ (see [6]):

$$L_q(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+a}}{[n+a]_q^s}, \quad (Re(s) > 1; a \notin \mathbb{Z}_0^-),$$

and:

$$l_q(s) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^s}, \quad (Re(s) > 1).$$

Choi and Srivastava [8] made the multiple Hurwitz–Euler eta function $\eta_r(s,a)$ and got some results about the multiple Hurwitz–Euler eta function $\eta_r(s,a)$, which follows the r-ple series:

$$\eta_r(s,a) = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{(k_1+\cdots+k_r+a)^s}, \quad (Re(s) > 0; a > 0; r \in \mathbb{N}).$$

The (p,q)-number is:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2 q^{n-3} + pq^{n-2} + q^{n-1}, p \neq q.$$

Note that this number is the q-number when $p = 1$. By substituting q by $\frac{q}{p}$ in the q-number, we cannot obtain the (p,q)-number. Therefore, many research works have been developed in the area of special numbers and polynomials, as well as functions by using the (p,q)-number (see [1–5]).

Kim introduced the Barnes-type multiple q-zeta function and q-Euler polynomials (see [9]). In [10], Simsek introduced the twisted (h,q)-Bernoulli numbers and polynomials of the twisted (h,q)-zeta function and L-function.

Inspired by their work, the multiple twisted (p,q)-L-function can be defined as follows: For $s, x \in \mathbb{C}$ with $Re(x) > 0$, the multiple twisted (p,q)-L-function $L_{p,q,\zeta}^{(r)}(s,x)$ is defined by:

$$L_{p,q,\zeta}^{(r)}(s,x) = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]_{p,q}^s}.$$

The goal of this paper is the investigation of new generalizations of the Carlitz-type higher order twisted q-Euler numbers and polynomials, multiple Hurwitz–Euler eta function, and q-L-function.

It is called the Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials, which is the multiple twisted (p,q)-L-function. In Section 2, we define the Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials and get some properties involving the distribution relation, and so on. In Section 3, we define the multiple twisted (p,q)-L-function used by the higher order-type twisted (p,q)-Euler numbers and polynomials. We also study some connected formulae between the Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials and the multiple twisted (p,q)-L-function. In Section 4, we study a few symmetric identities of the multiple twisted (p,q)-L-function and Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials. Throughout the paper, let ζ be the r^{th} root of 1 and $\zeta \neq 1$.

Definition 1. *The classical higher order twisted Euler numbers $E_{n,\zeta}^{(r)}$ and twisted Euler polynomials $E_{n,\zeta}^{(r)}(x)$ are the following:*

$$\left(\frac{2}{\zeta e^t+1}\right)^r = \sum_{n=0}^{\infty} E_{n,\zeta}^{(r)}\frac{t^n}{n!}, \quad (|t+\log\zeta| < \pi),$$

and:

$$\left(\frac{2}{\zeta e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,\zeta}^{(r)}(x)\frac{t^n}{n!}, \quad (|t+\log\zeta| < \pi).$$

respectively.

When $\zeta = 1$, $E_n^{(r)}(x)$ are called the classical high order Euler polynomials $E_n^{(r)}(x)$.

2. Carlitz's Type Higher Order Twisted (p,q)-Euler Numbers and Polynomials

First, we make the Carlitz-type higher order twisted (p,q)-Euler numbers and polynomials as follows:

Definition 2. *Let $0 < q < p \leq 1$ and $r \in \mathbb{N}$. The high order twisted (p,q)-Euler polynomials $E_{n,p,q,\zeta}^{(r)}(x)$ are defined by the following:*

$$\sum_{n=0}^{\infty} E_{n,p,q,\zeta}^{(r)}(x)\frac{t^n}{n!} = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_{p,q}t}. \tag{1}$$

When $x=0$, $E_{n,p,q,\zeta}^{(r)} = E_{n,p,q,\zeta}^{(r)}(0)$ are called the high order twisted (p,q)-Euler numbers $E_{n,p,q,\zeta}^{(r)}$. Observe that if $p=1$, then $\lim_{q\to 1} E_{n,p,q,\zeta}^{(r)} = E_{n,\zeta}^{(r)}$ and $\lim_{q\to 1} E_{n,p,q,\zeta}^{(r)}(x) = E_{n.\zeta}^{(r)}(x)$.

