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# The Inverse-Power Logistic-Exponential Distribution: Properties, Estimation Methods, and Application to Insurance Data

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**Abstract:** The present paper proposes a new distribution called the inverse power logistic exponential distribution that extends the inverse Weibull, inverse logistic exponential, inverse Rayleigh, and inverse exponential distributions. The proposed model accommodates symmetrical, right-skewed, left-skewed, reversed-J-shaped, and J-shaped densities and increasing, unimodal, decreasing, reversed-J-shaped, and J-shaped hazard rates. We derive some mathematical properties of the proposed model. The model parameters were estimated using five estimation methods including the maximum likelihood, Anderson–Darling, least-squares, Cramér–von Mises, and weighted least-squares estimation methods. The performance of these estimation methods was assessed by a detailed simulation study. Furthermore, the flexibility of the introduced model was studied using an insurance real dataset, showing that the proposed model can be used to fit the insurance data as compared with twelve competing models.

**Keywords:** logistic exponential distribution; Cramér–von Mises estimation; insurance data; parameter estimation; maximum likelihood estimation

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## 1. Introduction

Reliability and survival analysis has several applications, as an important branch of statistics, in different applied fields, such as actuarial science, engineering, demography, biomedical studies, and industrial reliability. Several lifetime distributions have been proposed in the statistical literature to model data in many applied sciences.

The exponential distribution is used in modeling real-life data due to its lack of memory property, and it is also analytically tractable. On the other hand, its applicability was limited because it has only a constant hazard rate and decreasing density function. Hence, many researcher have been interested in proposing modified forms of the exponential distribution to increase its flexibility. Some recent extensions of the exponential distribution include the exponentiated exponential [1], beta exponential [2], beta generalized exponential [3], transmuted generalized exponential [4], Harris extended exponential [5], Kumaraswamy transmuted exponential [6], Marshall–Olkin Nadarajah–Haghighi [7], modified exponential [8], alpha power exponential [9,10], odd exponentiated half-logistic exponential [11], Marshall–Olkin logistic exponential [12], generalized odd log-logistic exponential [13], Marshall–Olkin alpha power exponential [14], extended odd Weibull exponential [15], odd inverse Pareto exponential [16], modified Kies exponential [17], Topp–Leone moment exponential [18], heavy-tailed exponential [19], and odd log-logistic Lindley exponential distributions [20].

One of the important extensions of the exponential distribution is called the logistic exponential distribution, which was proposed by Lan and Leemis [21]. In this paper, we propose a flexible distribution called the inverse power logistic exponential (IPLE) distribution. The IPLE can provide more flexibility and accuracy in fitting actuarial data. The IPLE distribution also generalizes the inverse Weibull, inverse Rayleigh, inverse logistic exponential, and inverse exponential distributions. The proposed model was generated based on the inverse power transformation.

Let $X$ and $T$ be random variables. The inverse transformation, denoted by $X = T^{-1}$, or inverse power transformation, denoted by $X = T^{\frac{-1}{\beta}}$ have been used in generating inverted distributions. For example, the inverse two parameter Lindley distribution by Alkarni [22], the generalized inverse gamma distribution by Mead [23], the reverse Lindley distribution by Sharma et al. [24], and the inverse Lindley distribution using by Barco et al. [25].

The proposed IPLE model is motivated by some properties as follows.

- The IPLE model includes the inverse Weibull, inverse logistic exponential, inverse Rayleigh, and inverse exponential distributions as special sub-models.
- The IPLE distribution can provide symmetrical, right-skewed, left-skewed, reversed-J-shaped, and J-shaped densities and increasing, unimodal, decreasing, reversed-J-shaped, and J-shaped hazard rates.
- The probability density function (PDF), as well as the cumulative distribution function (CDF) of the IPLE model have simple closed forms, and hence, it can be adopted in analyzing censored data.
- The IPLE model has been used to model a heavy-tailed insurance dataset from actuarial science, and it provides adequate fits compared to other competing distributions.

The main aim of this paper is to study a new extension of the logistic exponential model based on the inverse power transformation and derive some of its distributional properties. We are also interested in exploring the estimation of the IPLE parameters by five classical estimation methods including the maximum likelihood estimators (MLEs), Anderson–Darling estimators (ADEs), least-squares estimators (LSEs), Cramér–von Mises estimators (CVMEs), and weighted least-squares estimators (WLSEs). These estimation methods were compared using an extensive simulation study to assess their performances and to provide a guideline for choosing the best estimation method that gives better estimates for the IPLE parameters. This would be of deep interest to applied statisticians, actuaries, or engineers.

Parameter estimation using several classical methods of estimation were studied by several statisticians. For example, the alpha logarithmic transformed Weibull distribution [26], Weibull–Marshall–Olkin–Lindley distribution [27], quasi xgamma-geometric distribution [28], logarithmic transformed Weibull distribution [29], generalized Ramos–Louzada distribution [30], and alpha power exponential distribution [10], among many others.

The paper is organized as follows. We define the IPLE distribution and its special sub-models in Section 2. Its mathematical properties are derived in Section 3. Five methods of estimation are discussed in Section 4. The performance of these estimation methods is explored using a simulation study in Section 5. A real dataset with a heavy tail from insurance science is analyzed to show the usefulness and importance of the IPLE distribution in Section 6. We present some conclusions in Section 7.

## 2. The IPLE Distribution

Based on the inverse power transformation and the logistic exponential (LE) (Lan and Leemis [21]) distribution, we generate the IPLE distribution. The CDF and PDF of the LE distributions are given by:

$$G(t) = 1 - \frac{1}{\left(e^{\lambda t} - 1\right)^{\alpha} + 1}, \; t > 0, \, \alpha, \lambda > 0,$$

and:

$$g(t) = \frac{\alpha \; \lambda \, e^{\lambda t} \; \left(e^{\lambda t} - 1\right)^{\alpha - 1}}{\left[\left(e^{\lambda t} - 1\right)^{\alpha} + 1\right]^{2}}, \; t > 0, \, \alpha, \lambda > 0,$$

where $\alpha$ and $\lambda$ are respectively the shape and scale parameters. For $\alpha = 1$, the exponential distribution follows as a special sub-model from the LE model.

Consider the inverse power transformation, $X = T^{\frac{-1}{\beta}}$, where $T \sim$LE$(\alpha, \lambda)$, then the resulting IPLE distribution of $X$ can be specified by the CDF:

$$F(x) = \frac{1}{\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha} + 1}, \ x > 0, \ \alpha, \beta, \lambda > 0. \tag{1}$$

The PDF of the IPLE distribution reduces to:

$$f(x) = \frac{\alpha\beta\lambda x^{-\beta-1} e^{\lambda x^{-\beta}} \left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha-1}}{\left[\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha} + 1\right]^2}, \ x > 0, \ \alpha, \beta, \lambda > 0, \tag{2}$$

where $\beta$ and $\alpha$ are the shape parameters and $\lambda$ is a scale parameter. By setting $\beta = 1$, we obtain the inverse logistic exponential distribution.

