mathematics

MDPI

Article

Global Stability of Fractional Order Coupled Systems with Impulses via a Graphic Approach

Bei Zhang [1], Yonghui Xia [2,*], Lijuan Zhu [2], Haidong Liu [3] and Longfei Gu [4]

1 School of Mathematics Science, Huaqiao University, Quanzhou 362000, China
2 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
3 School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
4 Department of Mathematics, Linyi University, Linyi 276000, China
* Correspondence: yhxia@zjnu.cn

Received: 7 July 2019; Accepted: 13 August 2019; Published: 15 August 2019

Abstract: Based on the graph theory and stability theory of dynamical system, this paper studies the stability of the trivial solution of a coupled fractional-order system. Some sufficient conditions are obtained to guarantee the global stability of the trivial solution. Finally, a comparison between fractional-order system and integer-order system ends the paper.

Keywords: neural networks; global stability; impulse

1. Introduction

Due to the great significance in applied science (e.g., signal and image processing, artificial intelligence, pattern classification), the neural networks have attracted many scholars' attention. There are a large amount of scientific research results on the stability and synchronization of both integer-order and fractional-order differential equations. For examples, one can refer to [1–15]. Besides, there are many results about fractional equations such as [16–22]. However, in the real world, at certain moments, many behaviors in neural networks may experience a sudden change. They are affected by short-term perturbations whose duration is particularly short comparing to the process with no change. We can use impulsive differential equations to describe the phenomena. Some works considered the impulsive effects on the neural networks (e.g., see [23–28]). It is worthwhile to mention that the fractional-order impulsive differential equations were studied recently (see e.g., [29–36]). Among them, Stamov and Stamova [31–34] studied the almost periodicity of the fractional-order impulsive differential equations. It is difficult to get less conservative conditions to guarantee the global stability of a system. Recently, a new powerful tool is to apply graph theory to study the stability and synchronization of neural networks (see e.g., [37–42]. Inspired by the previous works, we consider the global stability of fractional-order coupled systems with impulses on digraph $\mathcal{G}$.

$$\begin{cases} D^{\mu}x_p = -w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{n} a_{pq}(x_p(t) - x_q(t)), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x_p(t_k) = I_k(x_p(t_k)), \\ x(t_k^-) = x(t_k), \quad k = 1, 2, \cdots, \end{cases}$$

where $\Delta x_p(t_k) = x_p(t_k^+) - x_p(t_k^-)$ are the impulses at moments t_k and $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$ as $k \to \infty$ (see e.g., [30–32,43–46]). $I_k : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and $I_k = 0$ when the impulses are absent. For the fractional order systems, the criteria to determine the stability for the integer order differential systems may not be applicable because fractional derivative may not maintain the properties of the integer derivative. (e.g., see [47,48]). The difficulty comes from the following facts.

1. For the integer derivative, the sign of the first order derivative implies the monotonicity of a function. However, this is not valid for the fractional derivative (see [47]). This difference results in great difficulties to deal with the impulses at moment t_k.
2. For the integer-order system $\frac{dx}{dt} = f(x,t)$, the first derivative $\frac{dV(x)}{dt} \leq -\omega(x) < 0$ implies the **asymptotically stability** in the sense of Lyapunov. However, this classical Lyapunov stability result is not valid for fractional-order system. The derivative $D^\alpha V(x) \leq -\omega(x) < 0$ does not imply the **asymptotically stability** (see Lemma 2 in next section). It can only guarantee the stability.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, main results of this paper is presented by employing graph theory. In Section 4, an example and its simulations are presented to verify the feasibility of the obtained results. Finally, Conclusions and Discussion end the paper.

2. Preliminaries

There are a lot of different definitions of fractional derivative (e.g., Riemann-Liouville, Caputo, the conformable fractional derivative, [49–51]). In this paper, we employ Caputo fractional integral and derivative.

Definition 1. [50] *The fractional integral with noninteger order $\mu > 0$ for a function $x(t)$ is defined as*

$$I^\mu x(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t-\tau)^{\mu-1} x(\tau)\, d\tau,$$

where $t \geq t_0$, t_0 is the initial time, $\Gamma(\cdot)$ is the gamma function, given by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$.

Definition 2. [50] *The Caputo fractional derivative of order μ for a function $x(t)$ is defined as*

$$D^\mu x(t) = \frac{1}{\Gamma(n-\mu)} \int_{t_0}^{t} (t-\tau)^{n-\mu-1} x^{(n)}(\tau)\, d\tau,$$

in which $t \geq t_0$, t_0 is the initial time, $n-1 < \mu < n \in Z^+$.

