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Article

Improved Oscillation Results for Functional Nonlinear Dynamic Equations of Second Order

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Abstract: In this paper, the functional dynamic equation of second order is studied on an arbitrary time scale under milder restrictions without the assumed conditions in the recent literature. The Nehari, Hille, and Ohriska type oscillation criteria of the equation are investigated. The presented results confirm that the study of the equation in this formula is superior to other previous studies. Some examples are addressed to demonstrate the finding.

Keywords: time scales; functional dynamic equations; second order; oscillation criteria

1. Introduction

In order to combine continuous and discrete analysis, the theory of dynamic equations on time scales was proposed by Stefan Hilger in [1]. There are different types of time scales applied in many applications (see [2]). The cases when the time scale $\mathbb{T}$ as an arbitrary closed subset is equal to the reals or to the integers represent the classical theories of differential and of difference equations. The theory of dynamic equations includes the classical theories for the differential equations and difference equations cases and other cases in between these classical cases. That is, we are eligible to consider the q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0 \text{ for } q > 1\}$ which has significant applications in quantum theory (see [3]) and different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ (the set of the harmonic numbers) can also be applied. For more details of time scales calculus, see [2,4,5]. The study of nonlinear dynamic equations is considered in this work because these equations arise in various real-world problems like the turbulent flow of a polytrophic gas in a porous medium, non-Newtonian fluid theory, and in the study of $p-$Laplace equations. Therefore, we are interested in the oscillatory behavior of the nonlinear functional dynamic equation of second order with deviating arguments

$$\left[a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\right]^\Delta + q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right) = 0 \tag{1}$$

on an above-unbounded time scale $\mathbb{T}$, where $\varphi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$; a and q are positive rd-continuous functions on $\mathbb{T}$ such that

$$\int^\infty \frac{\Delta\varkappa}{a^{\frac{1}{\gamma}}(\varkappa)} = \infty; \tag{2}$$

and $\eta : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\lim_{\zeta\to\infty} \eta(\zeta) = \infty$.

By a solution of Equation (1) we mean a nontrivial real-valued function $z \in C^1_{rd}[\zeta_z, \infty)_{\mathbb{T}}$ for some $\zeta_z \geq \zeta_0$ with $\zeta_0 \in \mathbb{T}$ such that $z^\Delta, a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right) \in C^1_{rd}[\zeta_z, \infty)_{\mathbb{T}}$ and $z(\zeta)$ satisfies Equation (1) on $[\zeta_z, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. It should be mentioned that in a particular case when $\mathbb{T} = \mathbb{R}$ then

$$\sigma(\zeta) = \zeta,\ \mu(\zeta) = 0,\ g^\Delta(\zeta) = g'(\zeta),\ \int_a^b g(\zeta)\Delta\zeta = \int_a^b g(\zeta)d\zeta,$$

and (1) turns as the nonlinear functional differential equation

$$\left[a(\zeta)\varphi_\gamma\left(z'(\zeta)\right)\right]' + q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right) = 0. \tag{3}$$

The oscillation properties of Equation (3) and special cases were investigated by Nehari [6], Fite [7], Hille [8], Wong [9], Erbe [10], and Ohriska [11] as follows: The oscillatory behavior of the linear differential equation of second order

$$z''(\zeta) + q(\zeta)z(\zeta) = 0, \tag{4}$$

is investigated in Nehari [6] and showed that if

$$\liminf_{\zeta\to\infty} \frac{1}{\zeta}\int_{\zeta_0}^{\zeta} \varkappa^2 q(\varkappa)\mathrm{d}\varkappa > \frac{1}{4}, \tag{5}$$

then all solutions of (4) are oscillatory. Fite [7] proved that if

$$\int_{\zeta_0}^{\infty} q(\varkappa)\mathrm{d}\varkappa = \infty, \tag{6}$$

then all solutions of Equation (4) are oscillatory. Hille [8] developed the condition (6) and illustrated that if

$$\liminf_{\zeta\to\infty} \zeta\int_{\zeta}^{\infty} q(\varkappa)\mathrm{d}\varkappa > \frac{1}{4}, \tag{7}$$

then all solutions of Equation (4) are oscillatory. For the delay differential equation

$$z''(\zeta) + q(\zeta)z(\eta(\zeta)) = 0, \tag{8}$$

the Hille-type condition (7) is generalized by Wong [9], where $\eta(\zeta) \geq \gamma\zeta$ with $0 < \gamma < 1$, and showed that if

$$\liminf_{\zeta\to\infty} \zeta\int_{\zeta}^{\infty} q(\varkappa)\mathrm{d}\varkappa > \frac{1}{4\gamma}, \tag{9}$$

then all solutions of (8) are oscillatory. Erbe [10] enhanced the condition (9) and examined that if

$$\liminf_{\zeta\to\infty} \zeta\int_{\zeta}^{\infty} q(\varkappa)\frac{\eta(\varkappa)}{\varkappa}\mathrm{d}\varkappa > \frac{1}{4}, \tag{10}$$

then all solutions of (8) are oscillatory where $\eta(\zeta) \leq \zeta$. Ohriska [11] proved that, if

$$\limsup_{\zeta\to\infty} \zeta\int_{\zeta}^{\infty} q(\varkappa)\frac{\eta(\varkappa)}{\varkappa}\mathrm{d}\varkappa > 1, \tag{11}$$

then all solutions of (8) are oscillatory.

When $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\zeta) = \zeta + 1,\ \mu(\zeta) = 1,\ g^\Delta(\zeta) = \Delta g(\zeta),\ \int_a^b g(\zeta)\Delta\zeta = \sum_{\zeta=a}^{b-1} g(\zeta),$$

and (1) turns as the nonlinear functional difference equation

$$\Delta\left[a(\zeta)\varphi_\gamma\left(\Delta z(\zeta)\right)\right]+q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right)=0. \tag{12}$$

The oscillation of Equation (12) when $a(\zeta)=1$, $\eta(\zeta)=\zeta$, and $\gamma=\beta$ is the quotient of odd positive integers was elaborated by Thandapani et al. [12] in which $q(\zeta)$ is a positive sequence and showed that every solution of (12) is oscillatory, if

$$\sum_{k=k_0}^{\infty} q(k)=\infty.$$

We will examine that our results not only unite some of the known oscillation results for differential and difference equations but they also can be applied on other cases in which the oscillatory behavior of solutions for these equations on various types of time scales was not known. Note that, if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then

$$\sigma(\zeta)=\zeta+h,\ \mu(\zeta)=h,\ z^{\Delta}(\zeta)=\Delta_h z(\zeta)=\frac{z(\zeta+h)-z(\zeta)}{h},$$

$$\int_a^b g(\zeta)\Delta\zeta=\sum_{k=0}^{\frac{b-a-h}{h}} g(a+kh)h,$$

and (1) turns as the nonlinear functional difference equation

$$\Delta_h\left[a(\zeta)\varphi_\gamma\left(\Delta_h z(\zeta)\right)\right]+q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right)=0. \tag{13}$$

If

$$\mathbb{T}=q^{\mathbb{N}_0}=\{\zeta:\zeta=q^k,\ k\in\mathbb{N}_0,\ q>1\},$$

then

$$\sigma(\zeta)=q\,\zeta,\ \mu(\zeta)=(q-1)\zeta,\ z^{\Delta}(\zeta)=\Delta_q z(\zeta)=(z(q\,\zeta)-z(\zeta))/(q-1)\,\zeta,$$

$$\int_{\zeta_0}^{\infty} g(\zeta)\Delta\zeta=\sum_{k=n_0}^{\infty} g(q^k)\mu(q^k),$$

where $t_0=q^{n_0}$, and (1) turns as the second order $q-$nonlinear difference equation

$$\Delta_q\left[a(\zeta)\varphi_\gamma\left(\Delta_q z(\zeta)\right)\right]+q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right)=0. \tag{14}$$

If

$$\mathbb{T}=\mathbb{N}_0^2:=\{n^2:n\in\mathbb{N}_0\},$$

then

$$\sigma(\zeta)=(\sqrt{\zeta}+1)^2,\ \mu(\zeta)=1+2\sqrt{\zeta},\ \Delta_N z(\zeta)=\frac{z((\sqrt{\zeta}+1)^2)-z(\zeta)}{1+2\sqrt{\zeta}},$$

and (1) turns as the second order nonlinear difference equation

$$\Delta_N\left[a(\zeta)\varphi_\gamma\left(\Delta_N z(\zeta)\right)\right]+q(\zeta)\varphi_\beta\left(z(\eta(\zeta))\right)=0. \tag{15}$$

