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Article

Properties of Spiral-Like Close-to-Convex Functions Associated with Conic Domains

Hari M. Srivastava [1,2], Nazar Khan [3], Maslina Darus [4], Muhammad Tariq Rahim [3], Qazi Zahoor Ahmad [3,*] and Yousra Zeb [3]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan
4 School of Mathematical Sciences, Faculty of Sciences and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia
* Correspondence: zahoorqazi5@gmail.com; Tel.: +92-334-96-60-162

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Abstract: In this paper, our aim is to define certain new classes of multivalently spiral-like, starlike, convex and the varied Mocanu-type functions, which are associated with conic domains. We investigate such interesting properties of each of these function classes, such as (for example) sufficiency criteria, inclusion results and integral-preserving properties.

Keywords: analytic functions; multivalent functions; starlike functions; close-to-convex functions; uniformly starlike functions; uniformly close-to-convex functions; conic domains

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1. Introduction and Motivation

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk:

$$\mathbb{E} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write:

$$\mathcal{A}(1) = \mathcal{A}.$$

Furthermore, by $\mathcal{S} \subset \mathcal{A}$, we shall denote the class of all functions that are univalent in $\mathbb{E}$.

The familiar class of p-valently starlike functions in $\mathbb{E}$ will be denoted by $\mathcal{S}^*(p)$, which consists of functions $f \in \mathcal{A}(p)$ that satisfy the following conditions:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{E}).$$

One can easily see that:

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the well-known class of normalized starlike functions (see [1]).

We denote by $\mathcal{K}$ the class of close-to-convex functions, which consists of functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{E})$$

for some $g \in \mathcal{S}^*$.

For two functions f and g analytic in $\mathbb{E}$, we say that the function f is subordinate to the function g and write as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{E}$ with:

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that:

$$f(z) = g\big(w(z)\big).$$

Furthermore, if the function g is univalent in $\mathbb{E}$, then it follows that:

$$f(z) \prec g(z) \quad (z \in \mathbb{E}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{E}) \subset g(\mathbb{E}).$$

Next, for a function $f \in \mathcal{A}(p)$ given by (1) and another function $g \in \mathcal{A}(p)$ given by:

$$g(z) = z^p + \sum_{n=2}^{\infty} b_{n+p} z^{n+p} \qquad (\forall\, z \in \mathbb{E}),$$

the convolution (or the Hadamard product) of f and g is given by:

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

The subclass of $\mathcal{A}$ consisting of all analytic functions with a positive real part in $\mathbb{E}$ is denoted by $\mathcal{P}$. An analytic description of $\mathcal{P}$ is given by:

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (\forall\, z \in \mathbb{E}).$$

Furthermore, if:

$$\Re\{h(z)\} > \rho,$$

then we say that h is in the class $\mathcal{P}(\rho)$. Clearly, one see that:

$$\mathcal{P}(0) = \mathcal{P}.$$

Historically, in the year 1933, Spaček [2] introduced the β-spiral-like functions as follows.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\beta)$ if and only if:*

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{E})$$

for:

$$\beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2},$$

where $\mathbb{R}$ is the set of real numbers.

In the year 1967, Libera [3] extended this definition to the class of functions, which are spiral-like of order ρ denoted by $\mathcal{S}_{\rho}^{*}(\beta)$ as follows.

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\rho}^{*}(\beta)$ if and only if:*

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \rho \qquad (\forall\, z \in \mathbb{E})$$

$$\left(0 \leqq \rho < 1;\ \beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2}\right),$$

where $\mathbb{R}$ is the set of real numbers.

The above function classes $\mathcal{S}^{*}(\beta)$ and $\mathcal{S}_{\rho}^{*}(\beta)$ have been studied and generalized by different viewpoints and perspectives. For example, in the year 1974, a subclass $S_{\beta}^{\alpha}(\rho)$ of spiral-like functions was introduced by Silvia (see [4]), who gave some remarkable properties of this function class. Subsequently, Umarani [5] defined and studied another function class $SC(\alpha,\beta)$ of spiral-like functions. Recently, Noor et al. [6] generalized the works of Silvia [4] and Umarani [5] by defining the class $M(p,\alpha,\beta,\rho)$. Here, in this paper, we define certain new subclasses of spiral-like close-to-convex functions by using the idea of Noor et al. [6] and Umarani [5].