Lemma 1. [52] *Suppose that $x(t) \in \mathbb{R}$ is a continuous and differentiable vector-value function. Then for any time instant $t \geq t_0$, we have*

$$\frac{1}{2} D^\alpha x^2(t) \leq x(t) D^\alpha x(t)$$

when $0 < \alpha < 1$.

Lemma 2. [47] *Consider system $D^\alpha x = f(x,t)$, where $0 < \alpha \leq 1$, $f : (D \subset \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$. Let $V(x,t)$ be a continuously differentiable and positive definite function. Let $\omega(x)$ be a positive definite function continuous at $x = 0$ such that in the ball $B(r) \subseteq D$ around $x = 0$ with $x_0 \in B(r)$ we have*

$$D^\alpha V(x,t) \leq -\omega(x) \leq 0.$$

Then $\liminf_{t\to\infty} \|x\| = 0$ *and $x = 0$ is stable at $t = 0$. In particular, $x = 0 \in \bigcap_{x \in B(r)} \Omega(x)$. For $\alpha = 1$,* $\lim_{t\to\infty} \|x\| = 0$ *($x = 0$ is asymptotically stable at $t = 0$).*

Then in what follows, we recall some basic knowledge of graph theory [40,53].

A directed graph or digraph $\mathcal{G} = (V, E)$ contains a vertex set $V = \{1, 2, \ldots, n\}$ and a set E of arcs (p, q) from p to q. $\mathcal{H} \subseteq \mathcal{G}$ is said to be spanning if the vertex set of $\mathcal{H}$ is the same as $\mathcal{G}$. If each (p, q) is assigned a positive weight a_{pq}, then we say graph $\mathcal{G}$ is weighted. In our convention, $a_{pq} > 0$ if and only if there is an arc from p to q. The weight of a subgraph $\mathcal{H}$ is the product of the weight of each arc.

A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with vertices $\{p_1, p_2, \ldots, p_m\}$ such that its set of arcs is $\{(p_k, p_{k+1}) : k = 1, 2, \ldots, m-1\}$. If the arc (p_m, p_1) exists, then we call $\mathcal{P}$ a directed cycle. If there does not exist any cycle in the connected subgraph $\mathcal{T}$, then we call $\mathcal{T}$ a tree. For a tree $\mathcal{T}$, if there does not exist any arc to vertex p, then $\mathcal{T}$ is rooted at vertex p. If a subgraph $\mathcal{Q}$ is a disjoint union of some rooted trees and the roots of these trees can form a directed cycle, then we say $\mathcal{Q}$ is unicyclic.

For a given weighted digraph $\mathcal{G}$ with n vertices, $A = (a_{pq})_{n\times n}$ is the weight matrix whose entry a_{pq} is the weight of (p, q) if it exists, and 0 otherwise. For our purpose, we write a weighted digraph as $(\mathcal{G}, A)$. If for any pair of vertices there exists a directed arc from one to the other, then $\mathcal{G}$ is strongly connected. The we define the Laplacian matrix of $(\mathcal{G}, A)$ as

$$L = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{pmatrix}. \tag{1}$$

Let c_p be the cofactor of the p-th diagonal element of L. Then we have the following results.

Lemma 3. [40] *Assume $n \geq 2$. Then*

$$c_p = \sum_{\mathcal{T}\in\mathbb{T}_p} w(\mathcal{T}), \quad p = 1, 2, \cdots, n, \tag{2}$$

where $\mathbb{T}_p$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex p, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_p > 0$ for $1 \leq p \leq n$.

For the coupled system on a directed graph $\mathcal{G}$:

$$D^\alpha u_p = f_p(t, u_p) + \sum_{q=1}^{n} g_{pq}(t, u_p, u_q), \quad p = 1, 2, \cdots, n, \tag{3}$$

where $u_p \in \mathbb{R}^{m_p}$, $f_p : \mathbb{R} \times \mathbb{R}^{m_p} \to \mathbb{R}^{m_p}$, $g_{pq} : \mathbb{R} \times \mathbb{R}^{m_p} \times \mathbb{R}^{m_q} \to \mathbb{R}^{m_p}$ represent the influence from vertex p to vertex q, and $g_{pq} = 0$ if there does not exist arc from p to q in $\mathcal{G}$.

Motivated by Theorem 3.4 in [40], for fractional-order systems, we have the following theorem.