The survival function (SF) and hazard rate function (HRF) of the IPLE distribution are, respectively, given by:

$$S(x) = 1 - F(x) = 1 - \frac{1}{\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha} + 1}, \tag{3}$$

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\beta\lambda x^{-\beta-1} e^{\lambda x^{-\beta}}}{\left(e^{\lambda x^{-\beta}} - 1\right)\left[\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha} + 1\right]}. \tag{4}$$

Some possible plots of the PDF and HRF of the IPLE distribution are depicted in Figures 1 and 2, to show the flexibility of both functions.

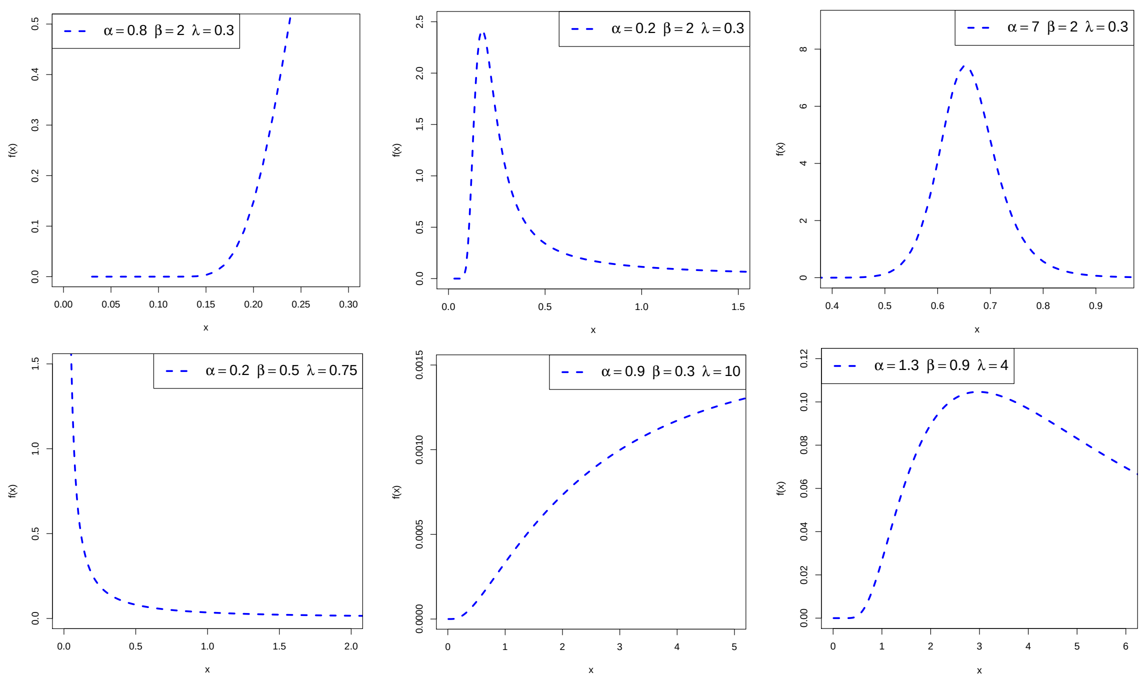


**Figure 1.** Plots of the inverse power logistic exponential (IPLE) PDF for different parametric values.

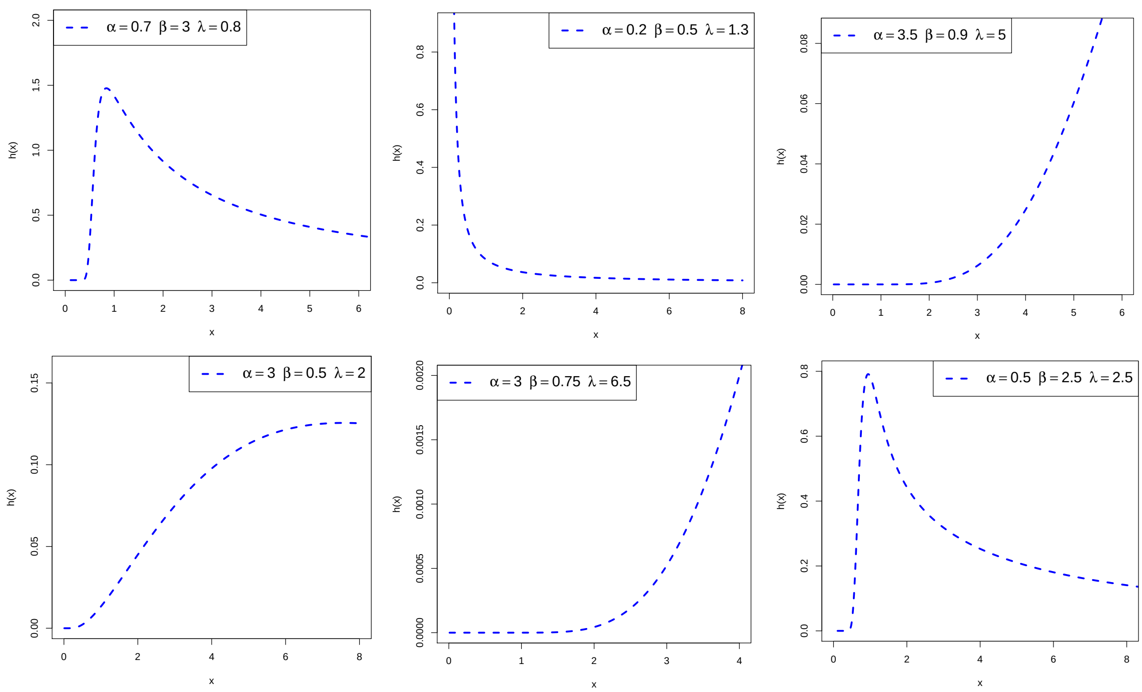


**Figure 2.** Plots of the IPLE hazard rate function (HRF) for different parametric values.

## 3. Mathematical Properties

### *3.1. Quantile Function*

The quantile function (QF) of the IPLE distribution is derived by the inverse function of the IPLE CDF (1). The QF of the IPLE distribution takes the form:

$$Q(p) = \lambda^{1/\beta} \left\{ \log \left[ \left( \frac{1}{p} - 1 \right)^{1/\alpha} + 1 \right] \right\}^{-1/\beta}, \quad 0 < p < 1, \tag{5}$$

where $p$ follow a uniform distribution $(0, 1)$. Setting $p = 0.25, 0.5$, and $0.75$, in (5), one can obtain the first, second, and third quartiles of the IPLE distribution.

Using the QF, we can determine the Bowley skewness measure and Moors kurtosis measure, respectively, as follows:

$$\begin{aligned} SK &= \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}, \\ KU &= \frac{Q(7/8) + Q(3/8) - Q(5/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \end{aligned}$$

The skewness and kurtosis of the IPLE model can be displayed graphically for some values of $\alpha$ and $\lambda$ in Figure 3.