We now recall that Kanas et al. (see [7,8]; see also [9]) defined the conic domains Ω_k $(k \geqq 0)$ as follows:

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}. \tag{2}$$

By using these conic domains Ω_k $(k \geqq 0)$, they also introduced and studied the corresponding class k-$\mathcal{ST}$ of k-starlike functions (see Definition 3 below).

Moreover, for fixed k, Ω_k represents the conic region bounded successively by the imaginary axis for $(k=0)$, for $k=1$ a parabola, for $0<k<1$ the right branch of a hyperbola, and for $k>1$ an ellipse. For these conic regions, the following functions $p_k(z)$, which are given by (3), play the role of extremal functions.

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots & (k=0) \\[2ex] 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1) \\[2ex] 1 + \dfrac{2}{1-k^2}\sinh^2\left\{\left(\dfrac{2}{\pi}\arccos k\right)\arctan(h\sqrt{z})\right\} & (0 \leqq k < 1) \\[2ex] 1 + \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2K(\kappa)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\kappa}}}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \dfrac{1}{k^2-1} & (k>1), \end{cases} \tag{3}$$

where:

$$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa z}} \qquad (\forall\, z \in \mathbb{E})$$

and $\kappa \in (0,1)$ is chosen such that:

$$k = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

Here, $K(\kappa)$ is Legendre's complete elliptic integral of the first kind and:

$$K'(\kappa) = K(\sqrt{1-\kappa^2}),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$.

These conic regions are being studied and generalized by several authors (see, for example, [10–13]).

The class k-$\mathcal{ST}$ is defined as follows.

Definition 3. *A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{ST}$ if and only if:*

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \qquad (\forall\, z \in \mathbb{E};\ k \geqq 0)$$

or, equivalently,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|.$$

The class of k-uniformly close-to-convex functions denoted by k-$\mathcal{UK}$ was studied by Acu [14].

Definition 4. *A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{UK}$ if and only if:*

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > k\left|\frac{zf'(z)}{g(z)} - 1\right|,$$

where $g \in k$-$\mathcal{ST}$.

In recent years, several interesting subclasses of analytic functions were introduced and investigated from different viewpoints (see, for example, [6,15–20]; see also [21–25]). Motivated and inspired by the recent and current research in the above-mentioned work, we here introduce and investigate certain new subclasses of analytic and p-valent functions by using the concept of conic domains and spiral-like functions as follows.

Definition 5. *Let $f \in \mathcal{A}(p)$. Then, $f \in k$-$\mathcal{K}(p,\lambda)$ for a real number λ with $|\lambda| < \frac{\pi}{2}$ if and only if:*

$$\Re\left(\frac{e^{i\lambda}}{p}\frac{zf'(z)}{\psi(z)}\right) > k\left|\frac{zf'(z)}{\psi(z)} - p\right| + \rho\cos\lambda \quad (k \geqq 0;\ 0 \leqq \rho < 1)$$

for some $\psi \in \mathcal{S}^*$.

Definition 6. *Let $f \in \mathcal{A}(p)$. Then, $f \in k$-$\mathcal{Q}(p,\lambda)$ for a real λ with $|\lambda| < \frac{\pi}{2}$ if and only if:*

$$\Re\left(\frac{e^{i\lambda}}{p}\frac{zf'(z)}{\psi'(z)}\right) > k\left|\frac{(zf'(z))'}{\psi'(z)} - p\right| + \rho\cos\lambda \quad (k \geqq 0;\ 0 \leqq \rho < 1)$$

for some $\psi \in \mathcal{C}$.