Theorem 1. *Assume that the following assumptions hold.*

(i) *For the Lyapunov function $V_p(t, u_p)$ on each vertex. There exist $F_{pq}(t, u_p, u_q)$, $a_{pq} \geq 0$, and $b_p \geq 0$ such that*

$$D^\alpha V_p(t, u_p) \leq -b_p V_p(t, u_p) + \sum_{q=1}^{n} a_{pq} F_{pq}(t, u_p, u_q), \quad t > 0, \quad u_p \in D_p, \quad 1 \leq p \leq n$$

holds.

(ii) *Along each directed cycle $\mathcal{C}$ in the weighted digraph $(\mathcal{G}, A)$, $A = (a_{pq})$,*

$$\sum_{(s,r)\in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, \quad u_r \in D_r, \quad u_s \in D_s.$$

(iii) *c_p are constants which are given in Lemma 3.*
Then $V(t, u) = \sum_{p=1}^{n} c_p V_p(t, u_p)$ satisfies

$$D^\alpha V(t, u) \leq -bV(t, u), \quad t > 0, \quad u \in D,$$

where $b = \min\{b_1, b_2, \ldots, b_n\}$.

Proof. For a spanning tree $\mathcal{T}$ (see Figure 1) rooted at q, by adding an arc (p, q) from p to q, we obtain a unicyclic graph $\mathcal{Q}$ (see Figure 2).

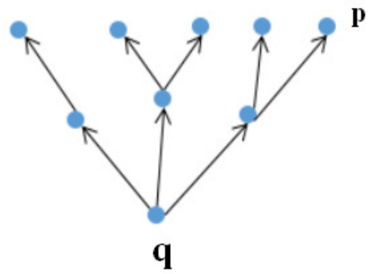

Figure 1. A rooted tree $\mathcal{T}$.

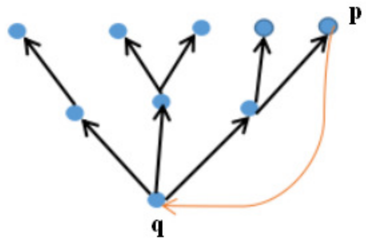

Figure 2. A unicyclic graph $\mathcal{Q}$.

According to the definition for the weight of a graph, we have $w(\mathcal{Q}) = w(\mathcal{T})a_{pq}$. As a result, $w(\mathcal{T})a_{pq}F_{pq}(t, u_p, u_q) = w(\mathcal{Q})F_{pq}(t, u_p, u_q)$, $(q, p) \in E(\mathcal{C}_\mathcal{Q})$. Here $F_{pq}(t, u_p, u_q)$, $1 \leq p, q \leq n$, are arbitrary functions, $\mathcal{C}_\mathcal{Q}$ denotes the directed cycle of $\mathcal{Q}$.

When we do this operation to all rooted spanning trees in diagraph $\mathcal{G}$ in all possible ways, we will derive all unicyclic graphs in $\mathcal{G}$. Then we get

$$\sum_{p,q=1}^{n} c_p a_{pq} F_{pq}(t, u_p, u_q) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_\mathcal{Q})} F_{rs}(t, u_r, u_s),$$

where $\mathbb{Q}$ is a set which includes all spanning unicyclic graphs of $(\mathcal{G}, A)$.

Based on the definition of the Caputo fractional order derivative, we know that $D^\alpha[lx(t) + my(t)] = lD^\alpha x(t) + mD^\alpha y(t)$ easily. Thus, for $V(t, u) = \sum_{i=p}^{n} c_p V_p(t, u_p)$, we have

$$\begin{aligned}
D^\alpha V(t, u) &= D^\alpha \sum_{p=1}^{n} c_p V_p(t, u_p) \\
&= \sum_{p=1}^{n} c_p D^\alpha V_p(t, u_p) \\
&\leq \sum_{p=1}^{n} c_p[-b_p V_p(t, u_p) + \sum_{q=1}^{n} a_{pq} F_{pq}(t, u_p, u_q)] \\
&= -\sum_{p=1}^{n} b_p c_p V_p(t, u_p) + \sum_{p,q=1}^{n} c_p a_{pq} F_{pq}(t, u_p, u_q) \\
&= -\sum_{p=1}^{n} b_p c_p V_p(t, u_p) + \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_\mathcal{Q})} F_{rs}(t, u_r, u_s).
\end{aligned}$$