Definition 3. *Let $0 < q < p \leq 1$, $r \in \mathbb{N}$, and $h \in \mathbb{Z}$. The high order twisted (h,p,q)-Euler polynomials $E_{n,p,q,\zeta}^{(r,h)}(x)$ are defined like this:*

$$\sum_{n=0}^{\infty} E_{n,p,q,\zeta}^{(r,h)}(x)\frac{t^n}{n!} = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-\zeta)^{m_1+\cdots+m_r} p^{h(m_1+\cdots+m_r)} e^{[m_1+\cdots+m_r+x]_{p,q}t}. \tag{2}$$

When $x=0$, $E_{n,p,q,\zeta}^{(r,h)} = E_{n,p,q,\zeta}^{(r,h)}(0)$ are called the high order twisted (h,p,q)-Euler numbers $E_{n,p,q,\zeta}^{(r)}$. WE remark that if $h=0$, then $E_{n,p,q,\zeta}^{(r,h)} = E_{n,p,q,\zeta}^{(r)}$ and $E_{n,p,q,\zeta}^{(r,h)}(x) = E_{n,p,q,\zeta}^{(r)}(x)$. Observe that if $p=1$, then $\lim_{q\to 1}\lim E_{n,p,q,\zeta}^{(r,h)} = E_{n,\zeta}^{(r)}$ and $\lim_{q\to 1} E_{n,p,q,\zeta}^{(r,h)}(x) = E_{n,\zeta}^{(r)}(x)$.

By (1) and (2), we see that:

$$\begin{aligned} E_{n,p,q,\zeta}^{(r)}(x+y) &= \sum_{i=0}^{n}\binom{n}{i} p^{(n-i)x} q^{yi} E_{i,p,q,\zeta}^{(r,n-i)}(x)[y]_{p,q}^{n-i}, \\ E_{n,p,q,\zeta}^{(r)}(x) &= \sum_{i=0}^{n}\binom{n}{i} q^{xi}[x]_{p,q}^{n-i} E_{i,p,q,\zeta}^{(r,n-i)}. \end{aligned} \tag{3}$$

Theorem 1. *Let $0 < q < p \leq 1$ and $r \in \mathbb{N}$. We get:*

$$\begin{aligned} E^{(r)}_{n,p,q,\zeta}(x) &= [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} [m_1+\cdots+m_r+x]^n_{p,q} \\ &= \frac{[2]^r_q}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \left(\frac{1}{1+\zeta q^l p^{n-l}} \right)^r. \end{aligned}$$

Proof. Using the Taylor series expansion of $e^{[x]_{p,q}t}$, we get:

$$\begin{aligned} \sum_{l=0}^{\infty} E^{(r)}_{l,p,q,\zeta}(x) \frac{t^l}{l!} &= [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_{p,q}t} \\ &= \sum_{l=0}^{\infty} \left([2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} [m_1+\cdots+m_r+x]^l_{p,q} \right) \frac{t^l}{l!}. \end{aligned}$$

The first part of the theorem follows when we compare the coefficients of $\frac{t^l}{l!}$ in the above equation. By using (p,q)-numbers and binomial expansion, we note that:

$$\begin{aligned} E^{(r)}_{n,p,q,\zeta}(x) &= [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} [m_1+\cdots+m_r+x]^n_{p,q} \\ &= [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} \left(\frac{p^{m_1+\cdots+m_r+x} - q^{m_1+\cdots+m_r+x}}{p-q} \right)^n \\ &= \frac{[2]^r_q}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \\ &\quad \times \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \zeta^{m_1+\cdots+m_r} q^{l(m_1+\cdots+m_r)} p^{(n-l)(m_1+\cdots+m_r)} \\ &= \frac{[2]^r_q}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \left(\frac{1}{1+\zeta q^l p^{n-l}} \right)^r. \end{aligned}$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $0 < q < p \leq 1$ and $r \in \mathbb{N}$. Then, we get:*

$$E^{(r)}_{n,p,q,\zeta}(x) = [2]^r_q \sum_{m=0}^{\infty} \binom{r+m-1}{m} (-1)^m \zeta^m [m+x]^n_{p,q}. \tag{4}$$