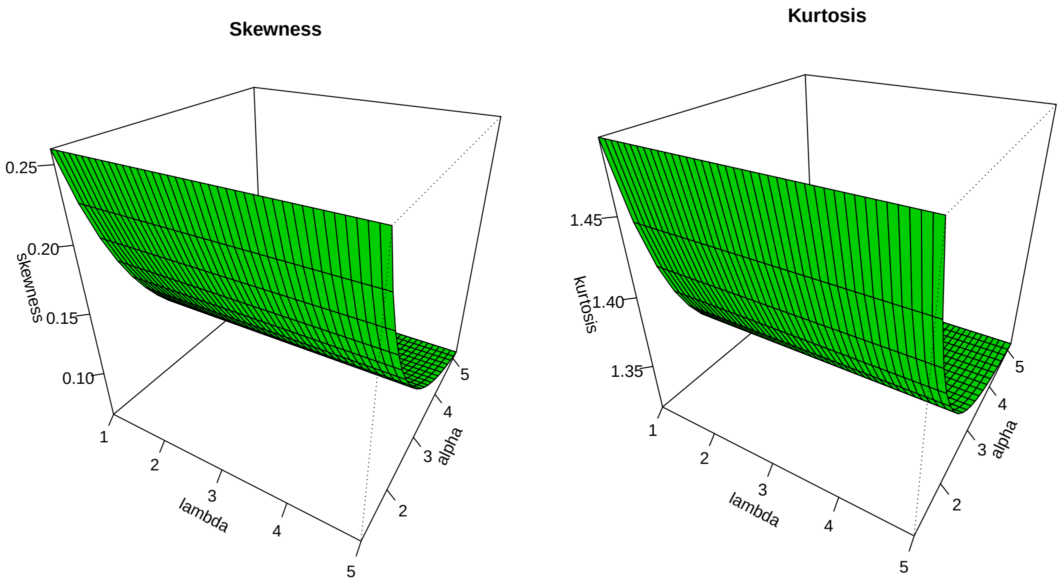


**Figure 3.** Shapes of the skewness and kurtosis of the IPLE distribution for different values of $\alpha$ and $\lambda$ with $\beta = 2$.

### *3.2. Moments*

The $r$th moments of the IPLE distribution takes the following form:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)dx = \lambda^{\frac{r}{\beta}} \int_1^\infty \frac{\left\{\log\left[1+(-1+y)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-r}{\beta}}}{y^2} dy,$$

The first four moments of the IPLE distribution follow respectively by setting $r = 1, 2, 3,$ and 4. The moment generating function of the IPLE distribution takes the form:

$$M(t) = \sum_{k=0}^{\infty} \frac{(t\lambda^{\frac{1}{\beta}})^k}{k!} \int_1^\infty \frac{\left\{\log\left[1+(-1+y)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-k}{\beta}}}{y^2} dy.$$

The characteristic function of the IPLE distribution can be obtained by replacing $t$ with $it$ in the last equation.

### *3.3. Inequality Curves*

The most important inequality curves are called Lorenz and Bonferroni, which have some applications in applied sciences such as economics, reliability, demography, and medicine.

The Lorenz and Bonferroni curves for the IPLE distribution take the forms:

$$\begin{aligned} L(p) &= \frac{1}{\mu}\int_0^{x_p} xf(x)dx = \frac{\lambda^{\frac{1}{\beta}}}{\mu} \int_A^\infty \frac{\left\{\log\left[1+(-1+y)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-1}{\beta}}}{y^2} dy, \\ B(p) &= \frac{L(p)}{p}, \end{aligned}$$

respectively, where $A = 1 + (e^{\lambda x_p^{-\beta}} - 1)^\alpha$ and $x_p$ is the QF of the IPLE distribution.

### *3.4. Moments of Residual Life*

The $m$th moment of residual life is defined by $M_m = \frac{1}{S(t)} \int_t^\infty (x-t)^m f(x)dx$

For the IPLE distribution, $m$th takes the form:

$$M_m = \frac{1}{S(t)(\alpha-1)} \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} t^{m-k} \lambda^{\frac{k}{\beta}} \int_1^{1+w} \frac{\left\{\log\left[1+(-1+y)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-k}{\beta}}}{y^2} dy,$$

$w = \left(e^{\lambda t^{-\beta}} - 1\right)^{\alpha}$.

For $m = 1$, we have the mean residual life, $M_1$, of the IPLE distribution, and by setting $t = 0$, the mean of the IPLE distribution follows from $M_1$.

## 4. Methods of Estimation

In this section, we explore the estimation of the IPLE parameters by different methods of estimation including the maximum likelihood estimators (MLEs), Anderson–Darling estimators (ADEs), least-squares estimators (LSEs), Cramér–von Mises estimators (CVMEs), and weighted least-squares estimators (WLSEs).

### 4.1. Maximum Likelihood Estimators

Let $x_1, x_2, \ldots, x_n$ be a random sample of size $n$ from the IPLE with PDF (2); hence, the log-likelihood function is specified by:

$$\begin{aligned} \ell =& n\log(\alpha) + n\log(\beta) + n\log(\lambda) - (\beta+1)\sum_{i=1}^{n}\log(x_i) + \lambda\sum_{i=1}^{n} x_i^{-\beta} + (\alpha-1)\sum_{i=1}^{n}\log\left(e^{\lambda x_i^{-\beta}} - 1\right) \\ & - 2\sum_{i=1}^{n}\log\left[\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha} + 1\right]. \end{aligned} \quad (6)$$

By differentiating Equation (6) with respect to $\alpha$, $\beta$, and $\lambda$, we can write:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(e^{\lambda x_i^{-\beta}} - 1\right) - 2\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha}\log\left(e^{\lambda x_i^{-\beta}} - 1\right)}{\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha} + 1} = 0,$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} =& \frac{n}{\beta} - \sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n} x_i^{-\beta}\log(x_i) - (\alpha-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\beta} e^{\lambda x_i^{-\beta}}\log(x_i)}{e^{\lambda x_i^{-\beta}} - 1} \\ & + 2\sum_{i=1}^{n}\frac{\alpha\lambda x_i^{-\beta} e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha-1}\log(x_i)}{\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha} + 1} = 0 \end{aligned}$$

and:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{-\beta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{-\beta} e^{\lambda x_i^{-\beta}}}{e^{\lambda x_i^{-\beta}} - 1} - 2\sum_{i=1}^{n}\frac{\alpha x_i^{-\beta} e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha-1}}{\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha} + 1} = 0.$$

We note that there were no explicit solutions for the above three equations, and hence, we require employing the nonlinear numerical techniques to obtain the MLEs of the IPLE parameters.

### 4.2. Anderson–Darling Estimation

Consider the order statistics of a random sample of size $n$ from the IPLE distribution denoted by $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$. The ADEs of the IPLE parameters, $\alpha$, $\beta$, and $\lambda$, are obtained by minimizing the following equation:

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}) + \log S(x_{n+1-i:n})],$$

with respect to $\alpha$, $\beta$, and $\lambda$. Using Equations (1) and (3), the above equation reduces to:

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left\{1-\left[\left(e^{\lambda x_{n+1-i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{-1}\right\} - \log\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]\right).$$

The ADEs can also be calculated by solving the following nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_{\kappa}(x_{i:n})}{F(x_{i:n})} - \frac{\Delta_{\kappa}(x_{n+1-i:n})}{S(x_{n+1-i:n})}\right] = 0,\ \kappa = \alpha, \beta, \lambda.$$

where:

$$\Delta_{\alpha}(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \alpha} = \frac{-\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}\log\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)}{\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{2}}, \tag{7}$$

$$\Delta_{\beta}(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \beta} = \frac{\alpha\ \lambda\ x_{i:n}^{-\beta}\ e^{\lambda x_{i:n}^{-\beta}}\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha-1}}{\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{2}}, \tag{8}$$

$$\Delta_{\lambda}(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \lambda} = \frac{-\ \alpha\ x_{i:n}^{-\beta}\ e^{\lambda x_{i:n}^{-\beta}}\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha-1}}{\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{2}}. \tag{9}$$

### 4.3. Cramér–von Mises Estimators

The CVMEs of IPLE parameters, $\alpha$, $\beta$, and $\lambda$, are obtained by minimizing the following equation:

$$\begin{aligned} CV &= \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^{2} \\ &= \frac{1}{12n} + \sum_{i=1}^{n}\left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{-1} - \frac{2i-1}{2n}\right\}^{2}, \end{aligned}$$

or by solving the following nonlinear equations with respect to $\alpha$, $\beta$, and $\lambda$:

$$\sum_{i=1}^{n}\left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}}-1\right)^{\alpha}+1\right]^{-1} - \frac{2i-1}{2n}\right\}\Delta_{\kappa}(x_{i:n}) = 0,$$

where $\Delta_{\kappa}(x_{i:n}), \kappa = \alpha, \beta, \lambda$ were defined in (7)–(9), respectively.