In view of the condition (ii) and $w(\mathcal{Q}) > 0$, we have

$$D^{\alpha}V(t,u) \leq -\sum_{p=1}^{n} b_p c_p V_p(t,u_p) \leq -\sum_{p=1}^{n} b c_p V_p(t,u_p) = -bV(t,u),$$

here $b = \min\{b_1, b_2, \ldots, b_n\}$. □

Remark 1. *To study the stability of the coupled systems, constructing a proper Lyapunov function is of great importance. Theorem 1 reveals that a global Lyapunov function for (3) can be the combination of the Lyapunov function V_i of each vertex system, which decreases the difficulty for us.*

3. Main Results

Given a network represented by a digraph $\mathcal{G}$ with n vertices. Assume that the dynamic of each vertex is described by the following impulsive differential equation:

$$\begin{cases} D^{\mu}x_p = -w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x_p(t_k) = I_k(x_p(t_k)), \\ x(t_k^-) = x(t_k), \quad k = 1, 2, \cdots, \end{cases} \tag{4}$$

$p, q = 1, 2, \ldots, n$, where $0 < \mu < 1$, $w_p > 0$ is the self-regulating parameters of the p-th vertex, a_{pq} represents the weight of the arc from vertex p to q. $f_q(x)$ is the neuron activation function satisfying Lipschitz condition: for all $x, y \in \mathbb{R}$, there exists a Lipschitz constant $l_j > 0$ such that $|f_q(x) - f_q(y)| \leq l_q|x - y|$. In addition, $f_q(0) = 0$.

Now we consider the following impulsive coupled system on digraph $\mathcal{G}$:

$$\begin{cases} D^{\mu}x_p = -w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{n} a_{pq}(x_p(t) - x_q(t)), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x_p(t_k) = I_k(x_p(t_k)), \\ x(t_k^-) = x(t_k), \quad k = 1, 2, \cdots. \end{cases} \tag{5}$$

Theorem 2. *Assume $(\mathcal{G}, A)$ is strongly connected. If the following conditions hold:*

(1) $b = \min\limits_{1 \leq p \leq n} (2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) > 0$;

(2) $I_k(x_{pk}(t_k)) = \delta_{pk} x_{pk}(t_k)$, *where* $-1 < \delta_{pk} < 0$;

(3) *In each interval,* $x_p(t)$ *satisfies* $|x_p(t_k)| < |x_p(t_{k-1}^+)|$.

Then the trivial solution of (5) is globally stable.

Proof. Construct a Lyapunov function $V_p = \frac{x_p^2}{2}$ and calculate the μ-order derivative of V_p along (5), we have

$$\begin{aligned}
D^\mu V_p = D^\mu \frac{x_p^2}{2} &\leq x_p D^\mu x_p \\
&= x_p[-w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q) + \sum_{q=1}^{n} a_{pq}(x_p - x_q)] \\
&\leq -w_p x_p^2 + \sum_{q=1}^{n} |x_p| a_{pq} |f_q(x_q)| + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p \\
&\leq -w_p x_p^2 + \sum_{q=1}^{n} |x_p| a_{pq} l_q |x_q| + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p \\
&\leq -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq}(x_p^2 + x_q^2) + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p \\
&= -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_q^2 + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p \\
&= -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_i a_{qp} x_p^2 + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p \\
&= -\frac{1}{2}(2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) x_p^2 + \sum_{q=1}^{n} a_{pq}(x_p - x_q) x_p
\end{aligned}$$

$$= -(2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) V_p + \sum_{q=1}^{n} a_{pq}(-\frac{1}{2}(x_p - x_q)^2 + \frac{1}{2}(x_q^2 - x_p^2))$$

$$\leq -(2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) V_p + \sum_{q=1}^{n} a_{pq}[\frac{1}{2}(x_q^2 - x_p^2)], \quad t \neq t_k.$$

Let $F_{pq} = \frac{1}{2}(x_q^2 - x_p^2)$, along every directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G}, A)$ we have $\sum_{(s,r)\in E(\mathcal{C})} F_{rs}(x_r, x_s) = \sum_{(s,r)\in E(\mathcal{C})} \frac{1}{2}(x_s^2 - x_r^2) = 0$.

Let $b_p = 2w_p - \sum_{q=1}^{n} l_q |a_{pq}| - \sum_{q=1}^{n} l_p |a_{qp}|$, $V = \sum_{q=1}^{n} c_p V_p$. In view of Theorem 1, we obtain

$$D^\mu V(t,x) \leq -bV(t,x) \quad t > 0, \quad t \neq t_k,$$

where $b = \min\{b_1, b_2, \ldots, b_n\}$. Now we select $\omega = bV(t,x)$, then ω is a positive definite function. From lemma 2, we know that the trivial solution is globally stable when $t \neq t_k$.