Definition 7. *Let $f \in \mathcal{A}(p)$ with:*

$$\frac{f'(z)\,f(z)}{pz} \neq 0$$

and for some real ϕ and λ with $|\lambda| < \frac{\pi}{2}$. Then, $f \in k$-$\mathcal{Q}(\phi,\lambda,\eta,f,\psi)$ if and only if:

$$\Re\left(\mathcal{M}(\phi,\lambda,\eta,f,\psi)\right) > k\left|\mathcal{M}(\phi,\lambda,\eta,f,\psi) - p\right| + \rho\cos\lambda,$$

where

$$\mathcal{M}(\phi,\lambda,\eta,f,\psi) = (e^{i\lambda} - \phi\cos\lambda)\frac{zf'(z)}{p\psi(z)} + \frac{\phi\cos\lambda}{p-\eta}\left(\frac{(zf'(z))'}{\psi'(z)} - \eta\right) \quad \left(\frac{-1}{2} \leqq \eta < 1\right). \tag{4}$$

2. A Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 1. (*see* [26] *p. 70*) *Let h be a convex function in* $\mathbb{E}$ *and:*

$$q : \mathbb{E} \implies \mathbb{C} \text{ and } \Re(q(z)) > 0 \qquad (z \in \mathbb{E}).$$

If p is analytic in $\mathbb{E}$ *with:*

$$p(0) = h(0),$$

then:

$$p(z) + q(z)\,zp'(z) \prec h(z) \quad \text{implies} \quad p(z) \prec h(z).$$

Lemma 2. (*see* [26] *p. 195*) *Let h be a convex function in* $\mathbb{E}$ *with:*

$$h(0) = 0 \quad \text{and} \quad A > 1.$$

Suppose that $j \geqq \frac{4}{h'(0)}$ *and that the functions* $B(z)$, $C(z)$ *and* $D(z)$ *are analytic in* $\mathbb{E}$ *and satisfy the following inequalities*:

$$\Re\{B(z)\} \geqq A + |C(z) - 1| - \Re(C(z) - 1) + jD(z), \quad z \in \mathbb{E}.$$

If p is analytic in $\mathbb{E}$ *with:*

$$p(z) = 1 + a_1 z + a_2 z^2 + \cdots$$

and the following subordination relation holds true:

$$Az^2 p''(z) + B(z)\,zp'(z) + C(z)\,p(z) + D(z) \prec h(z),$$

then:

$$p(z) \prec h(z).$$

3. Main Results and Their Demonstrations

In this section, we will prove our main results.

Theorem 1. *A function* $f \in \mathcal{A}$ *is in the class* k-$\mathcal{Q}(\phi,\lambda,\eta,f,\psi)$ *if:*

$$\sum_{n=2}^{\infty} \ddot{U}_n(p,\phi,\lambda,\eta,\xi) < p^2(p-\eta),$$

where:

$$\begin{aligned}\ddot{U}_n(p,\phi,\lambda,\eta,\xi) = (k+1)\,&[(e^{i\lambda} - \phi\cos\lambda)(p-\eta)p + p^4\phi\cos\lambda \\ &+ (e^{i\lambda} - \phi\cos\lambda)(p-\eta)(n+p)\left|a_{n+p}\right| + (n+p)^2\left|a_{n+p}\right| \\ &+ [(np\phi\cos\lambda + p^3(p-\eta)](n+p)\left|b_{n+p}\right| + np^2\phi\cos\lambda - p^3(p-\eta).\end{aligned} \tag{5}$$

Proof. Let us assume that the relation (4) holds true. It now suffices to show that:

$$k\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right|-\Re\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right|<1. \tag{6}$$

We first consider:

$$\begin{aligned}
&\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right| \\
&\quad=\left|\left(e^{i\lambda}-\phi\cos\lambda\right)\frac{zf'(z)}{p\psi(z)}+\frac{\phi\cos\lambda}{(p-\eta)}\left(\frac{(zf'(z))'}{\psi'(z)}-\eta\right)-p\right| \\
&\quad=\left|\frac{\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)f'(z)}{p\left(p-\eta\right)\psi'(z)}+\frac{p\phi\cos\lambda\left(zf'(z)\right)'}{p\left(p-\eta\right)\psi'(z)}-\right. \\
&\qquad\left.-\frac{\eta p\phi\cos\lambda\psi'(z)}{p\left(p-\eta\right)\psi'(z)}-\frac{p^2\left(p-\eta\right)\psi'(z)}{p(p-\eta)\psi'(z)}\right|.
\end{aligned}$$

