Logarithmic Coefficients For Univalent Functions Defined by Subordination

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Abstract: In this work, the bounds for the logarithmic coefficients γ_n of the general classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ were estimated. It is worthwhile mentioning that the given bounds would generalize some of the previous papers. Some consequences of the main results are also presented, noting that our method is more general than those used by others.

Keywords: starlike functions; convex functions; subordination; logarithmic coefficients

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. Then the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined with the following series expansion:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f)z^n, \ z \in \mathbb{U}. \tag{2}$$

These coefficients play an important role for various estimates in the theory of univalent functions. Note that we use γ_n instead of $\gamma_n(f)$. The idea of studying the logarithmic coefficients helped Kayumov [1] to solve Brennan's conjecture for conformal mappings.

Recall that we can rewrite (2) in the series form as follows:

$$
\begin{aligned}
2\sum_{n=1}^{\infty} \gamma_n z^n = {} & a_2 z + a_3 z^2 + a_4 z^3 + \cdots - \frac{1}{2}[a_2 z + a_3 z^2 + a_4 z^3 + \cdots]^2 \\
& + \frac{1}{3}[a_2 z + a_3 z^2 + a_4 z^3 + \cdots]^3 + \cdots.
\end{aligned}
$$

Now, considering the coefficients of z^n for $n = 1, 2, 3$, it follows that

$$
\begin{cases}
2\gamma_1 = a_2, \\
2\gamma_2 = a_3 - \dfrac{1}{2}a_2^2, \\
2\gamma_3 = a_4 - a_2 a_3 + \dfrac{1}{3}a_2^3.
\end{cases} \tag{3}
$$

For two functions f and g that are analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write $f(z) \prec g(z)$ if there exists a Schwarz function ω that is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$
f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).
$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Using subordination, different subclasses of starlike and convex functions were introduced by Ma and Minda [2], in which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. To this aim, they considered an analytic univalent function φ with positive real part in $\mathbb{U}$. $\varphi(\mathbb{U})$ is symmetric respecting the real axis and starlike considering $\varphi(0) = 1$ and $\varphi'(0) > 0$. They defined the classes consisting of several well-known classes as follows:

$$
\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z), \ z \in \mathbb{U} \right\},
$$

and

$$
\mathcal{K}(\varphi) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \ z \in \mathbb{U} \right\}.
$$

For example, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduce to the classes $\mathcal{S}^*[A, B] := \frac{1 + Az}{1 + Bz}$ and $\mathcal{K}[A, B] := \frac{1 + Az}{1 + Bz}$ of the well-known Janowski starlike and Janowski convex functions for $-1 \leq B < A \leq 1$, respectively. By replacing $A = 1 - 2\alpha$ and $B = -1$ where $0 \leq \alpha < 1$, we conclude the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of the starlike functions of order α and convex functions of order α, respectively. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the class of starlike functions and of convex functions in the unit disk $\mathbb{U}$, respectively. The Koebe function $k(z) = z/(1 - z)^2$ is starlike but not convex in $\mathbb{U}$. Thus, every convex function is starlike but not conversely; however, each starlike function is convex in the disk of radius $2 - \sqrt{3}$.

Lately, several researchers have subsequently investigated similar problems in the direction of the logarithmic coefficients, the coefficient problems, and differential subordination [3–11], to mention a few. For example, the rotation of Koebe function $k(z) = z(1 - e^{i\theta})^{-2}$ for each θ has logarithmic coefficients $\gamma_n = e^{i\theta n}/n$, $n \geq 1$. If $f \in \mathcal{S}$, then by using the Bieberbach inequality for the first equation of (3) it concludes $|\gamma_1| \leq 1$ and by utilizing the Fekete–Szegö inequality for the second equation of (3), (see [12] (Theorem 3.8)),

$$
|\gamma_2| = \frac{1}{2}\left| a_3 - \frac{1}{2}a_2^2 \right| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) = 0.635 \cdots.
$$

It was shown in [12] (Theorem 4) that the logarithmic coefficients γ_n of every function $f \in \mathcal{S}$ satisfy

$$
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6},
$$

and the equality is attained for the Koebe function. For $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \leq 1/n$ holds but is not true for the full class $\mathcal{S}$, even in order of magnitude (see [12] (Theorem 8.4)). In 2018, Ali and Vasudevarao [3] and Pranav Kumar and Vasudevarao [6] obtained the logarithmic coefficients γ_n for

certain subclasses of close-to-convex functions. Nevertheless, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for $n \geq 3$ is presumably still a concern.