When $t = t_k$, $\Delta x_p(t_k) = I_k(x_p(t_k)) = \delta_{pk} x_p(t_k)$. Besides, $\Delta x_p(t_k) = x_p(t_k^+) - x_p(t_k^-) = x_p(t_k^+) - x_p(t_k)$, then we can obtain

$$x_p(t_k^+) = (1 + \delta_{pk}) x_p(t_k).$$

Due to $-1 < \delta_{pk} < 0$, then $|x_p(t_k^+)| \leq |x_p(t_k)|$. In view of the third condition of this theorem, we derive

$$|x_p(t_k^+)| \leq |x_p(t_k)| < |x_p(t_{k-1}^+)| \leq |x_p(t_{k-1})|.$$

As a consequence, in each interval, we get $V(x_p(t_k^+)) < V(x_p(t_{k-1}))$. In view of $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$ as $k \to \infty$, then $V(x_p(t_k)) \to 0$ as $k \to \infty$.

This ends the proof. □

4. Example and Numerical Simulation

In this section, we study the following fractional impulsive system on a digraph with two vertices.

$$\begin{cases} D^{\mu}x_1 = -w_1x_1 + \sum_{j=1}^{2} a_{1j}f_j(x_j(t)) + \sum_{j=1}^{2} a_{1j}f_j(x_1(t) - x_j(t)), \quad t \geq 0, \quad t \neq 5, 10, 15 \ldots, \\ D^{\mu}x_2 = -w_2x_2 + \sum_{j=1}^{2} a_{2j}f_j(x_j(t)) + \sum_{j=1}^{2} a_{2j}f_j(x_2(t) - x_j(t)), \quad t \geq 0, \quad t \neq 5, 10, 15 \ldots, \\ \Delta x_i(t_k) = \delta_{ik}(x_i(t_k)), \\ x(t_k^-) = x(t_k), \quad t_k = 5, 10, 15, \ldots, \end{cases} \tag{6}$$

When $\mu = 0.92$, $\delta_{ik} = -\frac{1}{2}$, $w_1 = w_2 = 5$, $a_{11} = a_{22} = 4$, $a_{12} = a_{21} = 0$, $f_i(s) = \tanh(s)$. Obviously, we can take the Lipschitz constant $l_i = 1$. The initial conditions are assumed that $x_1(t)$ and $x_2(t)$ are $x_1(0) = 2$ and $x_2(0) = -2$. The simulation result for the above system is shown in Figure 3.

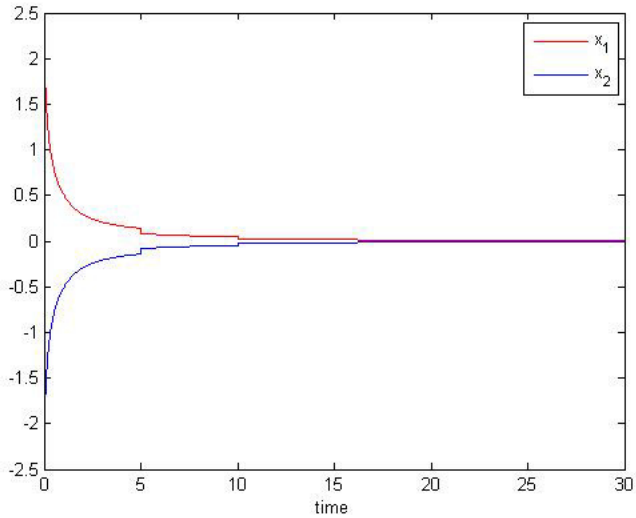

Figure 3. Dynamical behaviors of states $x_1(t)$ and $x_2(t)$ under above parameters.

When $\mu = \frac{1}{2}$, $\delta_{ik} = -\frac{1}{2}$, $w_1 = 15$, $w_2 = 14$, $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$, $f_i(s) = \tanh(s)$. Obviously, we can take the Lipschitz constant $l_i = 1$. The initial conditions are assumed that $x_1(t)$ and $x_2(t)$ are $x_1(0) = 1$ and $x_2(0) = -1$. The simulation result for the above system is shown in Figure 4.