Now, by using the series form of the functions f and ψ given by:

$$f(z)=z^p+\sum_{n=2}^{\infty}a_{n+p}z^{n+p}$$

and:

$$\psi(z)=z^p+\sum_{n=2}^{\infty}b_{n+p}z^{n+p}$$

in the above relation, we have:

$$\begin{aligned}
&\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right| \\
&\quad=\left|\frac{\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)\left(pz^{p-1}\right)+p\phi\cos\lambda(p^2z^{p-1})}{p(p-\eta)\left(pz^{p-1}+\sum_{n=2}^{\infty}(n+p)b_{n+p}z^{n+p-1}\right)}\right. \\
&\qquad\left.+\frac{\sum_{n=2}^{\infty}(n+p)a_{n+p}z^{n+p-1}\left[\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)+\left(n+p\right)\right]}{p\left(p-\eta\right)\left(pz^{p-1}+\sum_{n=2}^{\infty}(n+p)b_{n+p}z^{n+p-1}\right)}-\frac{n\phi\cos\lambda}{(p-\eta)}-p\right| \\
&\quad\leqq\frac{\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)\left(p\right)+p\phi\cos\lambda(p^2)}{p\left(p-\eta\right)\left(p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|\right)} \\
&\qquad+\frac{\sum_{n=2}^{\infty}(n+p)\left|a_{n+p}\right|\left\{\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)+\left(n+p\right)\right\}}{p\left(p-\eta\right)\left(p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|\right)}-\left\{\frac{n\phi\cos\lambda}{(p-\eta)}+p\right\}.
\end{aligned}$$

We now see that:

$$\begin{aligned}
&k\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right|-\Re\left\{\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right\} \\
&\quad\leqq(k+1)\left|\mathcal{M}\left(\phi,\lambda,\eta,f,\psi\right)-p\right| \\
&\quad\leqq(k+1)\left[\frac{\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)\left(p\right)+p\phi\cos\lambda(p^2)}{p\left(p-\eta\right)\left(p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|\right)}\right. \\
&\qquad\left.+\frac{\sum_{n=2}^{\infty}(n+p)\left|a_{n+p}\right|\left[\left(e^{i\lambda}-\phi\cos\lambda\right)\left(p-\eta\right)+\left(n+p\right)\right]}{p\left(p-\eta\right)\left(p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|\right)}-\left[\frac{n\phi\cos\lambda}{(p-\eta)}+p\right]\right].
\end{aligned}$$

The above inequality is bounded above by one, if:

$$\begin{aligned}
&(k+1)\left[\left(e^{i\lambda}-\phi\cos\lambda\right)(p-\eta)p\right]+(p\phi\cos\lambda)p^2 \\
&\quad+\left(\sum_{n=2}^{\infty}(n+p)\left|a_{n+p}\right|\right)\left\{(e^{i\lambda}-\phi\cos\lambda)(p-\eta)+(n+p)\right\}-\left[\frac{n\phi\cos\lambda}{(p-\eta)}-p\right] \\
&\quad\cdot\left\{p\,(p-\eta)\left(p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|\right)\right\} \\
&\leqq p(p-\eta)p+\sum_{n=2}^{\infty}(n+p)\left|b_{n+p}\right|.
\end{aligned}$$

Hence:

$$\sum_{n=2}^{\infty}\ddot{U}_n\,(p,\phi,\lambda,\eta,\xi)\leqq p^2(p-\eta),$$

where $\ddot{U}_n\,(p,\phi,\lambda,\eta,\xi)$ is given by (5) , which completes the proof of Theorem 1. □