Proof. By the Taylor–Maclaurin series expansion of $(1-a)^{-n}$, we have:

$$\left(\frac{1}{1+\zeta q^l p^{n-l}} \right)^r = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \zeta^m q^{ml} p^{m(n-l)}.$$

By Theorem 1 and the binomial expansion, we also get the desired result immediately. □

By Theorem 1, for $d \in \mathbb{N}$ and $d \equiv 1(\bmod 2)$, we can show:

$$\begin{aligned}E^{(r)}_{n,p,q,\zeta}(x) = &\frac{[2]^r_q}{(p-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\\ &\times\sum_{a_1,\cdots,a_r=0}^{d-1}\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{a_1+dm_1+\cdots+a_r+dm_r}\zeta^{a_1+dm_1+\cdots+a_r+dm_r}\\ &\times q^{l(a_1+dm_1+\cdots+a_r+dm_r)}p^{(n-l)(a_1+dm_1+\cdots+a_r+dm_r)}.\end{aligned}$$

Theorem 3. *(Distribution relation of higher order twisted (p,q)-Euler polynomials) For $d \in \mathbb{N}$ and $d \equiv 1(\bmod 2)$, we have:*

$$E^{(r)}_{n,p,q,\zeta}(x) = \frac{[2]^r_q}{[2]^r_{q^d}}[d]^n_{p,q}\sum_{a_1,\cdots,a_r=0}^{d-1}(-\zeta)^{a_1+\cdots+a_r}E^{(r)}_{n,p^d,q^d,\zeta^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right).$$

Proof. Since:

$$\begin{aligned}&E^{(r)}_{n,p^d,q^d,\zeta^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\\ &=\frac{[2]^r_{q^d}}{(p^d-q^d)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(a_1+\cdots+a_r+x)}p^{(n-l)(a_1+\cdots+a_r+x)}\left(\frac{1}{1+\zeta^d q^{dl}p^{d(n-l)}}\right)^r,\end{aligned}$$

we have:

$$\begin{aligned}&\sum_{a_1,\cdots,a_r=0}^{d-1}(-1)^{a_1+\cdots+a_r}\zeta^{a_1+\cdots+a_r}E^{(r)}_{n,p^d,q^d,\zeta^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\\ &=\frac{[2]^r_{q^d}}{(p^d-q^d)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}p^{(n-l)x}\\ &\times\sum_{a_1,\cdots,a_r=0}^{d-1}(-1)^{a_1+\cdots+a_r}\zeta^{a_1+\cdots+a_r}q^{l(a_1+\cdots+a_r)}p^{(n-l)(a_1+\cdots+a_r)}\left(\frac{1}{1+\zeta^d q^{dl}p^{d(n-l)}}\right)^r.\end{aligned}$$

By Theorem 1, we get:

$$\begin{aligned}&\frac{[2]^r_q}{[2]^r_{q^d}}[d]^n_{p,q}\sum_{a_1,\cdots,a_r=0}^{d-1}(-1)^{a_1+\cdots+a_r}\zeta^{a_1+\cdots+a_r}E^{(r)}_{n,p^d,q^d,\zeta^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\\ &=\frac{[2]^r_q}{(p-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\left(\frac{1}{1+\zeta q^l p^{n-l}}\right)^r\\ &=E^{(r)}_{n,p,q,\zeta}(x).\end{aligned}$$

This completes the proof of Theorem 3. □

3. Multiple Twisted (p,q)-L-Function

The multiple twisted (p,q)-L-function is defined in Section 3. This function interpolates the higher order twisted (p,q)-Euler polynomials at negative integers $-n$. Choi and Srivastava [9] defined the multiple Hurwitz–Euler eta function $\eta_r(s,a)$ by using:

$$\eta_r(s,a) = \sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}}{(k_1+\cdots+k_r+a)^s},\quad (Re(s)>0; a>0; r\in\mathbb{N}).$$