If $\mathbb{T}=\{H_n:n\in\mathbb{N}_0\}$ where H_n is the harmonic numbers defined by

$$H_0=0,\ H_n=\sum_{k=1}^{n}\frac{1}{k},\ n\in\mathbb{N},$$

then

$$\sigma(H_n) = H_{n+1},\ \mu(H_n) = \frac{1}{n+1},\ z^{\Delta}(t) = \Delta_{H_n} z(H_n) = (n+1)\Delta z(H_n),$$

and (1) turns as the second order nonlinear harmonic difference equation

$$\Delta_{H_n} \left[a(H_n)\varphi_{\gamma}\left(\Delta_{H_n} z(H_n)\right)\right] + q(H_n)\varphi_{\beta}\left(z(\eta(H_n))\right) = 0. \tag{16}$$

For dynamic equations, Erbe et al. in [13,14] expanded the Hille and Nehari oscillation criteria to the half-linear delay dynamic equation of second order

$$(a(\zeta)(z^{\Delta}(\zeta))^{\gamma})^{\Delta} + q(\zeta)z^{\gamma}(\eta(\zeta)) = 0, \tag{17}$$

where γ is a quotient of odd positive integers,

$$\eta(\zeta) \le \zeta,\ a^{\Delta}(\zeta) \ge 0,\ \int_{\zeta_0}^{\infty} \eta^{\gamma}(\zeta)q(\zeta)\Delta\zeta = \infty. \tag{18}$$

The authors showed that if either of the following conditions holds

$$\liminf_{\zeta\to\infty} \zeta^{\gamma} \int_{\sigma(\zeta)}^{\infty} q(\varkappa)\left(\frac{\eta(\varkappa)}{\sigma(\varkappa)}\right)^{\gamma} \Delta\varkappa > \frac{\gamma^{\gamma}}{l\gamma^2(\gamma+1)^{\gamma+1}}, \tag{19}$$

or

$$\liminf_{\zeta\to\infty} \zeta^{\gamma} \int_{\sigma(\zeta)}^{\infty} q(\varkappa)\left(\frac{\eta(\varkappa)}{\sigma(\varkappa)}\right)^{\gamma} \Delta\varkappa + \liminf_{\zeta\to\infty} \frac{1}{\zeta}\int_{\zeta_0}^{\zeta} \varkappa^{\gamma+1} q(\varkappa)\left(\frac{\eta(\varkappa)}{\sigma(\varkappa)}\right)^{\gamma} \Delta\varkappa > \frac{1}{l^{\gamma(\gamma+1)}},$$

where $l := \liminf_{\zeta\to\infty} \dfrac{\zeta}{\sigma(\zeta)}$, then all solutions of (17) are oscillatory. We refer the reader to related results [15–35] and the references cited therein.

A natural question now is: Do the oscillation criteria (5), (6), (7) and (11) for the differential equations of second order by Nehari, Fite, Hille and Ohriska extend to the nonlinear dynamic equation of second order (1) without the restrictive condition (18) in both cases $\eta(\zeta) \le \zeta$ and $\eta(\zeta) \ge \zeta$, and when $\beta \ge \gamma$ and $\beta \le \gamma$.

The aim of this paper is to propose an obvious answer to the above question. We will establish Nehari, Hille and Ohriska type oscillation criteria for (1) without imposing the restrictive condition (18), which generalize and improve the aforementioned results in the literature.

2. Oscillation Criteria of (1) when $\beta \ge \gamma$

In the subsequent results, we will use the subsequent notations

$$A(\zeta) := \int_{\zeta_0}^{\zeta} \frac{\Delta\varkappa}{a^{\frac{1}{\gamma}}(\varkappa)} \quad \text{and} \quad l := \liminf_{\zeta\to\infty} \frac{A(\zeta)}{A(\sigma(\zeta))} \le 1,$$

and

$$\phi(\zeta) := \begin{cases} 1, & \eta(\zeta) \ge \zeta, \\ \left(\dfrac{A(\eta(\zeta))}{A(\zeta)}\right)^{\beta}, & \eta(\zeta) \le \zeta. \end{cases}$$

Furthermore, $l > 0$ is assuming in the next results.

First, we derive Nehari type to the nonlinear dynamic equation of second order (1).

Theorem 1. *Let (2) holds, and*

$$\begin{aligned}&\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_T^{\zeta}A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa>\frac{1}{l^{\gamma(\gamma+1)}}\left(1-\frac{l^{\gamma}}{\gamma l^{\gamma}+1}\right),\quad 0<\gamma\le 1,\\&\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_T^{\zeta}A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa>\frac{\gamma}{l^{\gamma(\gamma+1)}(\gamma+l^{\gamma})},\qquad \gamma\ge 1,\end{aligned}\tag{20}$$

for enough large $T\in[\zeta_0,\infty)_{\mathbb{T}}$. Then all solutions of Equation (1) are oscillatory.

Proof. Assume $z(t)$ is a nonoscillatory solution of Equation (1) on $[\zeta_0,\infty)_{\mathbb{T}}$. Thus, without loss of generality, let $z(\zeta)>0$ and $z(\eta(\zeta))>0$ on $[\zeta_0,\infty)_{\mathbb{T}}$. Since $q\in C_{\mathrm{rd}}([\zeta_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and then

$$\left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta}<0\quad\text{for }\zeta\ge\zeta_0.$$

Hence $z^{\Delta}(\zeta)>0$, otherwise, it leads to a contradiction. Define

$$w(\zeta):=\frac{a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)}{z^{\gamma}(\zeta)}.$$

Using the product and quotient rules, we reach

$$\begin{aligned}w^{\Delta}(\zeta)&=\left(\frac{a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)}{z^{\gamma}(\zeta)}\right)^{\Delta}\\&=\frac{1}{z^{\gamma}(\zeta)}\left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta}\\&\quad+\left(\frac{1}{z^{\gamma}(\zeta)}\right)^{\Delta}\left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\sigma}\\&=\frac{\left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta}}{z^{\gamma}(\zeta)}-\frac{(z^{\gamma}(\zeta))^{\Delta}}{z^{\gamma}(\zeta)z^{\gamma}(\sigma(\zeta))}\left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\sigma}.\end{aligned}\tag{21}$$

From (1) and the definition of $w(\zeta)$, we have

$$w^{\Delta}(\zeta)=-\left(\frac{z(\eta(\zeta))}{z(\zeta)}\right)^{\beta}z^{\beta-\gamma}(\zeta)\,q(\zeta)-\frac{(z^{\gamma}(\zeta))^{\Delta}}{z^{\gamma}(\zeta)}w(\sigma(\zeta)).$$

Since $z^{\Delta}>0$, then $z(\zeta)\ge z(\zeta_0)$ for $\zeta\ge\zeta_0$ and so

$$z^{\beta-\gamma}(\zeta)\ge z^{\beta-\gamma}(\zeta_0)=:k>0\quad\text{for }\zeta\ge\zeta_0.$$

Therefore,

$$w^{\Delta}(\zeta)\le -k\left(\frac{z(\eta(\zeta))}{z(\zeta)}\right)^{\beta}q(\zeta)-\frac{(z^{\gamma}(\zeta))^{\Delta}}{z^{\gamma}(\zeta)}w(\sigma(\zeta)).$$

Let $\zeta\in[\zeta_0,\infty)_{\mathbb{T}}$ be fixed. If $\eta(\zeta)\ge\zeta$, then $z(\eta(\zeta))\ge z(\zeta)$ by the fact that $z^{\Delta}>0$. Now the case $\eta(\zeta)\le\zeta$ is considered. Since $\left(a\,\varphi_{\gamma}\left(z^{\Delta}\right)\right)^{\Delta}<0$ on $[\zeta_0,\infty)_{\mathbb{T}}$, we achieve

$$\begin{aligned}z(\zeta)&\ge z(\zeta)-z(\zeta_1)=\int_{\zeta_0}^{\zeta}z^{\Delta}(\varkappa)\Delta\varkappa\\&\ge a^{\frac{1}{\gamma}}(\zeta)z^{\Delta}(\zeta)\int_{\zeta_0}^{\zeta}\frac{\Delta\varkappa}{a^{\frac{1}{\gamma}}(\varkappa)}\\&=a^{\frac{1}{\gamma}}(\zeta)z^{\Delta}(\zeta)A(\zeta).\end{aligned}$$