Theorem 2. *A function $f\in\mathcal{A}(p)$ satisfies the condition:*

$$\left|\frac{1}{e^{ij}F(z)}-\frac{1}{2\rho}\right|<\frac{1}{2\rho}\qquad(0\leqq\rho<1;\ j\in\mathbb{R}) \tag{7}$$

if and only if $f\in 0\text{-}\mathcal{K}(p,\lambda)$, where

$$F(z)=\frac{zf'(z)}{p\psi(z)}.$$

Proof. Suppose that f satisfies (7). We then can write:

$$\begin{aligned}
&\left|\frac{2\rho-e^{ij}F(z)}{e^{ij}F(z)2\rho}\right|<\frac{1}{2\rho} \\
&\iff\left(\left|\frac{2\rho-e^{ij}F(z)}{e^{ij}F(z)2\rho}\right|\right)^2<\left(\frac{1}{2\rho}\right)^2 \\
&\iff\left(2\rho-e^{ij}F(z)\right)\left(\overline{2\rho-e^{ij}F(z)}\right)<e^{-ij}\overline{F(z)}e^{ij}F(z) \\
&\iff 4\rho^2-2\rho\left[e^{-ij}\overline{F(z)}+e^{ij}F(z)\right]+F(z)\overline{F(z)}<F(z)\overline{F(z)} \\
&\iff 4\rho^2-2\rho\left[e^{-ij}\overline{F(z)}+e^{ij}F(z)\right]<0 \\
&\iff 2\rho-2\Re\left[e^{ij}\overline{F(z)}\right]<0 \\
&\iff \Re\left[e^{ij}F(z)\right]>\rho \\
&\iff \Re\left(e^{ij}\frac{zf'(z)}{p\psi(z)}\right)>\rho.
\end{aligned}$$

This completes the proof of Theorem 2. □

Theorem 3. *For $0\leqq\varphi_1<\varphi_2$, it is asserted that:*

$$k\text{-}\mathcal{Q}\,(p,\varphi_2,\lambda,\eta)\subset 0\text{-}\mathcal{Q}\,(p,\varphi_1,\lambda,\eta)\,.$$

Proof. Let $f(z) \in k\text{-}\mathcal{Q}\left(p, \varphi_2, \lambda, \eta\right)$. Then:

$$\begin{aligned}
&\frac{1}{p-\eta}\left[\left(e^{i\lambda} - \phi_1 \cos\lambda\right)(p-\eta)\frac{zf'(z)}{p\psi(z)} + \varphi_1 \cos\lambda\left(\frac{(zf'(z))'}{\psi(z)'} - \eta\right)\right] \\
&= \frac{\varphi_1}{\varphi_2}\left[\left(e^{i\lambda} - \varphi_2 \cos\lambda\right)\frac{zf'(z)}{p\psi(z)} + \frac{\varphi_{2\cos\lambda}}{(p-\eta)}\left(\frac{(zf'(z))'}{p\psi(z)'} - \eta\right)\right] \\
&\quad - \left(\frac{\varphi_1 - \varphi_2}{\varphi_2}\right) e^{i\lambda}\frac{sf'(z)}{p\psi(z)} \\
&= \frac{\varphi_1}{\varphi_2} H_1(z) + \left(1 - \frac{\varphi_1}{\varphi_2}\right) H_2(z) = H(z),
\end{aligned}$$

where:

$$H_1(z) = \left(e^{i\lambda} - \varphi_2 \cos\lambda\right)\frac{zf'(z)}{p\psi(z)} + \frac{\varphi_2 \cos\lambda}{(p-\eta)}\left(\frac{(zf'z)'}{\psi'(z)} - \eta\right) \in \mathcal{P}\left(h_{k,\rho}\right) \subset \mathcal{P}(\rho)$$

and:

$$H_2(z) = e^{i\lambda}\frac{zf'(z)}{p\psi(z)} \in \mathcal{P}(\rho).$$

Since $\mathcal{P}(\rho)$ is a convex set (see [27]), we therefore have $H(z) \in \mathcal{P}(\rho)$. This implies that $f \in 0\text{-}\mathcal{Q}\left(p, \varphi_1, \lambda, \eta\right)$. Thus:

$$k\text{-}\mathcal{Q}\left(p, \varphi_2, \lambda, \eta\right) \subset 0\text{-}\mathcal{Q}\left(p, \varphi_1, \lambda, \eta\right).$$