It is known that $\eta_r(s,a)$ can be continued analytically in the whole complex s-plane (see [8]). The (p,q)-extension of the multiple Hurwitz–Euler eta function can be defined as follows:

Definition 4. *For $s, x \in \mathbb{C}$ with $Re(x) > 0$, the multiple twisted (p,q)-L-function $L^{(r)}_{p,q,\zeta}(s,x)$ is defined by:*

$$L^{(r)}_{p,q,\zeta}(s,x) = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]^s_{p,q}}.$$

Observe that if $\zeta = 1, p = 1$, then $\lim_{q\to1} L^{(r)}_{p,q,\zeta}(s,a) = 2^r\eta_r(s,a)$.

Let:

$$\begin{aligned} \widetilde{F}^{(r)}_{p,q,\zeta}(t,x) &= \sum_{n=0}^{\infty} E^{(r)}_{n,p,q,\zeta}(x)\frac{t^n}{n!} \\ &= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_{p,q}t}. \end{aligned} \tag{5}$$

Theorem 4. *For $r \in \mathbb{N}$, we have:*

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \widetilde{F}^{(r)}_{p,q,\zeta}(x,-t)t^{s-1}dt = L^{(r)}_{p,q,\zeta}(s,x), \tag{6}$$

where $\Gamma(s) = \int_0^{\infty} z^{s-1}e^{-z}dz$.

Proof. Apply the Mellin transformation to (5) and Definition 4. We have:

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^{\infty} \widetilde{F}^{(r)}_{p,q,\zeta}(x,-t)t^{s-1}dt \\ &= \frac{[2]_q^r}{\Gamma(s)} \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r} \int_0^{\infty} e^{-[m_1+\cdots+m_r+x]_{p,q}t}t^{s-1}dt \\ &= [2]_q^r \frac{1}{\Gamma(s)} \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]^s_{p,q}} \int_0^{\infty} z^{s-1}e^{-z}dz \\ &= [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\zeta^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]^s_{p,q}} \\ &= L^{(r)}_{p,q,\zeta}(s,x). \end{aligned}$$

This completes the proof of Theorem 4. □

The value of the multiple twisted (p,q)-L-function $L^{(r)}_{p,q,\zeta}(s,x)$ at negative integers $-n$ is given explicitly by the theorem below:

Theorem 5. *Let $n \in \mathbb{N}$. Then, we get:*

$$L^{(r)}_{p,q,\zeta}(-n,x) = E^{(r)}_{n,p,q,\zeta}(x).$$

Proof. By using (5) and (6), we have:

$$\begin{aligned} L^{(r)}_{p,q,\zeta}(s,x) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \widetilde{F}^{(r)}_{p,q,\zeta}(x,-t)t^{s-1}dt \\ &= \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} E^{(r)}_{m,p,q,\zeta}(x)\frac{(-1)^m}{m!} \int_0^{\infty} t^{m+s-1}dt. \end{aligned} \tag{7}$$

Observe that:

$$\begin{aligned}\Gamma(-n) &= \int_0^\infty e^{-z}z^{-n-1}dz \\ &= \lim_{z\to 0} 2\pi i \frac{1}{n!}\left(\frac{d}{dz}\right)^n (z^{n+1}e^{-z}z^{-n-1}) \\ &= 2\pi i \frac{(-1)^n}{n!}.\end{aligned} \tag{8}$$

Let us take $s = -n$ in (7) for $n \in \mathbb{N}$. We use (7), (8), and the Cauchy residue theorem. Then, we have:

$$\begin{aligned}L^{(r)}_{p,q,\zeta}(-n,x) &= \lim_{s\to -n}\frac{1}{\Gamma(s)}\sum_{m=0}^{\infty} E^{(r)}_{m,p,q,\zeta}(x)\frac{(-1)^m}{m!}\int_0^\infty t^{m-n-1}dt \\ &= 2\pi i\left(\lim_{s\to -n}\frac{1}{\Gamma(s)}\right)\left(E^{(r)}_{n,p,q,\zeta}(x)\frac{(-1)^n}{n!}\right) \\ &= 2\pi i\left(\frac{1}{2\pi i\frac{(-1)^n}{n!}}\right)\left(E^{(r)}_{n,p,q,\zeta}(x)\frac{(-1)^n}{n!}\right) \\ &= E^{(r)}_{n,p,q,\zeta}(x).\end{aligned}$$