Therefore

$$\begin{aligned}
\left[\frac{z(\zeta)}{A(\zeta)}\right]^{\Delta} &= \frac{A(\zeta)z^{\Delta}(\zeta) - z(\zeta)a^{-\frac{1}{\gamma}}(\zeta)}{A(\zeta)A^{\sigma}(\zeta)} \\
&= \frac{a^{-\frac{1}{\gamma}}(\zeta)}{A(\zeta)A^{\sigma}(\zeta)}\left(a^{\frac{1}{\gamma}}(\zeta)z^{\Delta}(\zeta)A(\zeta) - z(\zeta)\right) \\
&\leq 0, \quad \zeta \in (\zeta_0, \infty)_{\mathbb{T}}.
\end{aligned}$$

So there exists a $\zeta_1 \in (\zeta_0, \infty)_{\mathbb{T}}$ such that $\eta(\zeta) \in (\zeta_0, \infty)_{\mathbb{T}}$ for $\zeta \geq \zeta_1$ and so

$$\frac{z(\eta(\zeta))}{z(\zeta)} \geq \frac{A(\eta(\zeta))}{A(\zeta)} \quad \text{for } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}.$$

In both cases and from the definition of $\phi(\zeta)$ we have that

$$\left(\frac{z\left(\eta(\zeta)\right)}{z(\zeta)}\right)^{\beta} \geq \phi(\zeta), \tag{22}$$

and so

$$w^{\Delta}(\zeta) \leq -k\,\phi(\zeta)q(\zeta) - \frac{(z^{\gamma}(\zeta))^{\Delta}}{z^{\gamma}(\zeta)} w\left(\sigma(\zeta)\right), \quad \zeta \in [\zeta_1, \infty)_{\mathbb{T}}. \tag{23}$$

Then by using the Pötzsche chain rule ([2], Theorem 1.90), we get that

$$\begin{aligned}
(z^{\gamma}(\zeta))^{\Delta} &= \gamma\left(\int_0^1 \left[z(\zeta) + h\mu(\zeta)z^{\Delta}(\zeta)\right]^{\gamma-1} \mathrm{d}h\right) z^{\Delta}(\zeta) \\
&= \gamma\left(\int_0^1 \left[(1-h)\,z(\zeta) + hz\left(\sigma(\zeta)\right)\right]^{\gamma-1} \mathrm{d}h\right) z^{\Delta}(\zeta) \\
&\geq \begin{cases} \gamma z^{\gamma-1}\left(\sigma(\zeta)\right) z^{\Delta}(\zeta), & 0 < \gamma \leq 1, \\ \gamma z^{\gamma-1}(\zeta) z^{\Delta}(\zeta), & \gamma \geq 1. \end{cases}
\end{aligned}$$

If $0 < \gamma \leq 1$, then

$$w^{\Delta}(\zeta) < -k\,\phi(\zeta)q(\zeta) - \gamma\frac{z^{\Delta}(\zeta)}{z\left(\sigma(\zeta)\right)}\left(\frac{z\left(\sigma(\zeta)\right)}{z(\zeta)}\right)^{\gamma} w\left(\sigma(\zeta)\right);$$

and if $\gamma \geq 1$, then

$$w^{\Delta}(\zeta) \leq -k\,\phi(\zeta)q(\zeta) - \gamma\frac{z^{\Delta}(\zeta)}{z\left(\sigma(\zeta)\right)}\frac{z\left(\sigma(\zeta)\right)}{z(\zeta)} w\left(\sigma(\zeta)\right).$$

Note that $z^{\Delta} > 0$ and $\left(a\,\varphi_{\gamma}\left(z^{\Delta}\right)\right)^{\Delta} < 0$ on $[\zeta_1, \infty)_{\mathbb{T}}$, we see for $\gamma > 0$,

$$\begin{aligned}
w^{\Delta}(\zeta) &\leq -k\,\phi(\zeta)q(\zeta) - \gamma\frac{z^{\Delta}(\zeta)}{z\left(\sigma(\zeta)\right)} w\left(\sigma(\zeta)\right) \\
&\leq -k\,\phi(\zeta)q(\zeta) - \gamma a^{-\frac{1}{\gamma}}(\zeta) w^{1+\frac{1}{\gamma}}\left(\sigma(\zeta)\right), \quad \zeta \in [\zeta_1, \infty)_{\mathbb{T}}.
\end{aligned} \tag{24}$$

Multiplying both sides of (24) by $A^{\gamma+1}(\zeta)$ and integrating from ζ_2 to $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned}
\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)w^{\Delta}(\varkappa)\Delta\varkappa \leq\; & -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\
& -\gamma\int_{\zeta_2}^{\zeta} a^{-\frac{1}{\gamma}}(\varkappa)\left(A^{\gamma}(\varkappa)\,w\left(\sigma(\varkappa)\right)\right)^{\frac{\gamma+1}{\gamma}}\Delta\varkappa.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned} A^{\gamma+1}(\zeta)w(\zeta) &\leq A^{\gamma+1}(\zeta_2)w(\zeta_2) + \int_{\zeta_2}^{\zeta} \left(A^{\gamma+1}(\varkappa)\right)^{\Delta} w\left(\sigma(\varkappa)\right)\Delta\varkappa \\ &\quad -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad -\gamma\int_{\zeta_2}^{\zeta} a^{-\frac{1}{\gamma}}(\varkappa)\left(A^{\gamma}(\varkappa)w\left(\sigma(\varkappa)\right)\right)^{\frac{\gamma+1}{\gamma}}\Delta\varkappa. \end{aligned}$$

Using the Pötzsche chain rule, we arrive

$$\begin{aligned} \left(A^{\gamma+1}(\varkappa)\right)^{\Delta} &= (\gamma+1)\int_0^1 [A(\varkappa)+h\mu(\varkappa)A^{\Delta}(\varkappa)]^{\gamma}\mathrm{d}h\,\frac{1}{a^{1/\gamma}(\varkappa)} \\ &= (\gamma+1)\int_0^1 [(1-h)\,A(\varkappa)+hA\left(\sigma(\varkappa)\right)]^{\gamma}\mathrm{d}h\,\frac{1}{a^{1/\gamma}(\varkappa)} \\ &\leq (\gamma+1)\,\frac{A^{\gamma}\left(\sigma(\varkappa)\right)}{a^{1/\gamma}(\varkappa)}. \end{aligned} \tag{25}$$

Hence

$$\begin{aligned} A^{\gamma+1}(\zeta)w(\zeta) &\leq A^{\gamma+1}(\zeta_2)w(\zeta_2) - \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad +(\gamma+1)\int_{\zeta_2}^{\zeta}\frac{1}{a^{1/\gamma}(\varkappa)}\left[\frac{A\left(\sigma(\varkappa)\right)}{A(\varkappa)}\right]^{\gamma} A^{\gamma}(\varkappa)w\left(\sigma(\varkappa)\right)\Delta\varkappa \\ &\quad -\gamma\int_{\zeta_2}^{\zeta}\frac{1}{a^{1/\gamma}(\varkappa)}\left(A^{\gamma}(\varkappa)w\left(\sigma(\varkappa)\right)\right)^{\frac{\gamma+1}{\gamma}}\Delta\varkappa. \end{aligned}$$

It follows that $w^{\Delta}(\zeta) \leq 0$ on $[\zeta_1,\infty)_{\mathbb{T}}$. Let $\varepsilon > 0$, then we choose $\zeta_2 \in [\zeta_1,\infty)_{\mathbb{T}}$, enough large, so for $\zeta \in [\zeta_2,\infty)_{\mathbb{T}}$,

$$A^{\gamma}(\zeta)\,w\left(\sigma(\zeta)\right) \geq a_* - \varepsilon, \tag{26}$$

and

$$\frac{A(\zeta)}{A\left(\sigma(\zeta)\right)} \geq l - \varepsilon, \tag{27}$$

where a_* is defined by

$$a_* := \liminf_{\zeta\to\infty} A^{\gamma}(\zeta)w\left(\sigma(\zeta)\right) \leq 1. \tag{28}$$