Based on the results presented in previous research, in the current study, the bounds for the logarithmic coefficients γ_n of the general classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ were estimated. It is worthwhile mentioning that the given bounds in this paper would generalize some of the previous papers and that many new results are obtained, noting that our method is more general than those used by others. The following lemmas will be used in the proofs of our main results.

For this work, let Ω represent the class of all analytic functions ω in $\mathbb{U}$ that equips with conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$. Such functions are called Schwarz functions.

Lemma 1. *[13] (p. 172) Assume that ω is a Schwarz function so that $\omega(z) = \sum_{n=1}^{\infty} p_n z^n$. Then*

$$|p_1| \leq 1, \quad |p_n| \leq 1 - |p_1|^2 \quad n = 2, 3, \ldots.$$

Lemma 2. *[14] Let $\psi, \varpi \in \mathcal{H}$ be any convex univalent functions in $\mathbb{U}$. If $f(z) \prec \psi(z)$ and $g(z) \prec \varpi(z)$, then $f(z) * g(z) \prec \psi(z) * \varpi(z)$ where $f, g \in \mathcal{H}$.*

We observe that in the above lemma, nothing is assumed about the normalization of ψ and ϖ, and "$*$" represents the Hadamard (or convolution) product.

Lemma 3. *[12,15] (Theorem 6.3, p. 192; Rogosinski's Theorem II (i)) Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$, and suppose that $f \prec g$ where g is univalent in $\mathbb{U}$. Then*

$$\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2, \quad n = 1, 2, \ldots.$$

Lemma 4. *[12,15] (Theorem 6.4 (i), p. 195; Rogosinski's Theorem X) Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$, and suppose that $f \prec g$ where g is univalent in $\mathbb{U}$. Then*

(i)   If g is convex, then $|a_n| \leq |g'(0)| = |b_1|, \quad n = 1, 2, \ldots.$
(ii) If g is starlike (starlike with respect to 0), then $|a_n| \leq n|g'(0)| = n|b_1|, \quad n = 2, 3, \ldots.$

Lemma 5. *[16] If $\omega(z) = \sum_{n=1}^{\infty} p_n z^n \in \Omega$, then for any real numbers q_1 and q_2, the following sharp estimate holds:*

$$|p_3 + q_1 p_1 p_2 + q_2 p_1^3| \leq H(q_1; q_2),$$

where

$$H(q_1; q_2) = \begin{cases} 1 & \text{if} \quad (q_1, q_2) \in D_1 \cup D_2 \cup \{(2,1)\}, \\ |q_2| & \text{if} \quad (q_1, q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1| + 1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1, q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1, q_2) \in D_{10} \cup D_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|q_1| - 1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1, q_2) \in D_{12}. \end{cases}$$

While the sets D_k, $k = 1, 2, \ldots, 12$ are defined as follows:

$$D_1 = \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, \ |q_2| \leq 1 \right\},$$

$$D_2 = \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \ \frac{4}{27} \left((|q_1| + 1)^3 \right) - (|q_1| + 1) \leq |q_2| \leq 1 \right\},$$

$$D_3 = \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, \ |q_2| \leq -1 \right\},$$

$$D_4 = \left\{ (q_1, q_2) : |q_1| \geq \frac{1}{2}, \ |q_2| \leq -\frac{2}{3}(|q_1| + 1) \right\},$$

$$D_5 = \left\{ (q_1, q_2) : |q_1| \leq 2, \ |q_2| \geq 1 \right\},$$

$$D_6 = \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, \ |q_2| \geq \frac{1}{12}(q_1^2 + 8) \right\},$$

$$D_7 = \left\{ (q_1, q_2) : |q_1| \geq 4, \ |q_2| \geq \frac{2}{3}(|q_1| - 1) \right\},$$

$$D_8 = \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \ -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{4}{27} \left((|q_1| + 1)^3 \right) - (|q_1| + 1) \right\},$$