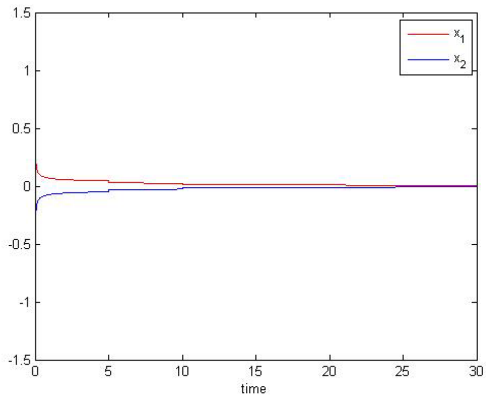

Figure 4. Dynamical behaviors of states $x_1(t)$ and $x_2(t)$ under above parameters.

5. Conclusions and Discussions

In this paper, we apply the graph theory and stability theory of dynamical system to study the stability of a coupled fractional-order system. This method can be extended to the other complex networks or multi-layer networks. In fact, many classical results for the integer-order system are not valid for the fractional-order system. We summarize the differences between fractional derivative and integer derivative as follows.

1. For the integer derivative, the sign of the first order derivative implies the monotonicity of a function. However, this is not valid for the fractional derivative (see [47]). This difference raises great difficulties for us to deal with the impulses at moment t_k. In order to ensure the stability of the trivial solution of (5), we have to add the condition $|x_p(t_k)| < |x_p(t_{k-1}^+)|$.
2. For the integer-order system $\frac{dx}{dt} = f(x,t)$, the first derivative $\frac{dV(x)}{dt} \leq -\omega(x) < 0$ implies the **asymptotically stability** in the sense of Lyapunov. However, this classical Lyapunov stability result is not valid for fractional-order system. The derivative $D^\alpha V(x) \leq -\omega(x) < 0$ does not imply the **asymptotically stability** in view of Lemma 2. It can only guarantee the stability.

Author Contributions: B.Z. carried out the computations in the proof. Y.X. conceived of the study, designed, drafted and edited the manuscript. L.Z. helped to make the figures. H.L. and L.G. participated in the discussion of the project. All authors read and approved the final manuscript.

Funding: This work was jointly supported by the National Natural Science Foundation of China under Grant (No. 11671176 and No. 11871251), Natural Science Foundation of Fujian Province under Grant (No. 2018J01001), start-up fund of Huaqiao University (Z16J00039).

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Ali, M.S.; Gunasekaran, N.; Cao, J. Sampled-data state estimation for neural networks with additive time-varying delays. *Acta Math. Sci.* **2019**, *39*, 195–213.
2. Ali, M.S.; Gunasekaran, N.; Agalya, R.; Joo, Y.H. Non-fragile synchronisation of mixed delayed neural networks with randomly occurring controller gain fluctuation. *Int. J. Syst. Sci.* **2018**, *49*, 3354–3364.
3. Balasubramaniam, P.; Chandran, R.; Jeeva, S. Synchronization of chaotic nonlinear continuous neural with time-varying delay. *Cogn. Neurodyn.* **2011**, *5*, 361–371. [CrossRef]
4. Bao, H.B.; Park, J.H.; Cao, J.D. Synchronization of fractional-order memristor-based neural networks with time delay. *Nonlinear Dynam.* **2015**, *158*, 1343–1354. [CrossRef]
5. Chen, H.; Sun, J.T. Stability analysis for coupled systems with time delay on networks. *Physica A* **2012**, *391*, 528–534. [CrossRef]
6. Chen, J.; Zeng, Z.; Jiang, P. Global Mittag-Lefflter stability and synchronization of memristor-based fractional-order neural networks. *Neural Netw.* **2014**, *51*, 1–8. [CrossRef]
7. Chen, L.; Huang, C.; Liu, H.; Xia, Y. Anti-synchronization of a class of chaotic systems with application to Lorenz system: a unified analysis of the integer order and fractional order. *Mathematics* **2019**, *7*, 559. [CrossRef]
8. Li, H.; Cao, J.; Hu, C.; Zhang, L.; Wang, Z. Global synchronization between two fractional-order complex networks with non-delayed and delayed coupling via hybrid impulsive control. *Neurocomputing* **2019**, *356*, 31–39. [CrossRef]
9. Li, H.; Wang, Y. On reachability and controllability of switched Boolean control networks. *Automatica* **2012**, *48*, 2917–2922. [CrossRef]
10. Liu, X.; Ho, D.W.; Yu, W.; Cao, J. A new switching design to finite-time stabilization of nonlinear systems with applications to neural networks. *Neural Netw.* **2014**, *57*, 94–102. [CrossRef]
11. Liu, X.; Su, H.; Chen, M. A switching approach to designing finite-time synchronizing controllers of couple neural networks. *Trans. Neural Netw. Learn. Syst.* **2016**, *27*, 471–482. [CrossRef]
12. Zhang, B.; Zhuang, J.; Liu, H.; Cao, J.; Xia, Y. Master-slave synchronization of a class of fractional-order Takagi-Sugeno fuzzy neural networks. *Adv. Differ. Equ.* **2018**, 473.10.1186/s13662-018-1918-y. [CrossRef]