By (27), we then get that

$$A^{\gamma+1}(\zeta)w(\zeta) \leq A^{\gamma+1}(\zeta_2)w(\zeta_2) - k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa$$

$$+\int_{\zeta_2}^{\zeta}\frac{1}{a^{1/\gamma}(\varkappa)}\left[\frac{\gamma+1}{(l-\varepsilon)^{\gamma}}A^{\gamma}(\varkappa)\,w\left(\sigma(\varkappa)\right) - \gamma\left(A^{\gamma}(\varkappa)\,w\left(\sigma(\varkappa)\right)\right)^{\frac{\gamma+1}{\gamma}}\right]\Delta\varkappa.$$

Using the inequality

$$Yu - Xu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{Y^{\gamma+1}}{X^{\gamma}} \tag{29}$$

with $X=\gamma$, $Y=\dfrac{\gamma+1}{(l-\varepsilon)^{\gamma}}$ and $u = A^{\gamma}(\varkappa)\,w\left(\sigma(\varkappa)\right)$, we get

$$\begin{aligned} A^{\gamma+1}(\zeta)w(\zeta) &\leq A^{\gamma+1}(\zeta_2)w(\zeta_2) - k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}\left[A(\zeta) - A(\zeta_2)\right]. \end{aligned}$$

Dividing both sides by $A(\zeta)$, we obtain

$$\begin{aligned} A^{\gamma}(\zeta)w(\zeta) &\leq \frac{A^{\gamma+1}(\zeta_2)w(\zeta_2)}{A(\zeta)} - \frac{k}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}\left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right]. \end{aligned}$$

Since $w^{\sigma}(\zeta) \leq w(\zeta)$ we get

$$\begin{aligned} A^{\gamma}(\zeta)w\left(\sigma(\zeta)\right) &\leq \frac{A^{\gamma+1}(\zeta_2)w(\zeta_2)}{A(\zeta)} - \frac{k}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}\left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right]. \end{aligned}$$

Taking the lim sup of both sides as $\zeta \to \infty$ we get

$$A_* \leq -\liminf_{\zeta\to\infty} \frac{k}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}.$$

where

$$A_* := \limsup_{\zeta\to\infty} A^{\gamma}(\zeta)w\left(\sigma(\zeta)\right).$$

Since $k,\ \varepsilon > 0$ are arbitrary constants, we obtain

$$A_* \leq -\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa + \frac{1}{l^{\gamma(\gamma+1)}}. \tag{30}$$

Now, multiplying both sides of (24) by $A^{\gamma+1}(\zeta)$, we get

$$\begin{aligned} A^{\gamma+1}(\zeta)\, w^{\Delta}(\zeta) &\leq -k\, A^{\gamma+1}(\zeta)\, \phi(\zeta)q(\zeta) - \gamma a^{-1/\gamma}(\zeta) A^{\gamma+1}(\zeta)\, w^{1+\frac{1}{\gamma}}\left(\sigma(\zeta)\right) \\ &= -A^{\gamma+1}(\zeta)\, \phi(\zeta)q(\zeta) \\ &\quad -\gamma a^{-1/\gamma}(\zeta) A^{\gamma}(\zeta)\, w\left(\sigma(\zeta)\right)\ A(\zeta)\, w^{\frac{1}{\gamma}}\left(\sigma(\zeta)\right). \end{aligned}$$

Using (26) gives

$$A^{\gamma+1}(\zeta)\, w^{\Delta}(\zeta) \leq -k\, A^{\gamma+1}(\zeta)\, \phi(\zeta)q(\zeta) - \vartheta a^{-1/\gamma}(\zeta), \quad \zeta \in [\zeta_2, \infty)_{\mathbb{T}}, \tag{31}$$

where $\vartheta = \gamma\left(a_* - \varepsilon\right)^{1+\frac{1}{\gamma}}$. Integrating the inequality (31) from ζ_2 to $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$, we get

$$\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, w^{\Delta}(\varkappa)\Delta\varkappa \leq -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta\int_{\zeta_2}^{\zeta} a^{-1/\gamma}(\varkappa)\Delta\varkappa.$$

Using integrating by parts, we get

$$\begin{aligned} A^{\gamma+1}(\zeta)\, w(\zeta) \quad &\leq \quad A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) + \int_{\zeta_2}^{\zeta} \left[A^{\gamma+1}(\varkappa)\right]^{\Delta} w(\sigma(\varkappa))\, \Delta\varkappa \\ &\quad -k \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa - \vartheta\left[A(\zeta) - A(\zeta_2)\right]. \end{aligned} \tag{32}$$

We consider the forthcoming two cases:

(I) When $0 < \gamma \leq 1$. Using the product rule, we have

$$\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} = \left[A^{\gamma}(\varkappa)\, A(\varkappa)\right]^{\Delta} = \left[A^{\gamma}(\varkappa)\right]^{\Delta} A(\varkappa) + A^{\gamma}(\sigma(\varkappa))\, A^{\Delta}(\varkappa).$$

Again use the Pötzsche chain rule, we get

$$\begin{aligned} (A^{\gamma}(\varkappa))^{\Delta} \quad &= \quad \gamma\left(\int_0^1 \left[A(\varkappa) + h\mu(\varkappa) A^{\Delta}(\varkappa)\right]^{\gamma-1} \mathrm{d}h\right) A^{\Delta}(\varkappa) \\ &= \quad \gamma\left(\int_0^1 \left[(1-h)\, A(\varkappa) + hA(\sigma(\varkappa))\right]^{\gamma-1} \mathrm{d}h\right) A^{\Delta}(\varkappa) \\ &\leq \quad \gamma A^{\gamma-1}(\varkappa)\, A^{\Delta}(\varkappa). \end{aligned}$$

Then

$$\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} \leq \left(\gamma A^{\gamma}(\varkappa) + A^{\gamma}(\sigma(\varkappa))\right) A^{\Delta}(\varkappa).$$

and so

$$\begin{aligned} A^{\gamma+1}(\zeta)\, w(\zeta) \quad &\leq \quad A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) \\ &\quad + \int_{\zeta_2}^{\zeta} \left(\gamma A^{\gamma}(\varkappa) + A^{\gamma}(\sigma(\varkappa))\right) A^{\Delta}(\varkappa)\, w(\sigma(\varkappa))\, \Delta\varkappa \\ &\quad -k \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa - \vartheta\left[A(\zeta) - A(\zeta_2)\right] \\ &= \quad A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) \\ &\quad + \int_{\zeta_2}^{\zeta} \left(\gamma + \left[\frac{A(\sigma(\varkappa))}{A(\varkappa)}\right]^{\gamma}\right) A^{\Delta}(\varkappa)\ \ A^{\gamma}(\varkappa)\, w(\sigma(\varkappa))\, \Delta\varkappa \\ &\quad -k \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa) - \vartheta\left[A(\zeta) - A(\zeta_2)\right] \\ &\leq \quad A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) + \left[\gamma + \frac{1}{(l-\varepsilon)^{\gamma}}\right](A_* + \varepsilon)\left[A(\zeta) - A(\zeta_2)\right] \\ &\quad -k \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa) - \vartheta\left[A(\zeta) - A(\zeta_2)\right]. \end{aligned}$$

Dividing both sides by $A(\zeta)$, we have

$$\begin{aligned} A^{\gamma}(\zeta)\, w(\sigma(\zeta)) \quad &\leq \quad A^{\gamma}(\zeta)\, w(\zeta) \leq \frac{A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2)}{A(\zeta)} \\ &\quad + \left[\gamma + \frac{1}{(l-\varepsilon)^{\gamma}}\right](A_* + \varepsilon)\left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right] \\ &\quad - \frac{k}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa - \vartheta\left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right]. \end{aligned}$$

Taking the lim sup of both sides as $\zeta \to \infty$ and using (2), we get

$$A_{*} \leq \left[\gamma + \frac{1}{(l-\varepsilon)^{\gamma}}\right](A_{*}+\varepsilon) - \liminf_{\zeta\to\infty} \frac{k}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa) q(\varkappa)\Delta\varkappa - \vartheta.$$

Since k and $\varepsilon > 0$ are arbitrary constants, we achieve the demanded inequality

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa) q(\varkappa)\Delta\varkappa \leq A_{*}\left[\gamma - 1 + \frac{1}{l^{\gamma}}\right] - \gamma a_{*}^{1+\frac{1}{\gamma}}. \tag{33}$$

From (30) and (33), we obtain

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa) q(\varkappa)\Delta\varkappa \leq \frac{1}{l^{\gamma(\gamma+1)}}\left(1 - \frac{l^{\gamma}}{\gamma l^{\gamma}+1}\right),$$

which contradicts the condition (20) if $0 < \gamma \leq 1$.