$$D_9 = \left\{ (q_1, q_2) : |q_1| \geq 2, \ -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{2|q_1|(|q_1| + 1|)}{q_1^2 + 2|q_1| + 4} \right\},$$

$$D_{10} = \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, \ \frac{2|q_1|(|q_1| + 1|)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{1}{12}(q_1^2 + 8) \right\},$$

$$D_{11} = \left\{ (q_1, q_2) : |q_1| \geq 4, \ \frac{2|q_1|(|q_1| + 1|)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{2|q_1|(|q_1| - 1|)}{q_1^2 - 2|q_1| + 4} \right\},$$

$$D_{12} = \left\{ (q_1, q_2) : |q_1| \geq 4, \ \frac{2|q_1|(|q_1| - 1|)}{q_1^2 - 2|q_1| + 4} \leq q_2 \leq \frac{2}{3}(|q_1| - 1) \right\}.$$

2. Main Results

Throughout this paper, we assume that φ is an analytic univalent function in the unit disk $\mathbb{U}$ satisfying $\varphi(0) = 1$ such that it has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 \neq 0. \tag{4}$$

Theorem 1. *Let the function $f \in \mathcal{S}^*(\varphi)$. Then the logarithmic coefficients of f satisfy the inequalities:*

(i) If φ is convex, then

$$|\gamma_n| \leq \frac{|B_1|}{2n}, \ n \in \mathbb{N}, \tag{5}$$

$$\sum_{n=1}^{k} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{k} \frac{|B_n|^2}{n^2}, \ k \in \mathbb{N}, \tag{6}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2}. \tag{7}$$

(ii) If φ is starlike with respect to 1, then

$$|\gamma_n| \leq \frac{|B_1|}{2}, \ n \in \mathbb{N}. \tag{8}$$

All inequalities in (5), (7), and (8) are sharp such that for any $n \in \mathbb{N}$, there is the function f_n given by $\frac{z f_n'(z)}{f_n(z)} = \varphi(z^n)$ and the function f given by $\frac{z f'(z)}{f(z)} = \varphi(z)$, respectively.

Proof. Suppose that $f \in \mathcal{S}^*(\varphi)$. Then considering the definition of $\mathcal{S}^*(\varphi)$, it follows that

$$z \frac{d}{dz} \left(\log \left(\frac{f(z)}{z} \right) \right) = \frac{z f'(z)}{f(z)} - 1 \prec \varphi(z) - 1 =: \phi(z), \ z \in \mathbb{U},$$

which according to the logarithmic coefficients γ_n of f given by (1), concludes

$$\sum_{n=1}^{\infty} 2n \gamma_n z^n \prec \phi(z), \ z \in \mathbb{U}.$$

Now, for the proof of inequality (5), we assume that φ is convex in $\mathbb{U}$. This implies that $\phi(z)$ is convex with $\phi'(0) = B_1$, and so by Lemma 4(i) we get

$$2n|\gamma_n| \leq |\phi'(0)| = |B_1|, \ n \in \mathbb{N},$$

and concluding the result.

Next, for the proof of inequality (6), we define $h(z) := \dfrac{f(z)}{z}$, which is an analytic function, and it satisfies the relation

$$\frac{z h'(z)}{h(z)} = \frac{z f'(z)}{f(z)} - 1 \prec \phi(z), \ z \in \mathbb{U}, \tag{9}$$

as ϕ is convex in $\mathbb{U}$ with $\phi(0) = 0$.

On the other hand, it is well known that the function (see [17])

$$b_0(z) = \log \left(\frac{1}{1-z} \right) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

belongs to the class $\mathcal{K}$, and for $f \in \mathcal{H}$,

$$f(z) * b_0(z) = \int_0^z \frac{f(t)}{t} dt. \tag{10}$$

Now, by Lemma 2 and from (9), we obtain

$$\frac{z h'(z)}{h(z)} * b_0(z) \prec \phi(z) * b_0(z).$$

Considering (10), the above relation becomes

$$\log \left(\frac{f(z)}{z} \right) \prec \int_0^z \frac{\phi(t)}{t} dt.$$

In addition, it has been proved in [18] that the class of convex univalent functions is closed under convolution. Therefore, the function $\displaystyle\int_0^z \frac{\phi(t)}{t} dt$ is convex univalent. In addition, the above relation considering the logarithmic coefficients γ_n of f given by (1) is equivalent to

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{B_n z^n}{n}.$$

Applying Lemma 3, from the above subordination this gives

$$4 \sum_{n=1}^{k} |\gamma_n|^2 \leq \sum_{n=1}^{k} \frac{|B_n|^2}{n^2},$$

which yields the inequality in (6). Supposing that $k \to \infty$, we deduce that

$$4 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2},$$

and it concludes the inequality (7).