13. Zhang, Y.; Zhuang J.; Xia, Y.; Bai, Y.; Cao, J.; Gu, L. Fixed-time synchronization of the impulsive memristor-based neural networks. *Commun. Nonlinear Sci. Numer. Simulat.* **2019**, *77*, 40–53. [CrossRef]
14. Zhang, W.; Wu, R.; Cao, J.; Alsaedi, A. Synchronization of a class of fractional-order neural networks with multiple time delays by comparison. *Nonlinear Anal. Model. Control* **2017**, *22*, 636–645. [CrossRef]
15. Zhuang, J.; Cao, J.; Tang, L.; Xia, Y.; Prec, M. Synchronization analysis for stochastic delayed multi-layer network with additive coupling. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**, *99*, 1–10. [CrossRef]
16. Anderson, J.; Kim, E.; Moradi, S. A fractional Fokker-Planck model for anomalous diffusion. *Phys. Plasmas* **2014**, *21*, 122109. [CrossRef]
17. Anderson, J.; Moradi, S.; Rafiq, T. Non-linear Langevin and Fractional Fokker-Planck equations for anomalous diffusion by Lévy Stable Processes. *Entropy* **2018**, *20*, 760. [CrossRef]
18. Del-Castillo-Negrete, D. Non-diffusive, non-local transport in fluids and plasmas. *Nonlinear Process. Geophys.* **2010**, *17*, 795–807. [CrossRef]
19. Da Fonseca, J.D.; del-Castillo-Negrete, D.; Caldas, I.L. Area-preserving maps models of gyroaveraged $E \times B$ Chaotic transport. *Phys. Plasmas* **2014**, *21*, 092310. [CrossRef]
20. Tarasov, V.E. Fractional Liouville and BBGKI equations. *J. Phys. Conf. Ser.* **2005**, *7*, 17–33. [CrossRef]
21. Tarasov, V.E. Fractional statistical mechanics. *Chaos* **2006**, *16*, 331081–331087. [CrossRef]
22. Zaslavsky, G.M. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.* **2002**, *371*, 461–580. [CrossRef]
23. Li, X.; Ho, D.W.; Cao, J. Finite-time stability and settling-time estimation of nonlinear impulsive systems. *Automatica* **2019**, *99*, 361–368. [CrossRef]
24. Li, X.; Song, S. Stabilization of delay systems: Delay-dependent impulsive control. *Trans. Autom. Control* **2017**, *62*, 406–411. [CrossRef]
25. Li, X.; Wu, J. Sufficient stability conditions of nonlinear differential systems under impulsive control with state-dependent delay. *Trans. Autom. Control* **2018**, *63*, 306–311. [CrossRef]
26. Lu, J.; Ho, D.W.; Cao, J.; Kurths, J. Exponential synchronization of linearly coupled neural networks with impulsive disturbances. *Trans. Neural Netw.* **2011**, *22*, 329–336. [CrossRef]
27. Lu, J.; Kurths, J.; Cao, J.; Mahdavi, N. Synchronization control for nonlinear stochastic dynamical networks: pinning impulsive strategy. *Trans. Neural Netw. Learn. Syst.* **2012**, *23*, 285–292.
28. Xia, Y.; Cao, J.; Cheng, S.S. Global exponential stability of delayed cellular neural networks with impulses. *Neurocomputing* **2007**, *70*, 2495–2501. [CrossRef]
29. Guo, H.; Li, M.Y.; Shuai, Z. Global dynamics of a general class of multistage models for infectious diseases. *SIAM J. Appl. Math.* **2012**, *72*, 261–279. [CrossRef]
30. Nieto, J.; Stamov, G.; Stamova, I. A fractional-order impulsive delay model of price fluctuations in commodity markets: Almost periodic solutions. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3811–3825. [CrossRef]
31. Stamov, G.; Stamova, I. Impulsive fractional functional differential systems and Lyapunov method for the existence of almost periodic solutions. *Rep. Math. Phys.* **2015**, *75*, 73–84. [CrossRef]
32. Stamov, G.; Stamova, I.; Cao, J. Uncertain impulsive functional differential systems of fractional order and almost periodicity. *J. Frankl. Inst.* **2018**, *355*, 5310–5323. [CrossRef]
33. Stamova, I. Global Mittag-Leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays. *Nonlinear Dynam.* **2014**, *77*, 1251–1260. [CrossRef]
34. Stamova, I. Global stability of impulsive fractional differential equations. *Appl. Math. Comput.* **2014**, *237*, 605–612. [CrossRef]
35. Wang, F.; Yang, Y.; Hu, M. Asymptotic stability of delayed fractional-order neural networks with impulsive effects. *Neurocomputing* **2015**, *154*, 239–244. [CrossRef]
36. Wang, J.; Zhou, Y.; Fečkan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **2012**, *64*, 3389–3405. [CrossRef]
37. Guo, H.; Li, M.Y.; Shuai, Z. A graph-theoretic approach to the method of global Lyapunov functions. *Proc. Am. Math. Soc.* **2008**, *136*, 2793–2802. [CrossRef]
38. Guo, B.; Xiao, Y.; Zhang, C. Graph-theoretic approach to exponential synchronization of coupled systems on networks with mixed time-varying delays. *J. Frankl. Inst.* **2017**, *354*, 5067–5090. [CrossRef]
39. Guo, B.; Xiao, Y.; Zhang, C. Synchronization analysis of stochastic coupled systems with time delay on networks by periodically intermittent control and graph-theoretic method. *Nonlinear Anal. Hybrid Syst.* **2018**, *30*, 118–133. [CrossRef]