(II) When $\gamma \geq 1$. Using the product rule, we have

$$\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} = \left[A^{\gamma}(\varkappa)\,A(\varkappa)\right]^{\Delta} = \left[A^{\gamma}(\varkappa)\right]^{\Delta} A(\sigma(\varkappa)) + A^{\gamma}(\varkappa)\,A^{\Delta}(\varkappa).$$

Again by the Pötzsche chain rule we obtain

$$\begin{aligned} \left(A^{\gamma}(\varkappa)\right)^{\Delta} &= \gamma\left(\int_0^1 \left[A(\varkappa) + h\mu(\varkappa)A^{\Delta}(\varkappa)\right]^{\gamma-1} \mathrm{d}h\right) A^{\Delta}(\varkappa) \\ &= \gamma\left(\int_0^1 \left[(1-h)\,A(\varkappa) + hA(\sigma(\varkappa))\right]^{\gamma-1} \mathrm{d}h\right) A^{\Delta}(\varkappa) \\ &\leq \gamma A^{\gamma-1}(\sigma(\varkappa))\,A^{\Delta}(\varkappa). \end{aligned}$$

Then

$$\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} \leq \left(\gamma A^{\gamma}(\sigma(\varkappa)) + A^{\gamma}(\varkappa)\right) A^{\Delta}(\varkappa).$$

and so

$$\begin{aligned} A^{\gamma+1}(\zeta)\,w(\zeta) &\leq A^{\gamma+1}(\zeta_2)\,w^{\Delta}(\zeta_2) \\ &\quad + \int_{\zeta_2}^{\zeta} \left(\gamma A^{\gamma}(\sigma(\varkappa)) + A^{\gamma}(\varkappa)\right) A^{\Delta}(\varkappa)\,w(\sigma(\varkappa))\,\Delta\varkappa \\ &\quad - k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa) q(\varkappa)\Delta\varkappa - \vartheta\left[A(\zeta) - A(\zeta_2)\right] \\ &= A^{\gamma+1}(\zeta_2)\,w^{\Delta}(\zeta_2) \\ &\quad + \int_{\zeta_2}^{\zeta} \left(\gamma\left[\frac{A(\sigma(\varkappa))}{A(\varkappa)}\right]^{\gamma} + 1\right) A^{\Delta}(\varkappa)\;A^{\gamma}(\varkappa)\,w(\sigma(\varkappa))\,\Delta\varkappa \\ &\quad - k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa) q(\varkappa)\Delta\varkappa) - \vartheta\left[A(\zeta) - A(\zeta_2)\right] \\ &\leq A^{\gamma+1}(\zeta_2)\,w^{\Delta}(\zeta_2) + \left(\frac{\gamma}{(l-\varepsilon)^{\gamma}} + 1\right)(A_{*}+\varepsilon)\left[A(\zeta) - A(\zeta_2)\right] \\ &\quad - k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa) q(\varkappa)\Delta\varkappa - \vartheta\left[A(\zeta) - A(\zeta_2)\right]. \end{aligned}$$

Dividing both sides by $A(\zeta)$, we have

$$\begin{aligned} A^{\gamma}(\zeta)\, w(\zeta) &\leq \frac{A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2)}{A(\zeta)} + \left(\frac{\gamma}{(l-\varepsilon)^{\gamma}}+1\right)(A_*+\varepsilon)\left[1-\frac{A(\zeta_2)}{A(\zeta)}\right] \\ &\quad -\frac{k}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta\left[1-\frac{A(\zeta_2)}{A(\zeta)}\right]. \end{aligned}$$

Taking the $\limsup$ of both sides as $\zeta \to \infty$ and by (2), we obtain

$$A_* \leq \left(\frac{\gamma}{(l-\varepsilon)^{\gamma}}+1\right)(A_*+\varepsilon) - \liminf_{\zeta\to\infty}\frac{k}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta.$$

Since $k,\ \varepsilon > 0$ are arbitrary constants, we reach the demanded inequality

$$\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa \leq \gamma\left(\frac{A_*}{l^{\gamma}} - a_*^{1+\frac{1}{\gamma}}\right). \tag{34}$$

From (30) and (34), we get

$$\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa \leq \frac{\gamma}{l^{\gamma(\gamma+1)}(\gamma+l^{\gamma})},$$

which is in contrast to the condition (20) if $\gamma \geq 1$. The proof is accomplished. □

Theorem 2. *Let (2) holds, and*

$$\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_{T}^{\zeta} A^{\gamma+1}(\varkappa)\phi(\varkappa)q(\varkappa)\Delta\varkappa > \frac{1}{l^{\gamma(\gamma+1)}}\left(1-\frac{l^{\gamma}}{\gamma+1}\right), \tag{35}$$

for enough large $T \in [\zeta_0,\infty)_{\mathbb{T}}$. Then all solutions of Equation (1) are oscillatory.

Proof. Assume z is a nonoscillatory solution of Equation (1) on $[\zeta_0,\infty)_{\mathbb{T}}$. Thus, without loss of generality, let $z(\zeta) > 0$ and $z(\eta(\zeta)) > 0$ on $[\zeta_0,\infty)_{\mathbb{T}}$. As shown in the proof of Theorem 1, we obtain

$$\begin{aligned} A^{\gamma+1}(\zeta)\, w(\zeta) &\leq A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) + \int_{\zeta_2}^{\zeta}\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} w(\sigma(\varkappa))\,\Delta\varkappa \\ &\quad -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta\left[A(\zeta)-A(\zeta_2)\right], \end{aligned} \tag{36}$$

where $\vartheta = \gamma\,(a_*-\varepsilon)^{1+\frac{1}{\gamma}}$. In addition, we have

$$\left[A^{\gamma+1}(\varkappa)\right]^{\Delta} \leq (\gamma+1)\,A^{\gamma}(\sigma(\varkappa))\ a^{-1/\gamma}(\varkappa). \tag{37}$$

Substituting (37) into (36) we get

$$\begin{aligned} A^{\gamma+1}(\zeta)\, w(\zeta) &\leq A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) \\ &\quad +(\gamma+1)\int_{\zeta_2}^{\zeta}\left[\frac{A(\sigma(\varkappa))}{A(\varkappa)}\right]^{\gamma} a^{-1/\gamma}(\varkappa)A^{\gamma}(\varkappa)\, w(\sigma(\varkappa))\,\Delta\varkappa \\ &\quad -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta\left[A(\zeta)-A(\zeta_2)\right] \\ &\leq A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2) + \frac{\gamma+1}{(l-\varepsilon)^{\gamma}}(a_*+\varepsilon)\left[A(\zeta)-A(\zeta_2)\right] \\ &\quad -k\int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\,\phi(\varkappa)q(\varkappa)\Delta\varkappa - \vartheta\left[A(\zeta)-A(\zeta_2)\right]. \end{aligned}$$

Dividing both sides by $A(\zeta)$, we have

$$\begin{aligned} A^{\gamma}(\zeta)\, w(\sigma(\zeta)) \quad \leq \quad & A^{\gamma}(\zeta)\, w(\zeta) \leq \frac{A^{\gamma+1}(\zeta_2)\, w^{\Delta}(\zeta_2)}{A(\zeta)} \\ & + \frac{(\gamma+1)}{(l-\varepsilon)^{\gamma}}(a_* + \varepsilon)\left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right] \\ & - \frac{k}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa - \vartheta \left[1 - \frac{A(\zeta_2)}{A(\zeta)}\right] \end{aligned}$$

Taking the lim sup of both sides as $\zeta \to \infty$ and by (2), we obtain

$$a_* \leq \frac{(\gamma+1)}{(l-\varepsilon)^{\gamma}}(a_* + \varepsilon) - \liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta_2}^{\zeta} a^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa - \vartheta.$$

Since $k,\ \varepsilon > 0$ are arbitrary, we get the required inequality

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa)\, \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq a_* \left[\frac{\gamma+1}{l^{\gamma}} - 1\right] - \gamma a_*^{1+\frac{1}{\gamma}}. \tag{38}$$