### 4.4. Least-Squares and Weighted Least-Squares Estimators

The LSEs of the IPLE parameters, $\alpha$, $\beta$, and $\lambda$, by minimizing the following equation:

$$\begin{aligned} LS &= \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}} - 1\right)^{\alpha} + 1\right]^{-1} - \frac{i}{n+1}\right\}^2. \end{aligned}$$

Furthermore, the LSEs of $\alpha$, $\beta$, and $\lambda$ follow also by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}} - 1\right)^{\alpha} + 1\right]^{-1} - \frac{i}{n+1}\right\} \Delta_{\kappa}(x_{i:n}) = 0, \quad \kappa = \alpha, \beta, \lambda,$$

where $\Delta_{\kappa}(x_{i:n})$ are defined in (7)–(9), respectively.

The WLSEs of the parameters $\alpha$, $\beta$, and $\lambda$ can be determined by minimizing the following equation:

$$\begin{aligned} W =& C \left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 \\ =& C \left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}} - 1\right)^{\alpha} + 1\right]^{-1} - \frac{i}{n+1}\right\}^2, \end{aligned}$$

where $C = \sum_{i=1}^{n} (n+1)^2 \; (n+2) / \; i(n-i+1)$.

Furthermore, the WLSEs of the parameters $\alpha$, $\beta$, and $\lambda$ are obtained by solving the following nonlinear equations:

$$C \left\{\left[\left(e^{\lambda x_{i:n}^{-\beta}} - 1\right)^{\alpha} + 1\right]^{-1} - \frac{i}{n+1}\right\} \Delta_{\kappa}(x_{i:n}) = 0,$$

where $\Delta_{\kappa}(x_{i:n}), \kappa = \alpha, \beta, \lambda$ are given in Equations (7)–(9), respectively.

## 5. Simulation Study

The performance of the five estimation methods in estimating the IPLE parameters based on simulation results is explored in this section. We considered various sample sizes, $n = \{20, 50, 100, 200, 500\}$, and different parametric values of $\alpha, \beta$, and $\lambda$, $\alpha = \{0.25, 0.50, 0.75, 1.0, 2.0\}$, $\beta = \{0.50, 0.75, 1.0, 1.5, 2.0\}$, and $\lambda = \{0.5, 0.75, 1.0, 1.5, 3.0\}$. We generate $n = 1000$ random samples from the IPLE distribution using its QF given in Equation (5). We calculate the average values of the estimates (AVEs) along with their associated average mean squared error (MSEs), average absolute biases, and average mean relative estimates (MREs) for the studied sample sizes and different parameter combinations, using the `R` software, to assess the performance of the proposed five estimation methods.

The MSEs, bias, and MREs are calculated by equations:

$$MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi})^2, \quad Bias = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}|, \quad MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}|/\boldsymbol{\phi},$$

where $\boldsymbol{\phi} = (\alpha, \beta, \lambda)'$.

Tables 1–5 report the simulation results including the AVEs, bias, MSEs, and MREs of the IPLE parameters using the five estimation approaches. It is noted that the estimates of the IPLE parameters obtained using the five estimation methods are quite reliable and very close to the true values, showing small MSEs, biases, and MREs in all studied cases. The five estimators are consistent, where the MSEs, biases, and MREs decrease as the sample size increases, for all studied cases. We can conclude that the MLE, ADE, CVME, WLSE, and LSE methods perform very well in estimating the IPLE parameters. In summary, the MLEs provide the best estimates for the parameters of the IPLE distribution; hence, the MLEs are adopted in the application section to estimate the IPLE parameters and the parameters of the compared models.

**Table 1.** Simulation values of the averages (AVEs), biases, MSEs, and mean relative estimates (MREs) for $(\alpha = 0.25,\ \beta = 0.75,\ \lambda = 0.5)$. ADEs, Anderson–Darling estimators; CVMEs, Cramér–von Mises estimators; WLSEs, weighted least-squares estimators.