This completes the proof of Theorem 5. □

If we use (4), then we have:

$$\sum_{n=0}^{\infty} E^{(r)}_{n,p,q,\zeta}\frac{t^n}{n!} = [2]_q^r \sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\zeta^m e^{[m]_{p,q}t}.$$

If we use the Taylor series of $e^{[m]_{p,q}t}$ in the above equation, we get:

$$\sum_{n=0}^{\infty} E^{(r)}_{n,p,q,\zeta}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\zeta^m[m]^n_{p,q}\right)\frac{t^n}{n!}.$$

Compare the coefficients $\frac{t^n}{n!}$ in the above equation. We have:

$$E^{(r)}_{n,p,q,\zeta} = [2]_q^r\sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\zeta^m[m]^n_{p,q}. \tag{9}$$

This is defined as the multiple twisted (p,q)-Euler zeta function in the definition below by (9):

Definition 5. *For $s \in \mathbb{C}$, we define:*

$$\zeta^{(r)}_{p,q,\zeta}(s) = [2]_q^r\sum_{m=1}^{\infty}\binom{m+r-1}{m}\frac{(-1)^m\zeta^m}{[m]^s_{p,q}}. \tag{10}$$

The function $\zeta^{(r)}_{p,q,\zeta}(s)$ interpolates the number $E^{(r)}_{n,p,q,\zeta}$ at negative integers. Substitute $s = -n$ instead of $n \in \mathbb{N}$ into (10), and use (9), then we get the following theorem:

Corollary 1. *Let $n \in \mathbb{N}$. We obtain:*

$$\zeta^{(r)}_{p,q,\zeta}(-n) = E^{(r)}_{n,p,q,\zeta}.$$

4. Some Identities for the Multiple Twisted (p,q)-L-Function

If we have $w_1, w_2 \in \mathbb{N}$ and $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$ and for $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we get symmetric identities for the multiple twisted (p,q)-L-function.

Theorem 6. *For $w_1, w_2 \in \mathbb{N}$ and $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we obtain:*

$$\begin{aligned}
&[w_2]_{p,q}^s [2]_{q^{w_2}}^r \sum_{j_1,\cdots,j_r=0}^{w_1-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_2 \sum_{l=1}^r j_l} \\
&\qquad \times L_{p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)} \left(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right) \\
&= [w_1]_{p,q}^s [2]_{q^{w_1}}^r \sum_{j_1,\cdots,j_r=0}^{w_2-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_1 \sum_{l=1}^r j_l} \\
&\qquad \times L_{p^{w_2},q^{w_2},\zeta^{w_2}}^{(r)} \left(s, w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_r) \right).
\end{aligned} \tag{11}$$

Proof. Note that $[xy]_q = [x]_q [y]_q$ for any $x, y \in \mathbb{C}$. In Definition 4, substitute $w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r)$ instead of x and replace q^{w_1}, p^{w_1}, and ζ^{w_1} instead of q, p, and ζ, respectively. We get the next result:

$$\begin{aligned}
&\frac{1}{[2]_{q^{w_1}}^r} L_{p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)} \left(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right) \\
&= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{w_1 m_1 + \cdots + w_1 m_r}}{[m_1 + \cdots + m_r + w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r)]_{p^{w_1},q^{w_1}}^s} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{w_1 m_1 + \cdots + w_1 m_r}}{\left[\frac{w_1(m_1 + \cdots + m_r) + w_1 w_2 x + w_2(j_1 + \cdots + j_r)}{w_1} \right]_{p^{w_1},q^{w_1}}^s} \\
&= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{w_1 m_1 + \cdots + w_1 m_r}}{\frac{[w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_{p,q}^s}{[w_1]_{p,q}^s}} \\
&= [w_1]_{p,q}^s \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{w_1 m_1 + \cdots + w_1 m_r}}{[w_1(m_1 + \cdots + m_r) + w_1 w_2 x + w_2(j_1 + \cdots + j_r)]_{p,q}^s} \\
&= [w_1]_{p,q}^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \frac{(-1)^{m_1+\cdots+m_r} \zeta^{w_1 m_1 + \cdots + w_1 m_r}}{[w_1(m_1 + \cdots + m_r) + w_1 w_2 x + w_2(j_1 + \cdots + j_r)]_{p,q}^s} \\
&= [w_1]_{p,q}^s \sum_{m_1,\cdots,m_r=0}^{\infty} \sum_{i_1,\cdots,i_r=0}^{w_2-1} (-1)^{\sum_{j=1}^r (w_2 m_j + i_j)} \zeta^{w_1 \sum_{j=1}^r (w_2 m_j + i_j)} \\
&\quad \times \left([w_1(w_2 m_1 + i_1) + \cdots + w_1(w_2 m_r + i_r) + w_1 w_2 x + w_2(j_1 + \cdots + j_r)]_{p,q}^s \right)^{-1} \\
&= [w_1]_{p,q}^s \sum_{m_1,\cdots,m_r=0}^{\infty} \sum_{i_1,\cdots,i_r=0}^{w_2-1} (-1)^{\sum_{j=1}^r m_j} (-1)^{\sum_{j=1}^r i_j} \zeta^{w_1 w_2 \sum_{j=1}^r m_j} \zeta^{w_1 \sum_{j=1}^r i_j} \\
&\quad \times \left([w_1 w_2 (x + m_1 + \cdots + m_r) + w_1(i_1 + \cdots + i_r) + w_2(j_1 + \cdots + j_r)]_{p,q}^s \right)^{-1}.
\end{aligned} \tag{12}$$

We get the following equation from (12).

$$\begin{aligned}
&\frac{[w_2]_{p,q}^s}{[2]_{q^{w_1}}^r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{j_1+\cdots+j_r}\zeta^{w_2(j_1+\cdots+j_r)}L_{p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)}\left(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)\right)\\
&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m_1,\cdots,m_r=0}^{\infty}\sum_{i_1,\cdots,i_r=0}^{w_2-1}\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r(j_l+i_l+m_l)}\zeta^{w_1w_2\sum_{l=1}^r m_l}\\
&\quad\times\zeta^{w_1\sum_{l=1}^r i_l}\zeta^{w_2\sum_{l=1}^r j_l}\\
&\quad\times\left([w_1w_2(x+m_1+\cdots+m_r)+w_1(i_1+\cdots+i_r)+w_2(j_1+\cdots+j_r)]_{p,q}^s\right)^{-1}
\end{aligned}\tag{13}$$

We have the following result from the same method like (13):

$$\begin{aligned}
&\frac{[w_1]_{p,q}^s}{[2]_{q^{w_2}}^r}\sum_{j_1,\cdots,j_r=0}^{w_2-1}(-1)^{j_1+\cdots+j_r}\zeta^{w_1(j_1+\cdots+j_r)}L_{p^{w_2},q^{w_2},\zeta^{w_2}}^{(r)}\left(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_r)\right)\\
&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{j_1,\cdots,j_r=0}^{w_2-1}\sum_{i_1,\cdots,i_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r(j_l+i_l+m_l)}\\
&\quad\times\zeta^{w_1w_2\sum_{l=1}^r m_l}\zeta^{w_2\sum_{l=1}^r i_l}\zeta^{w_1\sum_{l=1}^r j_l}\\
&\quad\times\left([w_1w_2(x+m_1+\cdots+m_r)+w_1(j_1+\cdots+j_r)+w_2(i_1+\cdots+i_r)]_{p,q}^s\right)^{-1}
\end{aligned}\tag{14}$$

Therefore, we have Theorem 6 from Equations (13) and (14). □

We obtain the below corollary when we take $w_2=1$ in Theorem 6.