From (30) and (38), we obtain

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta_2}^{\zeta} A^{\gamma+1}(\varkappa) \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq \frac{1}{l^{\gamma(\gamma+1)}} \left(1 - \frac{l^{\gamma}}{\gamma+1}\right),$$

which is in contrast to the condition (35). The proof is accomplished. $\square$

Example 1. *Consider the nonlinear dynamic equation of second order*

$$\left[\zeta^{\gamma-1} \varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta} + \frac{\delta \zeta^{\frac{1-\gamma}{\gamma}}}{\phi(\zeta) A^{\gamma+1}(\zeta)} \varphi_{\beta}(z(\eta(\zeta))) = 0, \tag{39}$$

where γ, β, and δ are positive constants with $\beta \geq \gamma$. Here $a(\zeta) = \zeta^{\gamma-1}$, and $q(\zeta) = \frac{\delta \zeta^{-(\gamma+1)}}{\phi(\zeta) A^{\gamma+1}(\zeta)}$, then the condition (2) holds since

$$\int^{\infty} \frac{\Delta\varkappa}{a^{\frac{1}{\gamma}}(\varkappa)} = \int^{\infty} \frac{\Delta\varkappa}{\varkappa^{1-\frac{1}{\gamma}}} = \infty$$

by Example 5.60 in [5]. In addition, a straightforward computation yields that

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta}^{\zeta} A^{\gamma+1}(\varkappa) \phi(\varkappa) q(\varkappa) \Delta\varkappa = \delta \liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_{\zeta}^{\zeta} \frac{\Delta\varkappa}{\varkappa^{\gamma+1}} = \delta.$$

By Theorem 2, every solution of (39) is oscillatory if

$$\delta > \frac{1}{l^{\gamma(\gamma+1)}} \left(1 - \frac{l^{\gamma}}{\gamma+1}\right).$$

We present a Fite–Wintner type oscillation criterion for (1). The proof is similar to that in [7], and hence is omitted.

Theorem 3. *Let (2) holds, and*

$$\int_{\zeta_0}^{\infty} q(\varkappa) \Delta\varkappa = \infty. \tag{40}$$

Then every solution of Equation (1) is oscillatory.

From Theorem 3, we assume without loss of generality that

$$\int_{\zeta_0}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa < \infty.$$

Otherwise, we have that (40) holds due to $\phi(\zeta) \leq 1$, which implies that Equation (1) is oscillatory by Theorem 3. The next theorem is generalized Hille type to the second order nonlinear dynamic Equation (1).

Theorem 4. *Let* (2) *holds, and*

$$\liminf_{\zeta\to\infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa > \frac{\gamma^{\gamma}}{l\gamma^2(\gamma+1)^{\gamma+1}}. \tag{41}$$

Then every solutions of Equation (1) *is oscillatory.*

Proof. Assume $z(t)$ be a nonoscillatory solution of Equation (1) on $[\zeta_0,\infty)_{\mathbb{T}}$. Thus, without loss of generality, let $z(\zeta) > 0$ and $z(\eta(\zeta)) > 0$ on $[\zeta_0,\infty)_{\mathbb{T}}$. As depicted in the proof of Theorem 1, we obtain (24) for $\zeta \geq \zeta_1$, for some $\zeta_1 \in (\zeta_0,\infty)_{\mathbb{T}}$ such that $\eta(\zeta) \in (\zeta_0,\infty)_{\mathbb{T}}$ for $\zeta \geq \zeta_1$. Also for $\varepsilon > 0$, then we can pick $\zeta_2 \in [\zeta_1,\infty)_{\mathbb{T}}$, sufficiently large, so that (26) and (27) for $\zeta \in [\zeta_2,\infty)_{\mathbb{T}}$. Replacing ζ by $\varkappa$ in the inequality (24) and then integrating it from $\sigma(\zeta) \geq \zeta_2$ to $v \in [\zeta,\infty)_{\mathbb{T}}$ and using the fact $w > 0$, we have

$$\begin{aligned} -w(\sigma(\zeta)) &\leq w(v) - w(\sigma(\zeta)) \\ &\leq -k\int_{\sigma(\zeta)}^{v} \phi(\varkappa)q(\varkappa)\Delta\varkappa - \gamma\int_{\sigma(\zeta)}^{v} a^{-\frac{1}{\gamma}}(\varkappa)\, w^{1+\frac{1}{\gamma}}(\sigma(\varkappa))\Delta\varkappa. \end{aligned}$$

Taking $v \to \infty$ we obtain

$$-w(\sigma(\zeta)) \leq -k\int_{\sigma(\zeta)}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa - \gamma\int_{\sigma(\zeta)}^{\infty} a^{-1/\gamma}(\varkappa)\, w^{1+1/\gamma}(\sigma(\varkappa))\Delta\varkappa. \tag{42}$$

Multiplying both sides of (42) by $A^{\gamma}(\zeta)$, we obtain

$$\begin{aligned} -A^{\gamma}(\zeta)\,w(\sigma(\zeta)) &\leq -k\,A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad -\gamma A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} a^{-1/\gamma}(\varkappa)\, w^{1+\frac{1}{\gamma}}(\sigma(\varkappa))\Delta\varkappa \\ &= -k\,A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad -\gamma A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} \frac{A^{\Delta}(\varkappa)}{A^{\gamma+1}(\varkappa)}\left[A^{\gamma}(\varkappa)w(\sigma(\varkappa))\right]^{1+\frac{1}{\gamma}}\Delta\varkappa. \end{aligned}$$

It follows from (26) that

$$\begin{aligned} -A^{\gamma}(\zeta)\,w(\sigma(\zeta)) &\leq -k\,A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} \phi(\varkappa)q(\varkappa)\Delta\varkappa \\ &\quad -(a_*-\varepsilon)^{1+\frac{1}{\gamma}}A^{\gamma}(\zeta)\int_{\sigma(\zeta)}^{\infty} \gamma\frac{A^{\Delta}(\varkappa)}{A^{\gamma+1}(\varkappa)}\Delta\varkappa. \end{aligned} \tag{43}$$

By Pötzsche chain rule, we reach

$$\left(\frac{-1}{A^{\gamma}}\right)^{\Delta} = \gamma\int_0^1 \frac{1}{[A+h\mu(\varkappa)A^{\Delta}]^{\gamma+1}}\mathrm{d}h\,A^{\Delta} \leq \gamma\frac{A^{\Delta}}{A^{\gamma+1}}. \tag{44}$$

Then from (43) and (44), we have

$$\begin{aligned} -A^{\gamma}(\zeta)\, w(\sigma(\zeta)) &\leq -k\, A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa - (a_* - \varepsilon)^{1+\frac{1}{\gamma}} \left[\frac{A(\zeta)}{A(\sigma(\zeta))}\right]^{\gamma} \\ &\leq -k\, A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa - (l-\varepsilon)^{\gamma} (a_* - \varepsilon)^{1+\frac{1}{\gamma}}, \end{aligned}$$

which yields

$$k\, A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq A^{\gamma}(\zeta)\, w(\sigma(\zeta)) - (l-\varepsilon)^{\gamma} (a_* - \varepsilon)^{1+\frac{1}{\gamma}}.$$

By taking the lim inf of both sides as $\zeta \to \infty$ we obtain that

$$\liminf_{\zeta \to \infty} k\, A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq a_* - (l-\varepsilon)^{\gamma} (a_* - \varepsilon)^{1+\frac{1}{\gamma}}.$$

Since k and $\varepsilon > 0$ are arbitrary, we achieve the following inequality

$$\liminf_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq a_* - l^{\gamma}\, a_*^{1+\frac{1}{\gamma}}.$$

Using the inequality (29) with $z = l^{\gamma}$, $\Upsilon = 1$ and $u = a_*$, we get the desired inequality

$$\liminf_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq \frac{\gamma^{\gamma}}{l\gamma^2 (\gamma+1)^{\gamma+1}},$$

which is in contrast to the condition (41). The proof is accomplished in Theorem 4. □

Example 2. *Consider the nonlinear second order dynamic equation*

$$\left[\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta} + \frac{\kappa\gamma}{L\zeta^{\gamma+1}} \varphi_{\beta}\left(z(\eta(\zeta))\right) = 0, \tag{45}$$

where γ, β, κ are positive constants, and $L = \liminf_{\zeta \to \infty} \left(\frac{\zeta}{\sigma(\zeta)}\right)^{\gamma}$ with $\beta \geq \gamma$. Here $a(\zeta) = 1$, $\eta(\zeta) \geq \zeta$ and $q(\zeta) = \frac{\eta\gamma}{L\zeta^{\gamma+1}}$, then the condition (2) holds, $A(\zeta) = \zeta - \zeta_0$ and $\phi(\zeta) = 1$. In addition,

$$\begin{aligned} \liminf_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa &= \frac{\kappa}{L} \liminf_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \frac{\gamma \Delta\varkappa}{\varkappa^{\gamma+1}} \\ &\geq \frac{\kappa}{L} \liminf_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\sigma(\zeta)}^{\infty} \left(\frac{-1}{\varkappa^{\gamma}}\right)^{\Delta} \Delta\varkappa \\ &= \frac{\kappa}{L} \liminf_{\zeta \to \infty} \left(\frac{\zeta}{\sigma(\zeta)} - \frac{\zeta_0}{\sigma(\zeta)}\right)^{\gamma} = \kappa \end{aligned}$$

if $\kappa > \frac{\gamma^{\gamma}}{l\gamma^2 (\gamma+1)^{\gamma+1}}$. Then by Theorem 4, all solutions of (45) are oscillatory if $\kappa > \frac{\gamma^{\gamma}}{l\gamma^2 (\gamma+1)^{\gamma+1}}$.