The proof of Theorem 3 is now completed. □

Theorem 4. *Let $\phi > 0$ and $\lambda < \frac{\pi}{2}$. Then:*

$$k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi) \subset k\text{-}\mathcal{K}(p, 0, \xi).$$

Proof. Let $f \in k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi)$, and suppose that:

$$\frac{f'(z)}{\psi'(z)} = p(z), \tag{8}$$

where $p(z)$ is analytic and $p(0) = 1$. Now, by differentiating both sides of (8) with respect to z, we have:

$$\frac{(zf'(z))'}{\psi'(z)} = zp'(z) + p(z)\varepsilon(z), \tag{9}$$

where:

$$\varepsilon(z) = \frac{(z\psi'(z))'}{\psi'(z)}.$$

By using (8) and (9) in (4), we arrive at:

$$\begin{aligned}
\mathcal{M}(\phi, \lambda, \eta, f, \psi) &= \left(e^{i\lambda} - \phi\cos\lambda\right)\frac{p(z)}{p} + \frac{\phi\cos\lambda}{p-\eta}\left(zp'(z) + p(z)\varepsilon(z) - \eta\right) \\
&= \frac{\phi\cos\lambda}{p-\eta} zp'(z) + \left(\frac{e^{i\lambda}}{p} - \frac{\phi\cos\lambda}{p-\varepsilon(z)}\right)\left(\frac{\phi\cos\lambda}{p-\eta}\right) p(z) - \frac{\eta\phi\cos\lambda}{p-\eta} \\
&= B(z)\, zp'(z) + C(z)\, p(z) + D(z),
\end{aligned} \tag{10}$$

where:

$$B(z) = \frac{\phi\cos\lambda}{p-\eta},$$

$$C(z) = \frac{e^{i\lambda}(p-\eta) - \phi\cos\lambda\,(p-\eta) + \phi\cos\lambda\varepsilon(z)p}{p(p-\eta)}$$

and:

$$D(z) = \frac{\eta\phi\cos\lambda}{p-\eta}.$$

Now, since $f \in k\text{-}\mathcal{Q}(p,\phi,\lambda,\eta,\xi)$, we have:

$$B(z)\,zp'(z) + C(z)\,p(z) + D(z) \prec p_k(z), \tag{11}$$

which, upon replacing $p(z)$ by:

$$p_*(z) = p(z) - 1,$$

and $p_k(z)$ by:

$$p_k^*(z) = p_k(z) - 1,$$

shows that the above subordination in (11) becomes as follows:

$$B(z)\,zp_x'(z) + C(z)\,p_x(z) + D_*(z) \prec p_k^*(z), \tag{12}$$

where:

$$D_*(z) = C(z) + D(z) - 1.$$

We now apply Lemma 2 with:

$$A = 0$$

and

$$p_*(z) \prec p_k^*(z).$$

We thus find that:

$$\frac{f'(z)}{\psi'(z)} = p(z) \prec p_k^*(z). \tag{13}$$

This complete the proof of Theorem 4. □

For $f \in \mathcal{A}$, we next consider the integral operator defined by:

$$F(z) = I_m[f] = \frac{m+1}{z^m}\int_0^z t^{m-1} f(t)\,dt. \tag{14}$$

This operator was given by Bernardi [28] in the year 1969. In particular, the operator I_1 was considered by Libera [29]. We prove the following result.

Theorem 5. *Let $f(z) \in k\text{-}\mathcal{Q}(p,\phi,\lambda,\eta,\xi)$. Then, $I_m[f] \in \mathcal{K}(p,0,\xi)$.*

Proof. Let the function $\psi(z)$ be such that:

$$\mathcal{M}(\phi,\lambda,\eta,f,\psi) = \left(e^{i\lambda} - \phi\cos\lambda\right)\frac{zf'(z)}{p\psi(z)} + \frac{\phi\cos\lambda}{(p-\eta)}\left(\frac{(zf'(z))'}{\psi'(z)} - \eta\right).$$