40. Li, M.Y.; Shuai, Z.S. Global-stability problem for coupled systems of differential equations on networks. *J. Differ. Equ.* **2010**, *248*, 1–20. [CrossRef]
41. Suo, J.; Sun, J.; Zhang, Y. Stability analysis for impulsive coupled systems on networks. *Neurocomputing* **2013**, *99*, 172–177. [CrossRef]
42. Zhang, C.; Li, W.; Wang, K. Graph-theoretic method on exponential synchronization of stochastic coupled networks with Markovian switching. *Nonlinear Anal. Hybrid Syst.* **2015**, *15*, 37–51. [CrossRef]
43. Guo, Y.; Ding, X. Razumikhin method to global exponential stability for coupled neutral stochastic delayed systems on networks. *Math. Meth. Appl. Sci.* **2017**, *40*, 5490–5501. [CrossRef]
44. Li, X.; Yang, X.; Huang, T. Persistence of delayed cooperative models: Impulsive control method. *Appl. Math. Comput.* **2019**, *342*, 130–146. [CrossRef]
45. Li, X.; Shen, J.; Rakkiyappan, R. Persistent impulsive effects on stability of functional differential equations with finite or infinite delay. *Appl. Math. Comput.* **2018**, *329*, 14–22. [CrossRef]
46. Shen, J.; Chen, L.; Yuan, X. Lagrange stability for impulsive Duffing equations. *J. Differ. Equ.* **2019**, *266*, 6924–6962. [CrossRef]
47. Gallegos, J.A.; Duarte-Mermoud, M.A. On the Lyapunov theory for fractional order systems. *Appl. Math. Comput.* **2016**, *287*, 161–170. [CrossRef]
48. Lakshmikantham, V.; Leela, S.; Sambandham, M. Lyapunov theory for fractional differential equations. *Commun. Appl. Anal.* **2008**, *12*, 365–376.
49. Liang, J.; Tang, L.; Xia, Y.; Zhang, Y. Bifurcations and exact solutions for a class of mKdV equation with the conformable fractional derivative via dynamical system method. *Int. J. Bifur. Chaos* **2020**, to appear in the first issue.
50. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
51. Zhu, W.; Xia, Y.; Zhang, B.; Bai, Y. Exact traveling wave solutions and bifurcations of the time fractional differential equations with applications. *Int. J. Bifur. Chaos* **2019**, *29*, 1950041. [CrossRef]
52. Aguila-Camacho, N.; Duarte-Mermoud, M.A.; Gallegos, J.A. Lyapunov functions for fractional order systems. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2951–2957. [CrossRef]
53. Diestel, R. *Graph Theory*; Springer: New York, NY, USA, 2000.