Remark 1. *We could refer to the recent results due to [13,14] and others do not apply to Equations (39) and (45).*

Theorem 5. *Let (2) hold, and*

$$\limsup_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\zeta}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa > 1. \tag{46}$$

Then all solutions of Equation (1) oscillate.

Proof. Assume $z(t)$ is a nonoscillatory solution of Equation (1) on $[\zeta_0, \infty)_{\mathbb{T}}$. Thus, without loss of generality, let $z(\zeta) > 0$ and $z(\eta(\zeta)) > 0$ on $[\zeta_0, \infty)_{\mathbb{T}}$. Integrating both sides of the dynamic Equation (1) from ζ to $v \in [\zeta_0, \infty)_{\mathbb{T}}$, we obtain

$$\int_{\zeta}^{v} q(\varkappa) z^{\beta}(\eta(\varkappa)) \Delta\varkappa = a(\zeta)(z^{\Delta}(\zeta))^{\gamma} - a(v)(z^{\Delta}(v))^{\gamma} \leq a(\zeta)(z^{\Delta}(\zeta))^{\gamma}. \tag{47}$$

As shown in the proof of Theorem 1, there exists $\zeta_1 \in (\zeta_0, \infty)_{\mathbb{T}}$ satisfying $\eta(\zeta) \in (\zeta_0, \infty)_{\mathbb{T}}$ for $\zeta \geq \zeta_1$ such that for $\zeta \geq \zeta_1$

$$z^{\beta}(\eta(\zeta)) \geq k\, \phi(\zeta) z^{\gamma}(\zeta) \tag{48}$$

and

$$z^{\gamma}(\zeta) \geq a(\zeta) \left(z^{\Delta}(\zeta)\right)^{\gamma} A^{\gamma}(\zeta). \tag{49}$$

From (47) and (48), we obtain

$$k \int_{\zeta}^{v} \phi(\varkappa) q(\varkappa) z^{\gamma}(\varkappa) \Delta\varkappa \leq a(\zeta)(z^{\Delta}(\zeta))^{\gamma}.$$

Since $z^{\Delta}(\zeta) > 0$, we get that

$$k\, z^{\gamma}(\zeta) \int_{\zeta}^{v} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq a(\zeta)(z^{\Delta}(\zeta))^{\gamma}. \tag{50}$$

From (49) and (50), we get

$$k\, A^{\gamma}(\zeta) \int_{\zeta}^{v} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq 1.$$

Taking $v \to \infty$, we have

$$k\, A^{\gamma}(\zeta) \int_{\zeta}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq 1.$$

Since $k > 0$ is arbitrary, we have

$$A^{\gamma}(\zeta) \int_{\zeta}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq 1,$$

which gives us the contradiction

$$\limsup_{\zeta \to \infty} A^{\gamma}(\zeta) \int_{\zeta}^{\infty} \phi(\varkappa) q(\varkappa) \Delta\varkappa \leq 1.$$

The proof of Theorem 5 is accomplished. □

3. Oscillation Criteria of (1) when $\beta \leq \gamma$

Assume that

$$z(\zeta) > 0,\ z(\eta(\zeta)) > 0,\ z^{\Delta}(\zeta) > 0,\ \left[a(\zeta)\varphi_{\gamma}\left(z^{\Delta}(\zeta)\right)\right]^{\Delta} < 0$$

eventually. Integrating Equation (1) from ζ to $v \in [\zeta, \infty)_{\mathbb{T}}$ and then using (22) and the fact that $z^{\Delta} > 0$, we obtain

$$
\begin{aligned}
-a(v)\varphi_\gamma\left(z^\Delta(v)\right)+a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right) &= \int_\zeta^v q(\varkappa)\,\varphi_\beta(z(\eta(\varkappa)))\,\Delta\varkappa \\
&\geq \int_\zeta^v \phi(\varkappa)q(\varkappa)\,\varphi_\beta(z(\varkappa))\,\Delta\varkappa \\
&\geq \varphi_\beta(z(\zeta))\int_\zeta^v \phi(\varkappa)q(\varkappa)\,\Delta\varkappa,
\end{aligned}
$$

and $a(v)\varphi_\gamma\left(z^\Delta(v)\right)>0$ gives

$$a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\geq \varphi_\beta(z(\zeta))\int_\zeta^v \phi(\varkappa)q(\varkappa)\,\Delta\varkappa.$$

Hence by taking limits as $v\to\infty$ we have

$$a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\geq \varphi_\beta(z(\zeta))\int_\zeta^\infty \phi(\varkappa)q(\varkappa)\,\Delta\varkappa. \tag{51}$$

Since $\left[a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\right]^\Delta<0$ eventually, then

$$a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\leq a(\zeta_2)\varphi_\gamma\left(z^\Delta(\zeta_2)\right)=:b \quad \text{for } \zeta\geq\zeta_2,$$

and hence from (51), we have

$$b\geq a(\zeta)\varphi_\gamma\left(z^\Delta(\zeta)\right)\geq \varphi_\beta(z(\zeta))\int_\zeta^\infty \phi(\varkappa)q(\varkappa)\,\Delta\varkappa,$$

and so

$$z^{\beta-\gamma}(\zeta)=\left[\varphi_\beta(z(\zeta))\right]^{\frac{\beta-\gamma}{\beta}}\geq c\left[\int_\zeta^\infty \phi(\varkappa)q(\varkappa)\,\Delta\varkappa\right]^{\frac{\gamma-\beta}{\beta}},$$

where $c:=b^{\frac{\beta-\gamma}{\beta}}>0$. Combining all these we see that for every arbitrary $c>0$,

$$z^{\beta-\gamma}(\zeta)\geq c\left[\int_\zeta^\infty \phi(\varkappa)q(\varkappa)\,\Delta\varkappa\right]^{\frac{\gamma-\beta}{\beta}}, \tag{52}$$

eventually. Let

$$Q(\zeta):=q(\zeta)\left[\int_\zeta^\infty \phi(\varkappa)q(\varkappa)\,\Delta\varkappa\right]^{\frac{\gamma-\beta}{\beta}}.$$

Therefore, by (52) and the definition of $Q(\zeta)$, as direct consequence of Theorems 1, 2, 4 and 5, we get oscillation criteria for Equation (1) with $\beta\leq\gamma$.

Theorem 6. *Let* (2) *hold, and*

$$
\begin{aligned}
&\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_T^\zeta A^{\gamma+1}(\varkappa)\,\phi(\varkappa)Q(\varkappa)\Delta\varkappa>\frac{1}{l^{\gamma(\gamma+1)}}\left(1-\frac{l^\gamma}{\gamma l^\gamma+1}\right), && 0<\gamma\leq 1,\\
&\liminf_{\zeta\to\infty}\frac{1}{A(\zeta)}\int_T^\zeta A^{\gamma+1}(\varkappa)\,\phi(\varkappa)Q(\varkappa)\Delta\varkappa>\frac{\gamma}{l^{\gamma(\gamma+1)}\left(\gamma+l^\gamma\right)}, && \gamma\geq 1,
\end{aligned}
\tag{53}
$$

for enough large $T\in[\zeta_0,\infty)_{\mathbb{T}}$. Then all solutions of Equation (1) *oscillate.*

Theorem 7. *Let* (2) *holds, and*

$$\liminf_{\zeta\to\infty} \frac{1}{A(\zeta)} \int_T^\zeta A^{\gamma+1}(\varkappa)\phi(\varkappa)Q(\varkappa)\Delta\varkappa > \frac{1}{l^{\gamma(\gamma+1)}}\left(1-\frac{l^\gamma}{\gamma+1}\right),$$

for enough large $T \in [\zeta_0, \infty)_{\mathbb{T}}$. *Then all solutions of Equation* (1) *oscillate.*