Corollary 2. *Let $w_1\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We obtain:*

$$\begin{aligned}
L_{p,q,\zeta}^{(r)}(s,w_1x)&=\frac{[2]_q^r}{[2]_{q^{w_1}}^r[w_1]_{p,q}^s}\\
&\quad\times\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{\sum_{l=1}^r j_l}L_{p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)}\left(s,x+\frac{j_1+\cdots+j_r}{w_1}\right).
\end{aligned}\tag{15}$$

We get the corollary below when $p=1,\zeta=1$, and q approaches one in Corollary 2.

Corollary 3. *Let $m\in\mathbb{N}$ with $m\equiv 1 \pmod 2$. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We obtain:*

$$\eta_r(s,x)=\frac{1}{m^s}\sum_{j_1,\cdots,j_r=0}^{m-1}(-1)^{j_1+\cdots+j_r}\eta_r\left(s,\frac{x+j_1+\cdots+j_r}{m}\right).$$

Let us take $s=-n$ in Theorem 6. We obtain symmetric identities for high order twisted (p,q)-Euler polynomials for $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$.

Theorem 7. *Let $w_1, w_2 \in \mathbb{N}$, and let $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain:*

$$\begin{aligned}
&[w_1]_{p,q}^n [2]_{q^{w_2}}^r \sum_{j_1,\cdots,j_r=0}^{w_1-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_2 \sum_{l=1}^r j_l} \\
&\quad \times E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right) \\
&= [w_2]_{p,q}^n [2]_{q^{w_1}}^r \sum_{j_1,\cdots,j_r=0}^{w_2-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_1 \sum_{l=1}^r j_l} \\
&\quad \times E_{n,p^{w_2},q^{w_2},\zeta^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_r) \right).
\end{aligned} \tag{16}$$

Proof. We obtain the theorem by Theorems 5 and 6. □

We get the corollary below when we take $w_2 = 1$ in Theorem 7.

Corollary 4. *Let $w_1 \in \mathbb{N}$, and let $w_1 \equiv 1 \pmod 2$. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Then, we have:*

$$\begin{aligned}
E_{n,p,q,\zeta}^{(r)}(w_1 x) = \frac{[2]_q^r}{[2]_{q^{w_1}}^r} [w_1]_{p,q}^n \sum_{j_1,\cdots,j_r=0}^{w_1-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{\sum_{l=1}^r j_l} \\
\times E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)} \left(s, x + \frac{j_1 + \cdots + j_r}{w_1} \right).
\end{aligned} \tag{17}$$

We have the corollary below when $p = 1$, and q approaches one in (17).

Corollary 5. *Let $m \in \mathbb{N}$ with $m \equiv 1 \pmod 2$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain:*

$$E_{n,\zeta}^{(r)}(x) = m^n \sum_{j_1,\cdots,j_r=0}^{m-1} (-1)^{j_1+\cdots+j_r} \zeta^{j_1+\cdots+j_r} E_{n,\zeta^m}^{(r)} \left(\frac{x + j_1 + \cdots + j_r}{m} \right).$$

We obtain the following corollary if $\zeta = 1$ in Corollary 5.

Corollary 6. *Let $m \in \mathbb{N}$ with $m \equiv 1 \pmod 2$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain:*

$$E_n^{(r)}(x) = m^n \sum_{j_1,\cdots,j_r=0}^{m-1} (-1)^{j_1+\cdots+j_r} E_n^{(r)} \left(\frac{x + j_1 + \cdots + j_r}{m} \right). \tag{18}$$

We have the theorem below.

Theorem 8. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain:*

$$\begin{aligned}
&\sum_{j_1,\cdots,j_r=0}^{w_1-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_2 \sum_{l=1}^r j_l} \times E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^i [w_1]_{p,q}^{-i} p^{w_1 w_2 x i} E_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}^{(r,i)}(w_2 x) \\
&\times \sum_{j_1,\cdots,j_r=0}^{w_1-1} (-1)^{\sum_{l=1}^r j_l} \zeta^{w_2 \sum_{l=1}^r j_l} q^{w_2(n-i) \sum_{l=1}^r j_l} [j_1 \cdots + j_r]_{p^{w_2},q^{w_2}}^i.
\end{aligned}$$