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| MLEs | 20 | 0.29959 | 0.88811 | 0.50733 | 0.13269 | 0.26169 | 0.30401 | 1.80809 | 0.12442 | 0.15792 | 0.53074 | 0.34892 | 0.60802 |
| | 50 | 0.25991 | 0.79873 | 0.50798 | 0.05854 | 0.13806 | 0.19014 | 0.00783 | 0.03766 | 0.06174 | 0.23414 | 0.18409 | 0.38027 |
| | 100 | 0.25895 | 0.77429 | 0.50633 | 0.04156 | 0.08943 | 0.12938 | 0.00609 | 0.01801 | 0.03136 | 0.16625 | 0.11924 | 0.25877 |
| | 200 | 0.25873 | 0.75914 | 0.51297 | 0.03051 | 0.05831 | 0.08815 | 0.00459 | 0.01000 | 0.01983 | 0.12204 | 0.07775 | 0.17631 |
| | 500 | 0.25725 | 0.75171 | 0.50998 | 0.02027 | 0.03069 | 0.05142 | 0.00376 | 0.00413 | 0.01196 | 0.08110 | 0.04093 | 0.10285 |
| ADEs | 20 | 1.26579 | 0.79586 | 0.57588 | 1.08235 | 0.28551 | 0.34821 | 116.65009 | 0.13636 | 0.19635 | 4.32941 | 0.38068 | 0.69643 |
| | 50 | 0.27595 | 0.76685 | 0.54284 | 0.07209 | 0.17236 | 0.24400 | 0.01854 | 0.04866 | 0.09423 | 0.28836 | 0.22981 | 0.48799 |
| | 100 | 0.26022 | 0.76086 | 0.51886 | 0.04580 | 0.11980 | 0.17337 | 0.00380 | 0.02315 | 0.04820 | 0.18320 | 0.15973 | 0.34674 |
| | 200 | 0.25536 | 0.75225 | 0.51502 | 0.03015 | 0.08217 | 0.12481 | 0.00159 | 0.01091 | 0.02536 | 0.12058 | 0.10956 | 0.24962 |
| | 500 | 0.25218 | 0.74980 | 0.50695 | 0.01832 | 0.04998 | 0.07614 | 0.00057 | 0.00415 | 0.00959 | 0.07329 | 0.06664 | 0.15229 |
| CVMEs | 20 | 1.78919 | 0.85649 | 0.55473 | 1.61846 | 0.37561 | 0.35747 | 151.11781 | 0.25508 | 0.21699 | 6.47385 | 0.50082 | 0.71494 |
| | 50 | 0.30719 | 0.79458 | 0.53295 | 0.11283 | 0.22474 | 0.26263 | 1.77022 | 0.08639 | 0.10349 | 0.45134 | 0.29965 | 0.52526 |
| | 100 | 0.26469 | 0.76813 | 0.52365 | 0.05615 | 0.15042 | 0.19391 | 0.00587 | 0.03701 | 0.05812 | 0.22461 | 0.20056 | 0.38783 |
| | 200 | 0.25608 | 0.76069 | 0.51137 | 0.03669 | 0.10336 | 0.13915 | 0.00230 | 0.01761 | 0.03072 | 0.14677 | 0.13782 | 0.27829 |
| | 500 | 0.25250 | 0.75338 | 0.50575 | 0.02278 | 0.06464 | 0.08860 | 0.00084 | 0.00671 | 0.01238 | 0.09114 | 0.08618 | 0.17720 |
| LSEs | 20 | 1.35563 | 0.81818 | 0.57272 | 1.18807 | 0.35716 | 0.36193 | 122.09909 | 0.22213 | 0.21403 | 4.75226 | 0.47621 | 0.72386 |
| | 50 | 0.29041 | 0.77859 | 0.54445 | 0.09799 | 0.21584 | 0.26076 | 0.58039 | 0.07977 | 0.10480 | 0.39198 | 0.28779 | 0.52152 |
| | 100 | 0.26299 | 0.76297 | 0.52593 | 0.05524 | 0.14971 | 0.19431 | 0.00564 | 0.03658 | 0.05866 | 0.22095 | 0.19962 | 0.38863 |
| | 200 | 0.25413 | 0.75937 | 0.51083 | 0.03643 | 0.10359 | 0.13843 | 0.00221 | 0.01712 | 0.02968 | 0.14570 | 0.13812 | 0.27686 |
| | 500 | 0.25195 | 0.75325 | 0.50489 | 0.02293 | 0.06517 | 0.08886 | 0.00084 | 0.00669 | 0.01242 | 0.09172 | 0.08689 | 0.17771 |
| WLSEs | 20 | 1.01014 | 0.77884 | 0.58895 | 0.83437 | 0.30811 | 0.34971 | 71.84218 | 0.16052 | 0.19307 | 3.33749 | 0.41082 | 0.69943 |
| | 50 | 0.26952 | 0.77672 | 0.54040 | 0.07246 | 0.18758 | 0.25074 | 0.01117 | 0.05932 | 0.09816 | 0.28983 | 0.25010 | 0.50149 |
| | 100 | 0.25804 | 0.76201 | 0.52441 | 0.04725 | 0.12609 | 0.18030 | 0.00383 | 0.02634 | 0.05161 | 0.18898 | 0.16812 | 0.36061 |
| | 200 | 0.25450 | 0.75544 | 0.51125 | 0.03160 | 0.08658 | 0.12822 | 0.00164 | 0.01218 | 0.02601 | 0.12641 | 0.11544 | 0.25644 |
| | 500 | 0.25151 | 0.75334 | 0.50269 | 0.01945 | 0.05377 | 0.08022 | 0.00060 | 0.00460 | 0.01027 | 0.07779 | 0.07169 | 0.16044 |

**Table 2.** Simulation values of the AVEs, biases, MSEs, and MREs for $(\alpha = 2,\ \beta = 1.5,\ \lambda = 1)$.

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| MLEs | 20 | 1.98692 | 1.65613 | 1.05014 | 0.40172 | 0.32998 | 0.13721 | 0.18342 | 0.13625 | 0.03212 | 0.20086 | 0.21999 | 0.13721 |
| | 50 | 1.96919 | 1.62393 | 1.03293 | 0.39783 | 0.30905 | 0.09710 | 0.18012 | 0.12099 | 0.01568 | 0.19892 | 0.20604 | 0.09710 |
| | 100 | 1.96333 | 1.60204 | 1.02699 | 0.38290 | 0.28849 | 0.07974 | 0.16996 | 0.10569 | 0.01019 | 0.19145 | 0.19233 | 0.07974 |
| | 200 | 1.98399 | 1.57231 | 1.01797 | 0.34806 | 0.25748 | 0.06396 | 0.14809 | 0.08706 | 0.00634 | 0.17403 | 0.17165 | 0.06396 |
| | 500 | 2.00053 | 1.54050 | 1.00999 | 0.28207 | 0.20466 | 0.04962 | 0.10730 | 0.05913 | 0.00369 | 0.14103 | 0.13644 | 0.04962 |
| ADEs | 20 | 2.05895 | 1.53835 | 1.02221 | 0.41636 | 0.30581 | 0.12429 | 0.19326 | 0.12045 | 0.02598 | 0.20818 | 0.20388 | 0.12429 |
| | 50 | 2.03424 | 1.55152 | 1.01309 | 0.41798 | 0.29525 | 0.08847 | 0.19257 | 0.11049 | 0.01295 | 0.20899 | 0.19683 | 0.08847 |
| | 100 | 2.03559 | 1.54074 | 1.01178 | 0.39759 | 0.28508 | 0.07501 | 0.18036 | 0.10188 | 0.00887 | 0.19879 | 0.19005 | 0.07501 |
| | 100 | 2.02842 | 1.53773 | 1.01109 | 0.37028 | 0.26689 | 0.06550 | 0.16258 | 0.09023 | 0.00638 | 0.18514 | 0.17792 | 0.06550 |
| | 200 | 2.03281 | 1.52109 | 1.00526 | 0.31318 | 0.22445 | 0.05283 | 0.12747 | 0.06767 | 0.00406 | 0.15659 | 0.14963 | 0.05283 |
| CVMEs | 20 | 1.87826 | 1.72135 | 1.07690 | 0.38363 | 0.33235 | 0.07690 | 0.17189 | 0.13969 | 0.02028 | 0.19181 | 0.22156 | 0.07690 |
| | 50 | 1.85124 | 1.70679 | 1.05444 | 0.35344 | 0.30482 | 0.05444 | 0.15031 | 0.12129 | 0.01005 | 0.17672 | 0.20321 | 0.05444 |
| | 100 | 1.82886 | 1.69365 | 1.04599 | 0.32938 | 0.27898 | 0.04599 | 0.13554 | 0.10594 | 0.00672 | 0.16469 | 0.18599 | 0.04599 |
| | 200 | 1.83366 | 1.67706 | 1.04134 | 0.28956 | 0.24702 | 0.04134 | 0.11196 | 0.08856 | 0.00509 | 0.14478 | 0.16468 | 0.04134 |
| | 500 | 1.84133 | 1.65510 | 1.03588 | 0.23572 | 0.20102 | 0.03588 | 0.08330 | 0.06709 | 0.00362 | 0.11786 | 0.13402 | 0.03588 |
| LSEs | 10 | 2.00523 | 1.46603 | 1.00127 | 0.09759 | 0.24010 | 0.11181 | 0.00968 | 0.08205 | 0.02051 | 0.04880 | 0.16006 | 0.11181 |
| | 50 | 2.00391 | 1.49271 | 1.00076 | 0.09740 | 0.16174 | 0.07225 | 0.00965 | 0.04083 | 0.00846 | 0.04870 | 0.10783 | 0.07225 |
| | 100 | 2.00088 | 1.49953 | 1.00131 | 0.09702 | 0.12334 | 0.05038 | 0.00960 | 0.02360 | 0.00405 | 0.04851 | 0.08222 | 0.05038 |
| | 200 | 1.99873 | 1.50270 | 1.00131 | 0.09654 | 0.09461 | 0.03641 | 0.00954 | 0.01399 | 0.00212 | 0.04827 | 0.06307 | 0.03641 |
| | 500 | 2.00174 | 1.50034 | 0.99943 | 0.09424 | 0.07327 | 0.02476 | 0.00923 | 0.00804 | 0.00097 | 0.04712 | 0.04885 | 0.02476 |
| WLSEs | 20 | 2.06749 | 1.49826 | 1.00850 | 0.41887 | 0.30856 | 0.12001 | 0.19513 | 0.12231 | 0.02408 | 0.20943 | 0.20571 | 0.12001 |
| | 50 | 2.04871 | 1.52778 | 1.01135 | 0.41703 | 0.29708 | 0.08941 | 0.19303 | 0.11136 | 0.01299 | 0.20851 | 0.19805 | 0.08941 |
| | 100 | 2.04046 | 1.53485 | 1.01107 | 0.40579 | 0.28887 | 0.07680 | 0.18504 | 0.10366 | 0.00938 | 0.20289 | 0.19258 | 0.07680 |
| | 200 | 2.03033 | 1.53654 | 1.01142 | 0.37599 | 0.27231 | 0.06618 | 0.16631 | 0.09317 | 0.00650 | 0.18800 | 0.18154 | 0.06618 |
| | 500 | 2.02224 | 1.52710 | 1.00653 | 0.31262 | 0.22640 | 0.05315 | 0.12676 | 0.06940 | 0.00410 | 0.15631 | 0.15094 | 0.05315 |