Theorem 8. *Let* (2) *holds, and*

$$\liminf_{\zeta\to\infty} A^\gamma(\zeta) \int_{\sigma(\zeta)}^\infty \phi(\varkappa)Q(\varkappa)\Delta\varkappa > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}.$$

Then all solutions of Equation (1) *oscillate.*

Theorem 9. *Let* (2) *holds, and*

$$\limsup_{\zeta\to\infty} A^\gamma(\zeta) \int_\zeta^\infty \phi(\varkappa)Q(\varkappa)\Delta\varkappa > 1.$$

Then all solutions of Equation (1) *oscillate.*

4. Conclusions

(1) In this paper, several Nehari, Hille and Ohriska type oscillation criterion have been given. The applicability of these criteria for (1) on an arbitrary time scale is achieved. The reported results have extended related findings to the differential and dynamics equations of second order as follows:

(i) Condition (41) reduces to (7) in the case if $\mathbb{T}=\mathbb{R}$, $\gamma=\beta=1$, $a(\zeta)=1$, and $\eta(\zeta)=\zeta$;
(ii) Condition (41) reduces to (10) in the case when $\mathbb{T}=\mathbb{R}$, $\gamma=\beta=1$, $a(\zeta)=1$, and $g(\zeta)\le\zeta$;
(iii) Condition (41) reduces to (19) under the assumptions that $\gamma=\beta$, $a^\Delta(\zeta)\ge 0$, and $g(\zeta)\le\zeta$;
(iv) Conditions (46) reduces to (11) supposing that $\mathbb{T}=\mathbb{R}$, $\gamma=\beta=1$, $a(\zeta)=1$, and $g(\zeta)\le\zeta$.

(2) Several oscillation criteria for (1) have been derived in the cases: $\eta(\zeta)\le\zeta$, $\eta(\zeta)\ge\zeta$, $\beta\ge\gamma$, and $\beta\le\gamma$. In contrast to [13,14], the restrictive condition (18) is not imposed in the oscillation results of the presented case-study. This leads to a great improvement in comparison with the proceeding results.

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References

1. Hilger, S. Analysis on measure chains — A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [CrossRef]
2. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
3. Kac, V.; Chueng, P. *Quantum Calculus*; Universitext: Ann Arbor, MI, USA, 2002.
4. Agarwal, R.P.; Bohner, M.; O'Regan, D.; Peterson, A. Dynamic equations on time scales: A survey. *J. Comput. Appl. Math.* **2002**, *141*, 1–26. [CrossRef]

5. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
6. Nehari, Z. Oscillation criteria for second-order linear differential equations. *Trans. Am. Math. Soc.* **1957**, *85*, 428–445. [CrossRef]
7. Fite, W.B. Concerning the zeros of the solutions of certain differential equations. *Trans. Amer. Math. Soc.* **1918**, *19*, 341–352. [CrossRef]
8. Hille, E. Non-oscillation theorems. *Trans. Am. Math. Soc.* **1948**, *64*, 234–252. [CrossRef]
9. Wong, J.S. Second order oscillation with retarded arguments. In *Ordinary Differential Equations*; Academic Press: New York, NY, USA; London, UK, 1972; pp. 581–596.
10. Erbe, L. Oscillation criteria for second order nonlinear delay equations. *Can. Math. Bull.* **1973**, *16*, 49–56. [CrossRef]
11. Ohriska, J. Oscillation of second order delay and ordinary differential equations. *Czech. Math. J.* **1984**, *34*, 107–112. [CrossRef]
12. Thandapani, E.; Ravi, K.; Graef, J. Oscillation and comparison theorems for half-linear second order difference equations. *Comput. Math. Appl.* **2001**, *42*, 953–960. [CrossRef]
13. Erbe, L.; Hassan, T.S.; Peterson, A.; Saker, S.H. Oscillation criteria for half-linear delay dynamic equations on time scales. *Nonlinear Dyn. Syst. Theory* **2009**, *9*, 51–68.
14. Erbe, L.; Hassan, T.S.; Peterson, A.; Saker, S.H. Oscillation criteria for sublinear half-linear delay dynamic equations on time scales. *Int. J. Differ. Eq.* **2008**, *3*, 227–245.
15. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Hille and Nehari type criteria for third order delay dynamic equations. *J. Differ. Eq. Appl.* **2013**, *19*, 1563–1579. [CrossRef]
16. Baculikova, B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. *Appl. Math. Lett.* **2019**, *91*, 68–75. [CrossRef]
17. Bazighifan, O.; El-Nabulsi, E.M. Different techniques for studying oscillatory behavior of solution of differential equations. *Rocky Mountain Journal of Mathematics Volume forthcoming, Number Forthcoming (2020).* Available online: https://projecteuclid.org/euclid.rmjm/1596037174 (accessed on 19 October 2020).
18. Bohner, M.; Hassan, T.S. Oscillation and boundedness of solutions to first and second order forced functional dynamic equations with mixed nonlinearities. *Appl. Anal. Discret. Math.* **2009**, *3*, 242–252. [CrossRef]
19. Bohner, M.; Hassan, T.S.; Li, T. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indag. Math.* **2018**, *29*, 548–560. [CrossRef]
20. Džurina, J.; Jadlovská, I. A sharp oscillation result for second-order half-linear noncanonical delay differential equations. *Electron. J. Qual. Theory* **2020**, *46*, 1–14.
21. Elabbasy, E.M.; El-Nabulsi, R.A.; Moaaz, O.; Bazighifan, O. Oscillatory properties of solutions of even order differential equations. *Symmetry* **2020**, *12*, 212. [CrossRef]
22. Erbe, L.; Peterson, A. Oscillation criteria for second-order matrix dynamic equations on a time scale. *J. Comput. Appl. Math.* **2002**, *141*, 169–185. [CrossRef]
23. Erbe, L.; Peterson, A. Boundedness and oscillation for nonlinear dynamic equations on a time scale. *Proc. Am. Math. Soc.* **2003**, *132*, 735–744. [CrossRef]
24. Erbe, L.; Hassan, T.S. New oscillation criteria for second order sublinear dynamic equations. *Dyn. Syst. Appl.* **2013**, *22*, 49–63.
25. Erbe, L.; Peterson, A.; Saker, S.H. Hille and Nehari type criteria for third order dynamic equations. *J. Math. Anal. Appl.* **2007**, *329*, 112–131. [CrossRef]
26. Grace, S.R.; Bohner, M.; Agarwal, R.P. On the oscillation of second-order half-linear dynamic equations. *J. Differ. Eq. Appl.* **2009**, *15*, 451–460. [CrossRef]
27. Hassan, T.S.; Agarwal, R.P.; Mohammed,W. Oscillation criteria for third-order functional half-linear dynamic equations. *Adv. Differ. Eq.* **2017**, *2017*, 111. [CrossRef]
28. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation criteria for second order sublinear dynamic equations with damping term. *J. Differ. Eq. Appl.* **2011**, *17*, 505–523.
29. Leighton, W. The detection of the oscillation of solutions of asecond order linear differential equation. *Duke J. Math.* **1950**, *17*, 57–62. [CrossRef]
30. Karpuz, B. Hille–Nehari theorems for dynamic equations with a time scale independent critical constant. *Appl. Math. Comput.* **2019**, *346*, 336–351. [CrossRef]
31. Řehak, P. New results on critical oscillation constants depending on a graininess. *Dyn. Syst. Appl.* **2010**, *19*, 271–288.

32. Sun, S.; Han, Z.; Zhao, P.; Zhang, C. Oscillation for a class of second-order Emden-Fowler delay dynamic equations on time scales. *Adv. Differ. Eq.* **2010**, *2010*, 642356. [CrossRef]
33. Wintner, A. On the nonexistence of conjugate points. *Am. J. Math.* **1951**, *73*, 368–380. [CrossRef]
34. Sun, Y.; Hassan, T.S. Oscillation criteria for functional dynamic equations with nonlinearities given by Riemann-Stieltjes integral. *Abstr. Appl. Anal.* **2014**, *2014*, 697526. [CrossRef]
35. Zhang, Q.; Gao, L.; Wang, L. Oscillation of second-order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl.* **2011**, *61*, 2342–2348. [CrossRef]