Then, according to [14], the function $G = I_m[f] \in \mathcal{CD}(k,\delta)$. Furthermore, from (14), we deduce that:

$$(1+m)f(z) = (1+m)F(z) + z(F(z))' \tag{15}$$

and:

$$(1+m)g(z) = (1+m)G(z) + z(G(z))'. \tag{16}$$

If we now put:

$$p(z) = \frac{F'(z)}{G'(z)}$$

and:

$$q(z) = \frac{1}{(m+1) + \left(\frac{zG''(z)}{G'(z)}\right)},$$

then, by simple computations, we find that:

$$\frac{f(z)}{\psi(z)} = \frac{(1+m) F'(z) + zF''(z)}{(1+m) G'(z) + zG''(z)}$$

or, equivalently, that:

$$\frac{f'(z)}{\psi'(z)} = p(z) + zp'(z) q(z). \tag{17}$$

We now let:

$$\frac{f'(z)}{\psi'(z)} = p(z) + zp'(z) q(z) = h(z), \tag{18}$$

where the function $h(z)$ is analytic in $\mathbb{E}$ with $h(0) = 1$. Then, by using (18), we have:

$$\frac{(zf(z))'}{\psi'(z)} = zh'(z) + \varepsilon(z) h(z), \tag{19}$$

where:

$$\varepsilon(z) = \frac{(z\psi'(z))'}{\psi'(z)}.$$

Furthermore, by using (18) and (19) in (4), we obtain:

$$\begin{aligned}
\mathcal{M}(\alpha,\beta,\gamma,\lambda,\delta,f) &= \left(e^{i\lambda} - \theta\cos\lambda\right)\frac{zf'(z)}{\psi'(z)} + \frac{\phi\cos\lambda}{p-\eta}\left(\frac{(zf'(z))'}{\psi'(z)} - \eta\right) \\
&= \left(e^{i\lambda} - \theta\cos\lambda\right) + \frac{\phi\cos\lambda}{p-\eta} zh'(z) + \left[zh'(z) + \varepsilon(z) h(z) - \eta\right] \\
&= \frac{\phi\cos\lambda}{p-\eta} zh'(z) + \left(e^{i\lambda} - \phi\cos\lambda + \frac{\phi\cos\lambda}{p-\eta}\right) h(z) - \frac{\eta(\phi\cos\lambda)}{p-\eta} \\
&= B(z) zh'(z) + C(z) h(z) + D(z),
\end{aligned}$$

where:

$$B(z) = \frac{\phi\cos\lambda}{p-\eta},$$

$$C(z) = \frac{\left((p-\eta) e^{i\lambda} - (p-\eta)\phi\cos\lambda + \phi\cos\lambda\right)}{p-\eta}$$

and:

$$D(z) = \frac{\eta(\phi\cos\lambda)}{p-\eta}.$$

Now, if we apply Lemma 1 with $A = 0$, we get:

$$\frac{f'(z)}{\psi'(z)} = h(z) \prec p_k(z). \tag{20}$$

Furthermore, from (18), we have:

$$p(z) + zp'(z) q(z) \prec p_k(z).$$

By using Lemma 2 on (20), we obtain the desired result. This completes the proof of Theorem 5. □

4. Conclusions

Using the idea of spiral-like and close-to-convex functions, we have introduced Mocanu-type functions associated with conic domains. We have derived some interesting results such as sufficiency criteria, inclusion results, and integral-preserving properties. We have also proven that the our newly-defined function classes are closed under the famous Libera operator.