Proof. We have the following formula by (3):

$$\begin{aligned}
&\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{w_2\sum_{l=1}^r j_l}\times E^{(r)}_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{w_2\sum_{l=1}^r j_l}\\
&\times\sum_{i=0}^{n}\binom{n}{i}q^{w_2(n-i)(j_1+\cdots+j_r)}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_r)\right]^{i}_{p^{w_1},q^{w_1}}\\
&=\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{w_2\sum_{l=1}^r j_l}\\
&\times\sum_{i=0}^{n}\binom{n}{i}q^{w_2(n-i)\sum_{l=1}^r j_l}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left(\frac{[w_2]_{p,q}}{[w_1]_{p,q}}\right)^{i}[j_1+\cdots+j_r]^{i}_{p^{w_1},q^{w_1}}
\end{aligned}\tag{19}$$

□

For each integer $n\geq 0$, let:

$$\mathcal{A}^{(r)}_{n,i,p,q,\zeta}(w)=\sum_{j_1,\cdots,j_r=0}^{w-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{\sum_{l=1}^r j_l}q^{(n-i)\sum_{l=1}^r j_l}[j_1\cdots+j_k]^{i}_{p,q}.$$

The above sum $\mathcal{A}^{(k)}_{n,i,p,q,\zeta}(w)$ is called the alternating twisted (p,q)-power sums.

Theorem 9. *Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we obtain:*

$$\begin{aligned}
&[2]^{r}_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^{i}_{p,q}[w_2]^{n-i}_{p,q}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{A}^{(r)}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\\
&=[2]^{r}_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^{i}_{p,q}[w_1]^{n-i}_{p,q}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{A}^{(r)}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1).
\end{aligned}$$

Proof. If we use Theorem 8, then we have:

$$\begin{aligned}
&[2]^{r}_{q^{w_2}}[w_1]^{n}_{p,q}\sum_{j_1,\cdots,j_r=0}^{w_1-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{w_2\sum_{l=1}^r j_l}\\
&\qquad\times E^{(r)}_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)\right)\\
&=[2]^{r}_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^{i}_{p,q}[w_1]^{n-i}_{p,q}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{A}^{(r)}_{n,i,p^{w_2},q^{w_2},zeta^{w_2}}(w_1)
\end{aligned}\tag{20}$$

If we use the same method as the proof method of Formula (20), we have:

$$\begin{aligned}
&[2]^{r}_{q^{w_1}}[w_2]^{n}_{p,q}\sum_{j_1,\cdots,j_r=0}^{w_2-1}(-1)^{\sum_{l=1}^r j_l}\zeta^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times E^{(r)}_{n,p^{w_2},q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_r)\right)\\
&=[2]^{r}_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^{i}_{p,q}[w_2]^{n-i}_{p,q}p^{w_1w_2xi}E^{(r,i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{A}^{(r)}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)
\end{aligned}\tag{21}$$

Therefore, we have Theorem 9 by (20) and (21) and Theorem 7. □

We obtain the symmetric identity for the higher order twisted (h, p, q)-Euler numbers $E^{(r,h)}_{n,p,q,\zeta}$ in the complex field using Theorem 9.

Corollary 7. *Let $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$ for $w_1, w_2 \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. It follows that:*

$$[2]^r_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]^i_{p,q} [w_2]^{n-i}_{p,q} p^{w_1 w_2 x i} \mathcal{A}^{(r)}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2) E^{(r,i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}$$
$$= [2]^r_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]^i_{p,q} [w_1]^{n-i}_{p,q} p^{w_1 w_2 x i} \mathcal{A}^{(r)}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1) E^{(r,i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}.$$

If $\zeta = 1, p = 1, r = 1$, and q approaches one in Theorem 7, then we have the following theorem for Euler polynomials, which are symmetric in w_1 and w_2 (see [11]).

Corollary 8. *Let $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$ for $w_1, w_2 \in \mathbb{N}$. Then, we obtain:*

$$w_1^n \sum_{j=0}^{w_1-1} (-1)^j E_n \left(w_2 x + \frac{w_2}{w_1} j \right) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j E_n \left(w_1 x + \frac{w_1}{w_2} j \right).$$

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