Finally, we suppose that φ is starlike with respect to 1 in $\mathbb{U}$, which implies $\phi(z)$ is starlike, and thus by Lemma 4(ii), we obtain

$$2n|\gamma_n| \leq n|\phi'(0)| = n|B_1|, \ n \in \mathbb{N},$$

This implies the inequality in (8).

For the sharp bounds, it suffices to use the equality

$$z \frac{d}{dz} \left(\log \left(\frac{f(z)}{z} \right) \right) = \frac{z f'(z)}{f(z)} - 1,$$

and so these results are sharp in inequalities (5), (6), and (8) such that for any $n \in \mathbb{N}$, there is the function f_n given by $\frac{z f_n'(z)}{f_n(z)} = \varphi(z^n)$ and the function f given by $\frac{z f'(z)}{f(z)} = \varphi(z)$, respectively. This completes the proof. □

In the following corollaries, we obtain the logarithmic coefficients γ_n for two subclasses $\mathcal{S}^*(\alpha + (1-\alpha)e^z)$ and $\mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$, which were defined by Khatter et al. in [19], and $\alpha + (1-\alpha)e^z$ and $\alpha + (1-\alpha)\sqrt{1+z}$ are the convex univalent functions in $\mathbb{U}$. For $\alpha = 0$, these results reduce to the logarithmic coefficients γ_n for the subclasses $\mathcal{S}^*(e^z)$ and $\mathcal{S}^*(\sqrt{1+z})$ (see [20,21]).

Corollary 1. *For $0 \leq \alpha < 1$, let the function $f \in \mathcal{S}^*(\alpha + (1-\alpha)e^z)$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \leq \frac{1-\alpha}{2n}, \quad n \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{(1-\alpha)^2/(n!)^2}{n^2}.$$

These results are sharp such that for any $n \in \mathbb{N}$, there is the function f_n given by $\frac{z f_n'(z)}{f_n(z)} = \alpha + (1-\alpha)e^{z^n}$ and the function f given by $\frac{z f'(z)}{f(z)} = \alpha + (1-\alpha)e^z$.

Corollary 2. *For $0 \leq \alpha < 1$, let the function $f \in \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \leq \frac{1-\alpha}{4n}, \quad n \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{\left((1-\alpha)\binom{\frac{1}{2}}{n}\right)^2}{n^2}.$$

These results are sharp such that for any $n \in \mathbb{N}$, there is the function f_n given by $\frac{z f_n'(z)}{f_n(z)} = \alpha + (1 - \alpha)\sqrt{1+z^n}$ and the function f given by $\frac{z f'(z)}{f(z)} = \alpha + (1-\alpha)\sqrt{1+z}$.

The following corollary concludes the logarithmic coefficients γ_n for a subclass $\mathcal{S}^*(1 + \sin z)$ defined by Cho et al. in [22], in which considering the proof of Theorem 1 and Corollary 1, the convexity radius for $q_0(z) = 1 + \sin z$ is given by $r_0 \approx 0.345$.

Corollary 3. *Let the function* $f \in \mathcal{S}^*(1 + \sin z)$ *where* $q_0(z)$ *is a convex univalent function for* $r_0 \approx 0.345$ *in* $\mathbb{U}$*. Then the logarithmic coefficients of* f *satisfy the inequalities*

$$|\gamma_n| \leq \frac{1}{2n}, \quad n \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{((2n+1)!n)^2}.$$

These results are sharp such that for any $n \in \mathbb{N}$*, there is the function* f_n *given by* $\dfrac{z f_n'(z)}{f_n(z)} = q_0(z^n)$ *and the function* f *given by* $\dfrac{z f'(z)}{f(z)} = q_0(z)$.