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References

1. Duren, P.L. *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259; Springer-Verlag: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
2. Spaček, L. Prispevek *k* teorii funkei prostych. *Casŏpis. Pest. Mat.* **1933**, *62*, 12–19.
3. Libera, R. J. Univalent *a*-spiral functions, *Can. J. Math.* **1967**, *19*, 449-456.
4. Silvia, M.E. On a subclass of spiral-like functions, *Proc. Am. Math. Soc.* **1974**, *44*, 411–420. [CrossRef]
5. Umarani, P. On a subclass of spiral-like functions, *Indian J. Pure Appl. Math.* **1979**, *10*, 1292–1297.
6. Noor, K.I.; Khan, N.; Ahmad, Q.Z. Some properties of multivalent spiral-like functions. *Maejo Int. J. Sci. Technol.* **2018**, *3*, 353–364.
7. Kanas. S.; Wiśniowska, A. Conic regions and *k*-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
8. Kanas. S.; Wiśniowska, A. Conic domains and starlike functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
9. Kanas, S.; Srivastava, H. M.; Linear operators associated with *k*-uniformly convex functions. *Integral Transforms Spec. Funct.* **2000**, *9*, 121–132. [CrossRef]
10. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
11. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Math. Appl.* **2011**, *62*, 2209–2217. [CrossRef]
12. Shams, S.; Kulkarni, S.R.; Jahangiri, J.M. Classes of uniformly starlike and convex functions. *Int. J. Math. Math. Sci.* **2004**, *55*, 2959–2961. [CrossRef]
13. Srivastava, H.M.; Shanmugam, T.N.; Ramachandran, C.; Sivassurbramanian, S. A new subclass of *k*-uniformly convex functions with negative coefficients. *J. Inequal. Pure Appl. Math.* **2007**, *8*, 1–14.
14. Acu, M. On a subclass of *k*-uniformly close to convex functions. *Gen. Math.* **2006**, *14*, 55–64.
15. Arif, M.; Dziok, J.; Raza, M.; Sokól, J. On products of multivalent close-to-star functions, *J. Inequal. Appl.* **2015**, *2015*, 1–14. [CrossRef]
16. Khan, N.; Khan, B.; Ahmad, Q.Z.; Ahmad, S. Some Convolution properties of multivalent analytic functions. *AIMS Math.* **2017**, *2*, 260–268. [CrossRef]
17. Noor, K.I.; Khan, N.; Noor, M.A. On generalized spiral-like analytic functions. *Filomat* **2014**, *28*, 1493–1503. [CrossRef]
18. Raza, M.; Din, M.U.; Malik, S.N. Certain geometric properties of normalized Wright functions. *J. Funct. Spaces* **2016**, *2016*. [CrossRef]
19. Srivastava, H.M.; Eker, S.S. Some applications of a subordination theorem for a class of analytic functions. *Appl. Math. Lett.* **2008**, *21*, 394–399. [CrossRef]
20. Srivastava, H.M.; El-Ashwah, R.M.; Breaz, N. A certain subclass of multivalent functions involving higher-order derivatives. *Filomat* **2016**, *30*, 113–124. [CrossRef]

21. Aldweby, H.; Darus, M. A note on q-integral operators. *Electron. Notes Discret. Math.* **2018**, *67*, 25–30. [CrossRef]
22. Aldweby, H.; Darus, M. On Fekete-Szegö problems for certain subclasses defined by q-derivative. *J. Funct. Spaces* **2017**, 1–5. . [CrossRef]
23. Elhaddad, S.; Aldweby, H.; Darus, M. Some Properties on a class of harmonic univalent functions defined by q-analoque of Ruscheweyh operator. *J. Math. Anal.* **2018**, *9*, 28–35.
24. Hussain, S.; Khan, S.; Zaighum, M.A.; Darus, M. Applications of a q-Sălăgean type operator on multivalent functions. *J. Inequal. Appl.* **2018**, *301*, 1–12. [CrossRef]
25. Rasheed, A.; Hussain, S.; Zaighum, M.A.; Darus, M. Class of analytic function related with uniformly convex and Janowski's functions. *J. Funct. Spaces* **2018**, *2018*. [CrossRef]
26. Miller, S.S.; Mocanu, P.T. *Differential Subordinations*: *Theory and Applications*; Marcel Dekker: New York, NY, USA, 2000.
27. Kanas. S. Techniques of the differential subordination for domains bounded by conic sections. *Int. J. Math. Math. Sci.* **2003**, *38*, 2389–2400. [CrossRef]
28. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
29. Libera, R.J. Some classes of regular univalent functions. *Proc. Am. Math. Soc.* **1965**, *16*, 755–758. [CrossRef]