**Table 3.** Simulation values of the AVEs, biases, MSEs, and MREs for $(\alpha = 0.75,\ \beta = 1,\ \lambda = 1.5)$.

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| MLEs | 20 | 5.00407 | 1.33856 | 1.69462 | 4.58584 | 0.58300 | 0.61194 | 915.57192 | 0.82702 | 1.79847 | 6.11446 | 0.58300 | 0.40796 |
| | 50 | 0.80009 | 1.10950 | 1.56565 | 0.25042 | 0.26100 | 0.28367 | 2.77923 | 0.12885 | 0.17539 | 0.33390 | 0.26100 | 0.18911 |
| | 100 | 0.75820 | 1.04701 | 1.52647 | 0.14670 | 0.16658 | 0.18688 | 0.03663 | 0.04629 | 0.05870 | 0.19561 | 0.16658 | 0.12459 |
| | 200 | 0.75105 | 1.02483 | 1.51570 | 0.09973 | 0.11192 | 0.12771 | 0.01602 | 0.02034 | 0.02670 | 0.13298 | 0.11192 | 0.08514 |
| | 500 | 0.75042 | 1.01062 | 1.50764 | 0.06399 | 0.07115 | 0.07960 | 0.00655 | 0.00813 | 0.01008 | 0.08532 | 0.07115 | 0.05307 |
| ADEs | 20 | 9.38732 | 1.06719 | 1.57006 | 8.88929 | 0.50246 | 0.56440 | 1880.45673 | 0.47327 | 1.06650 | 11.85239 | 0.50246 | 0.37627 |
| | 50 | 1.47665 | 1.00523 | 1.49807 | 0.89390 | 0.27826 | 0.29899 | 61.89330 | 0.12985 | 0.15880 | 1.19187 | 0.27826 | 0.19933 |
| | 100 | 0.82120 | 1.00558 | 1.49631 | 0.20048 | 0.18789 | 0.19384 | 0.71525 | 0.05817 | 0.06173 | 0.26731 | 0.18789 | 0.12922 |
| | 200 | 0.77776 | 1.00022 | 1.49962 | 0.12315 | 0.13102 | 0.13796 | 0.02781 | 0.02756 | 0.03125 | 0.16420 | 0.13102 | 0.09198 |
| | 500 | 0.76053 | 0.99923 | 1.49858 | 0.07278 | 0.08147 | 0.08400 | 0.00888 | 0.01041 | 0.01126 | 0.09704 | 0.08147 | 0.05600 |
| CVMEs | 20 | 17.17150 | 1.20607 | 1.94422 | 16.72239 | 0.71332 | 0.99773 | 3834.81526 | 1.07678 | 20.16972 | 22.29651 | 0.71332 | 0.66515 |
| | 50 | 3.15400 | 1.03508 | 1.53326 | 2.60571 | 0.38905 | 0.39081 | 273.18805 | 0.26011 | 0.31904 | 3.47427 | 0.38905 | 0.26054 |
| | 100 | 0.98389 | 1.01821 | 1.50993 | 0.39120 | 0.26162 | 0.25295 | 7.71210 | 0.11351 | 0.11368 | 0.52160 | 0.26162 | 0.16863 |
| | 200 | 0.79328 | 1.00765 | 1.50508 | 0.15907 | 0.17208 | 0.16963 | 0.05690 | 0.04793 | 0.04686 | 0.21209 | 0.17208 | 0.11309 |
| | 500 | 0.76726 | 1.00035 | 1.50054 | 0.09351 | 0.10738 | 0.10484 | 0.01522 | 0.01817 | 0.01748 | 0.12467 | 0.10738 | 0.06989 |
| LSEs | 20 | 15.51388 | 1.11072 | 1.75752 | 15.06375 | 0.66087 | 0.84020 | 2945.08220 | 0.87576 | 5.92644 | 20.08500 | 0.66087 | 0.56013 |
| | 50 | 2.90779 | 1.00897 | 1.50329 | 2.36165 | 0.37711 | 0.37609 | 220.69516 | 0.24414 | 0.31469 | 3.14887 | 0.37711 | 0.25073 |
| | 100 | 0.95680 | 0.99714 | 1.49358 | 0.35849 | 0.25029 | 0.24414 | 5.02891 | 0.10431 | 0.10430 | 0.47799 | 0.25029 | 0.16276 |
| | 200 | 0.78921 | 1.00561 | 1.50073 | 0.15660 | 0.17016 | 0.16939 | 0.05349 | 0.04723 | 0.04736 | 0.20881 | 0.17016 | 0.11293 |
| | 500 | 0.76231 | 1.00428 | 1.50377 | 0.09246 | 0.10792 | 0.10395 | 0.01419 | 0.01836 | 0.01722 | 0.12328 | 0.10792 | 0.06930 |
| WLSEs | 20 | 10.83923 | 1.05075 | 1.61118 | 10.35417 | 0.56427 | 0.66276 | 1971.92149 | 0.63157 | 4.08047 | 13.80556 | 0.56427 | 0.44184 |
| | 50 | 1.40103 | 1.01544 | 1.50136 | 0.83711 | 0.31227 | 0.32132 | 67.46259 | 0.16992 | 0.20701 | 1.11615 | 0.31227 | 0.21421 |
| | 100 | 0.81149 | 1.00615 | 1.50063 | 0.19688 | 0.20227 | 0.20938 | 0.09490 | 0.06721 | 0.07468 | 0.26250 | 0.20227 | 0.13959 |
| | 200 | 0.77457 | 1.00254 | 1.50089 | 0.12427 | 0.13633 | 0.13784 | 0.02785 | 0.03006 | 0.03156 | 0.16569 | 0.13633 | 0.09189 |
| | 500 | 0.75926 | 1.00120 | 1.49797 | 0.07363 | 0.08283 | 0.08370 | 0.00882 | 0.01088 | 0.01110 | 0.09817 | 0.08283 | 0.05580 |