In the following result, we get the logarithmic coefficients γ_n for a subclass $\mathcal{S}^*(p_k(z))$ defined by Kanas and Wisniowska in [23] (see also [24,25]), in which

$$p_k(z) = 1 + P_1(k)z + P_2(k)z^2 + \cdots,$$

where $p_k(z)$ is a convex univalent function in $\mathbb{U}$ and

$$P_1(k) = \begin{cases} \dfrac{2A^2}{1-k^2} & \text{if} \quad 0 \leq k < 1, \\ \dfrac{8}{\pi^2} & \text{if} \quad k = 1, \\ \dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1+t)\sqrt{t}} & \text{if} \quad k > 1. \end{cases}$$

$A = \frac{2}{\pi} \arccos k$ and $\kappa(t)$ is the complete elliptic integral of the first kind.

Corollary 4. *For* $0 \leq k < \infty$*, let the function* $f \in \mathcal{S}^*(p_k(z))$*. Then the logarithmic coefficients of* f *satisfy the inequalities*

$$|\gamma_n| \leq \frac{P_1(k)}{2n}, \quad n \in \mathbb{N}.$$

This result is sharp such that for any $n \in \mathbb{N}$*, there is the function* f_n *given by* $\dfrac{z f_n'(z)}{f_n(z)} = p_k(z^n)$.

The following result concludes the logarithmic coefficients γ_n for a subclass $\mathcal{S}^*\left(\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right)$ defined by Mendiratta et al. in [26], in which

$$\varphi_0(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}} = 1 + \frac{5 - 3\sqrt{2}}{2}z + \frac{71 - 51\sqrt{2}}{8}z^2 + \cdots,$$

where φ_0 is a convex univalent function in $\mathbb{U}$.

Corollary 5. *Let the function* $f \in \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right)$*. Then the logarithmic coefficients of* f *satisfy the inequalities*

$$|\gamma_n| \leq \frac{5 - 3\sqrt{2}}{4n}, \quad n \in \mathbb{N}.$$

This result is sharp such that for any $n \in \mathbb{N}$*, there is the function* f_n *given by* $\dfrac{z f_n'(z)}{f_n(z)} = \varphi_0(z^n)$.

The following results conclude the logarithmic coefficients γ_n for two subclasses $\mathcal{S}^*(z + \sqrt{1+z^2})$ and $\mathcal{S}^*\left(1 + \frac{z}{(1-\alpha z^2)}\right)$ defined by Krishna Raina and Sokół in [27] and Kargar et al. in [28], where

$$z + \sqrt{1+z^2} = 1 + z + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n}^2 z^{2n},$$

and

$$1 + \frac{z}{(1-\alpha z^2)} = 1 + z + \sum_{n=1}^{\infty} \alpha^n z^{2n+1}, \quad (0 \le \alpha < 1),$$

respectively. These functions are univalent and starlike with respect to 1 in $\mathbb{U}$.

Corollary 6. *Let the function* $f \in \mathcal{S}^*(z + \sqrt{1+z^2})$. *Then the logarithmic coefficients of* f *satisfy the inequalities*

$$|\gamma_n| \le \frac{1}{2}, \quad n \in \mathbb{N}.$$

This result is sharp such that for any $n \in \mathbb{N}$, there is the function f_n given by $\dfrac{z f_n'(z)}{f_n(z)} = z^n + \sqrt{1+z^{2n}}$.

Corollary 7. *Let the function $f \in \mathcal{S}^*\left(1 + \frac{z}{(1-\alpha z^2)}\right)$, where $0 \le \alpha < 1$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \le \frac{1}{2}, \quad n \in \mathbb{N}.$$

This result is sharp such that for any $n \in \mathbb{N}$, there is the function f_n given by $\dfrac{z f_n'(z)}{f_n(z)} = 1 + \dfrac{z}{(1 - \alpha z^{2n})}$.

Remark 1. *1. Letting*

$$\begin{aligned}
\varphi(z) &= \frac{1 + Az}{1 + Bz} \\
&= 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \cdots \\
&= 1 + \begin{cases} \dfrac{A - B}{B} \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} B^n z^n, & \text{if } B \ne 0 \\ Az, & \text{if } B = 0, \end{cases} \quad (-1 \le B < A \le 1),
\end{aligned}$$

which is convex univalent in $\mathbb{U}$ in Theorem 1, then we get the results obtained by Ponnusamy et al. [7] (Theorem 2.1 and Corollary 2.3).

2. For $A = e^{i\alpha}(e^{i\alpha} - 2\beta\cos\alpha)$, where $\beta \in [0, 1)$ and $\alpha \in (-\pi/2, \pi/2)$ in the above expression, then we get the results obtained by Ponnusamy et al. [7] (Theorem 2.5).

3. Taking

$$\begin{aligned}
\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^{\alpha} &= 1 + 2\alpha z + 2\alpha^2 z^2 + \frac{8\alpha^3 + 4\alpha}{6}z^3 + \ldots \\
&= 1 + \sum_{n=1}^{\infty} A_n(\alpha) z^n, \quad (0 < \alpha \le 1),
\end{aligned}$$

which is convex univalent in $\mathbb{U}$, and $A_n(\alpha) = \sum\limits_{k=1}^{n} \binom{n-1}{k-1}\binom{\alpha}{k} 2^k$ in Theorem 1, then we get the results obtained by Ponnusamy et al. [7] (Theorem 2.6).

4. Setting

$$\varphi(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha} z}}{1 - z}\right) = 1 + \sum_{n=1}^{\infty} C_n z^n, \quad (\alpha > 1,\ \beta < 1),$$

which is convex univalent in $\mathbb{U}$, and $C_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right)$ in Theorem 1, then we get the results obtained by Kargar [5] (Theorems 2.2 and 2.3).

5. Letting

$$\varphi(z) = 1 + \frac{1}{2i \sin \delta} \log\left(\frac{1 + ze^{i\delta}}{1 + ze^{-i\delta}}\right) = 1 + \sum_{n=1}^{\infty} D_n z^n, \quad (\pi/2 \leq \delta < \pi),$$

which is convex univalent in $\mathbb{U}$, and $D_n = \frac{(-1)^{n-1} \sin n\delta}{n \sin \delta}$ in Theorem 1, then we get the results obtained by Kargar [5] (Theorems 2.5 and 2.6).

6. Letting

$$\varphi(z) = \left(\frac{1 + cz}{1 - z}\right)^{(\alpha_1 + \alpha_2)/2} = 1 + \sum_{n=1}^{\infty} \lambda_n z^n, \quad \left(0 < \alpha_1, \alpha_2 \leq 1,\ c = e^{\pi i\theta},\ \theta = \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1}\right),$$

which is convex univalent in $\mathbb{U}$, and

$$\lambda_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \binom{(\alpha_1 + \alpha_2)/2}{k} (1 + c)^k$$

in Theorem 1, then we get the results obtained for $|\gamma_n|$ by Kargar et al. [29] (Theorem 3.1). Moreover, for $\alpha_1 = \alpha_2 = \beta$, we get the result presented by Thomas in [30] (Theorem 1).

7. Let the function $f \in \mathcal{K}\left(1 - \frac{cz}{1 - z}\right) = \mathcal{K}(1 - cz - cz^2 - cz^3 + \ldots)$, where $c \in (0, 1]$. It is equivalent to

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{c}{2}.$$

Then we have (see e.g., [31] (Theorem 1))

$$\frac{zf'(z)}{f(z)} \prec \frac{(1 + c)(1 - z)}{1 + c - z},$$

where $\frac{(1 + c)(1 - z)}{1 + c - z}$ is a convex univalent function in $\mathbb{U}$, and

$$\frac{(1 + c)(1 - z)}{1 + c - z} = 1 - \frac{c}{c + 1} z - \frac{c}{(c + 1)^2} z^2 + \cdots = 1 - c \sum_{n=1}^{\infty} \frac{z^n}{(1 + c)^n}.$$

Thus, applying Theorem 1, we get the results obtained by Obradović et al. [4] (Theorem 2 and Corollary 2).