**Table 4.** Simulation values of the AVEs, biases, MSEs, and MREs for $(\alpha = 1,\ \beta = 0.5,\ \lambda = 0.75)$.

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| MLEs | 20 | 10.66634 | 0.69815 | 0.70434 | 10.15056 | 0.35036 | 0.20796 | 2560.59384 | 0.29186 | 0.10072 | 10.15056 | 0.70073 | 0.27728 |
| | 50 | 1.67072 | 0.55715 | 0.73867 | 0.97109 | 0.15862 | 0.10938 | 63.19275 | 0.04454 | 0.02101 | 0.97109 | 0.31725 | 0.14584 |
| | 100 | 1.02670 | 0.53081 | 0.74374 | 0.24073 | 0.09970 | 0.07498 | 0.51058 | 0.01665 | 0.00921 | 0.24073 | 0.19939 | 0.09997 |
| | 200 | 1.00792 | 0.51336 | 0.74574 | 0.15385 | 0.06640 | 0.04930 | 0.04167 | 0.00727 | 0.00395 | 0.15385 | 0.13280 | 0.06573 |
| | 500 | 1.00540 | 0.50468 | 0.74840 | 0.09867 | 0.04157 | 0.03114 | 0.01568 | 0.00274 | 0.00153 | 0.09867 | 0.08313 | 0.04152 |
| ADEs | 20 | 17.29534 | 0.53508 | 0.72786 | 16.65270 | 0.29523 | 0.16557 | 3593.12529 | 0.15502 | 0.06100 | 16.65270 | 0.59046 | 0.22076 |
| | 50 | 3.39687 | 0.50182 | 0.73745 | 2.65286 | 0.17299 | 0.09914 | 273.28185 | 0.04915 | 0.01733 | 2.65286 | 0.34597 | 0.13219 |
| | 100 | 1.35940 | 0.49382 | 0.74544 | 0.54678 | 0.11664 | 0.06810 | 18.76243 | 0.02211 | 0.00775 | 0.54678 | 0.23328 | 0.09080 |
| | 200 | 1.05195 | 0.50079 | 0.74686 | 0.19624 | 0.07866 | 0.04900 | 0.08054 | 0.00984 | 0.00393 | 0.19624 | 0.15732 | 0.06533 |
| | 500 | 1.02297 | 0.49892 | 0.74925 | 0.12031 | 0.05114 | 0.03129 | 0.02495 | 0.00412 | 0.00156 | 0.12031 | 0.10228 | 0.04172 |
| CVMEs | 20 | 28.80055 | 0.60443 | 0.72876 | 28.22486 | 0.42195 | 0.18950 | 6531.10923 | 0.35574 | 0.13008 | 28.22486 | 0.84390 | 0.25266 |
| | 50 | 6.61153 | 0.53382 | 0.73173 | 5.93043 | 0.24163 | 0.10781 | 752.14533 | 0.09964 | 0.02226 | 5.93043 | 0.48326 | 0.14374 |
| | 100 | 1.92448 | 0.50863 | 0.73822 | 1.16477 | 0.16055 | 0.07253 | 55.85547 | 0.04206 | 0.00923 | 1.16477 | 0.32110 | 0.09670 |
| | 200 | 1.10588 | 0.50426 | 0.74532 | 0.28650 | 0.10743 | 0.04998 | 0.26844 | 0.01880 | 0.00420 | 0.28650 | 0.21486 | 0.06664 |
| | 500 | 1.03365 | 0.50083 | 0.74742 | 0.15593 | 0.06740 | 0.03104 | 0.04528 | 0.00716 | 0.00155 | 0.15593 | 0.13480 | 0.04138 |
| LSEs | 20 | 24.81221 | 0.56144 | 0.73570 | 24.23066 | 0.38075 | 0.18041 | 5010.98830 | 0.29532 | 0.19139 | 24.23066 | 0.76149 | 0.24055 |
| | 50 | 6.35294 | 0.51441 | 0.73189 | 5.66583 | 0.22901 | 0.10271 | 695.64053 | 0.09017 | 0.02012 | 5.66583 | 0.45802 | 0.13694 |
| | 100 | 1.69405 | 0.50453 | 0.74100 | 0.93870 | 0.15992 | 0.07136 | 32.30933 | 0.04220 | 0.00901 | 0.93870 | 0.31984 | 0.09514 |
| | 200 | 1.15725 | 0.49988 | 0.74634 | 0.34226 | 0.11035 | 0.04861 | 1.99838 | 0.01957 | 0.00396 | 0.34226 | 0.22069 | 0.06482 |
| | 500 | 1.03342 | 0.49961 | 0.74872 | 0.15567 | 0.06693 | 0.03105 | 0.04564 | 0.00716 | 0.00156 | 0.15567 | 0.13386 | 0.04140 |
| WLSEs | 20 | 18.93983 | 0.53275 | 0.73201 | 18.32293 | 0.33507 | 0.16053 | 3681.49435 | 0.22646 | 0.07512 | 18.32293 | 0.67014 | 0.21404 |
| | 50 | 3.21300 | 0.49467 | 0.73714 | 2.47225 | 0.18704 | 0.09678 | 240.64885 | 0.05822 | 0.01700 | 2.47225 | 0.37407 | 0.12904 |
| | 100 | 1.18486 | 0.50121 | 0.74353 | 0.38732 | 0.12284 | 0.07002 | 2.46525 | 0.02467 | 0.00835 | 0.38732 | 0.24568 | 0.09336 |
| | 200 | 1.06726 | 0.49571 | 0.74710 | 0.20810 | 0.08176 | 0.05000 | 0.09630 | 0.01085 | 0.00405 | 0.20810 | 0.16353 | 0.06666 |
| | 500 | 1.01595 | 0.50088 | 0.74943 | 0.11884 | 0.05091 | 0.03099 | 0.02346 | 0.00408 | 0.00154 | 0.11884 | 0.10181 | 0.04132 |