Theorem 2. *Let the function $f \in \mathcal{K}(\varphi)$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{11}$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12} & \text{if} \quad |4B_2 + B_1^2| \leq 4|B_1| \\[4mm] \dfrac{|4B_2 + B_1^2|}{48} & \text{if} \quad |4B_2 + B_1^2| > 4|B_1| \end{cases}, \tag{12}$$

and if B_1, B_2, and B_3 are real values,

$$|\gamma_3| \leq \frac{|B_1|}{24} H(q_1; q_2), \tag{13}$$

where $H(q_1; q_2)$ is given by Lemma 5, $q_1 = \dfrac{B_1 + \frac{4B_2}{B_1}}{2}$, and $q_2 = \dfrac{B_2 + \frac{2B_3}{B_1}}{2}$. The bounds (11) and (12) are sharp.

Proof. Let $f \in \mathcal{K}(\varphi)$. Then by the definition of the subordination, there is a $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ so that

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \varphi(\omega(z)) \\ &= 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + (B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3) z^3 + \cdots. \end{aligned}$$

From the above equation, we get that

$$\begin{cases} 2a_2 = B_1 c_1 \\ 6a_3 - 4a_2^2 = B_1 c_2 + B_2 c_1^2 \\ 12a_4 - 18a_2 a_3 + 8a_2^3 = B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3. \end{cases} \tag{14}$$

By substituting values a_n (n = 1, 2, 3) from (14) in (3), we have

$$\begin{cases} 2\gamma_1 = \dfrac{B_1 c_1}{2} \\[3mm] 2\gamma_2 = \dfrac{8B_1 c_2 + (8B_2 + 2B_1^2) c_1^2}{48} \\[3mm] 2\gamma_3 = \dfrac{B_1}{12}\left[c_3 + \dfrac{B_1 + \frac{4B_2}{B_1}}{2} c_1 c_2 + \dfrac{B_2 + \frac{2B_3}{B_1}}{2} c_1^3 \right]. \end{cases}$$

Firstly, for γ_1, by applying Lemma 1 we get $|\gamma_1| \leq \dfrac{|B_1|}{4}$, and this bound is sharp for $|c_1| = 1$. Next, applying Lemma 1 for γ_2, we have

$$\begin{aligned} |\gamma_2| &\leq \frac{4|B_1|(1 - |c_1|^2) + |4B_2 + B_1^2||c_1|^2}{48} \\ &= \frac{4|B_1| + \left[|4B_2 + B_1^2| - 4|B_1|\right]|c_1|^2}{48} \\ &\leq \begin{cases} \dfrac{4|B_1|}{48} & \text{if} \quad |4B_2 + B_1^2| \leq 4|B_1| \\[4mm] \dfrac{|4B_2 + B_1^2|}{48} & \text{if} \quad |4B_2 + B_1^2| > 4|B_1|. \end{cases} \end{aligned}$$

These bounds are sharp for $c_1 = 0$ and $|c_1| = 1$, respectively.

Finally, using Lemma 5 for γ_3, we obtain

$$2|\gamma_3| \leq \frac{|B_1|}{12} \left| c_3 + \frac{B_1 + \frac{4B_2}{B_1}}{2} c_1 c_2 + \frac{B_2 + \frac{2B_3}{B_1}}{2} c_1^3 \right| \leq H(q_1; q_2),$$

where $q_1 = \frac{B_1 + \frac{4B_2}{B_1}}{2}$ and $q_2 = \frac{B_2 + \frac{2B_3}{B_1}}{2}$. Therefore, this completes the proof. □

Remark 2. *1. Letting*

$$\varphi(z) = 1 + \frac{cz}{1-z}$$
$$= 1 + cz + cz^2 + cz^3 + \ldots \quad (c \in (0,3])$$

in Theorem 2, (for $|\gamma_3|$ with respect to D_6) then we get the results obtained by Ponnusamy et al. [7] (Theorem 2.7 and Corollary 2.8).

2. Taking

$$\varphi(z) = 1 - \frac{cz}{1-z}$$
$$= 1 - cz - cz^2 - cz^3 + \ldots \quad (c \in (0,1])$$

in Theorem 2, (for $|\gamma_3|$ respect to D_2) then we get the results obtained by Ponnusamy et al. [7] (Theorem 2.10).

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