**Table 5.** Simulation values of the AVEs, biases, MSEs, and MREs for $(\alpha = 0.5,\ \beta = 2,\ \lambda = 3)$.

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| MLE | 20 | 1.88556 | 2.59014 | 3.99837 | 1.58797 | 0.96638 | 1.72681 | 392.11695 | 2.04166 | 37.32390 | 3.17594 | 0.48319 | 0.57560 |
| | 50 | 0.50674 | 2.18512 | 3.24298 | 0.12822 | 0.45987 | 0.73043 | 0.02901 | 0.38797 | 1.07395 | 0.25645 | 0.22994 | 0.24348 |
| | 100 | 0.50481 | 2.08042 | 3.10209 | 0.08770 | 0.29555 | 0.47206 | 0.01292 | 0.14872 | 0.38945 | 0.17541 | 0.14778 | 0.15735 |
| | 200 | 0.50165 | 2.03826 | 3.05662 | 0.06187 | 0.20242 | 0.32396 | 0.00612 | 0.06561 | 0.17325 | 0.12375 | 0.10121 | 0.10799 |
| | 500 | 0.50005 | 2.01613 | 3.02382 | 0.03799 | 0.12291 | 0.20007 | 0.00226 | 0.02407 | 0.06423 | 0.07597 | 0.06146 | 0.06669 |
| AD | 20 | 3.96553 | 2.11542 | 3.34479 | 3.61726 | 0.85448 | 1.44437 | 535.50972 | 1.33161 | 6.85513 | 7.23452 | 0.42724 | 0.48146 |
| | 50 | 0.60271 | 2.03972 | 3.06957 | 0.20560 | 0.46733 | 0.73634 | 2.72839 | 0.36640 | 1.03220 | 0.41120 | 0.23366 | 0.24545 |
| | 100 | 0.52541 | 2.01725 | 3.02193 | 0.10073 | 0.32557 | 0.49841 | 0.01861 | 0.17145 | 0.41353 | 0.20145 | 0.16279 | 0.16614 |
| | 200 | 0.50979 | 2.01459 | 3.01119 | 0.06788 | 0.22917 | 0.34138 | 0.00783 | 0.08375 | 0.19032 | 0.13575 | 0.11458 | 0.11379 |
| | 500 | 0.50379 | 2.00337 | 3.00178 | 0.04195 | 0.14183 | 0.21564 | 0.00281 | 0.03185 | 0.07349 | 0.08391 | 0.07092 | 0.07188 |

**Table 5.** *Cont.*

| Method | n | AVEs | | | Bias | | | MSEs | | | MREs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| CV | 20 | 7.57233 | 2.31764 | 7.46770 | 7.24915 | 1.19529 | 5.75209 | 1160.36446 | 2.98504 | 33,000.03944 | 14.49830 | 0.59764 | 1.91736 |
| | 50 | 1.00678 | 2.09441 | 3.26840 | 0.63006 | 0.63320 | 1.07383 | 56.85043 | 0.71505 | 3.56632 | 1.26012 | 0.31660 | 0.35794 |
| | 100 | 0.53923 | 2.04312 | 3.08599 | 0.13104 | 0.41775 | 0.64473 | 0.04556 | 0.28696 | 0.76081 | 0.26208 | 0.20887 | 0.21491 |
| | 200 | 0.51562 | 2.02613 | 3.04866 | 0.08297 | 0.28403 | 0.43812 | 0.01229 | 0.13022 | 0.33084 | 0.16594 | 0.14201 | 0.14604 |
| | 500 | 0.50685 | 2.00668 | 3.01755 | 0.05017 | 0.17464 | 0.26905 | 0.00421 | 0.04904 | 0.11532 | 0.10035 | 0.08732 | 0.08968 |
| LS | 20 | 6.74249 | 2.15412 | 4.02082 | 6.42494 | 1.10222 | 2.39731 | 970.48582 | 2.38090 | 111.44446 | 12.84988 | 0.55111 | 0.79910 |
| | 50 | 0.87003 | 2.06527 | 3.17638 | 0.49691 | 0.60725 | 1.00070 | 34.32248 | 0.65355 | 2.77405 | 0.99381 | 0.30362 | 0.33357 |
| | 100 | 0.56616 | 2.01473 | 3.04200 | 0.15686 | 0.40511 | 0.63364 | 2.37028 | 0.26999 | 0.74052 | 0.31371 | 0.20255 | 0.21121 |
| | 200 | 0.51501 | 2.01103 | 3.02743 | 0.08287 | 0.28648 | 0.43391 | 0.01198 | 0.12981 | 0.31715 | 0.16575 | 0.14324 | 0.14464 |
| | 500 | 0.50621 | 2.00366 | 3.00350 | 0.05139 | 0.17942 | 0.26784 | 0.00441 | 0.05089 | 0.11674 | 0.10279 | 0.08971 | 0.08928 |
| WLS | 20 | 3.88754 | 2.10038 | 3.54341 | 3.55136 | 0.94658 | 1.83658 | 680.43476 | 1.76249 | 26.14833 | 7.10272 | 0.47329 | 0.61219 |
| | 50 | 0.59300 | 2.03469 | 3.05918 | 0.20356 | 0.51373 | 0.78963 | 2.39966 | 0.45358 | 1.29793 | 0.40711 | 0.25687 | 0.26321 |
| | 100 | 0.52105 | 2.01904 | 3.03118 | 0.10138 | 0.33352 | 0.52531 | 0.01853 | 0.18159 | 0.48072 | 0.20276 | 0.16676 | 0.17510 |
| | 200 | 0.50895 | 2.01393 | 3.02204 | 0.06985 | 0.23696 | 0.35833 | 0.00822 | 0.08908 | 0.21367 | 0.13969 | 0.11848 | 0.11944 |
| | 500 | 0.50324 | 2.00546 | 3.00805 | 0.04257 | 0.14463 | 0.21777 | 0.00287 | 0.03274 | 0.07518 | 0.08514 | 0.07232 | 0.07259 |

## 6. Applications

This section is devoted to analyzing a real dataset from the insurance field. The dataset represents losses from a private passenger in the United Kingdom (U.K.) automobile insurance policies. It consists of four variables, and we studied the variable number three in particular. It is available in the R©software library. The aim of this section is to show that the proposed model can be provide the best fit to insurance data compared to other competitive extensions of the exponential distribution.

We compare the proposed IPLE distribution with some other competing distributions, including the beta exponential (BE) [31], transmuted generalized exponential (TGE) [4], odd inverse Pareto exponential (OIPE) [16], exponentiated exponential (EE) [1], alpha power exponentiated exponential (APEE) [19], generalized odd log-logistic exponential (GLLE) [13], logistic exponential (LE) [21], alpha power exponential (APE) [9], Marshall–Olkin exponential (MOE) [32], Weibull (W), transmuted exponential (TE) [33], and exponential (E) distributions.

The competing models were compared based on some discrimination measures namely the Akaike information (AI-C) Akaike [34], consistent Akaike information (CAI-C) Sugiura [35], Hannan–Quinn information (HQI-C) [36], and the Bayesian information (BI-C) [37] criteria. Further discrimination measures include the Cramér–von Mises (CVM), Anderson–Darling (AD) [38], and Kolmogorov–Smirnov (K-S) with its $p$-value. The formulae of these measures were mentioned in [39].

The simulation results show that the maximum likelihood method provides accurate estimates for the IPLE parameters. Hence, the maximum likelihood is adopted here to estimate the parameters of the IPLE model and other competing models. The MLEs and the goodness-of-fit measures are calculated using the Wolfram Mathematica software Version 10. Tables 6 and 7 report the analytical measures, MLEs, and their standard errors. The results in Tables 6 and 7 illustrate that the IPLE distribution provides the best fit to insurance data compared to other competing distributions, and hence, it can be an adequate distribution to analyze heavy-tailed insurance data. The results in these tables were obtained based on the real insurance data using the Wolfram Mathematica software.

The fitted PDF, CDF, SF, and P-Pplots of the IPLP distribution for the analyzed dataset are depicted in Figure 4.

**Table 6.** Discrimination measures of the IPLE model and other fitted models. CAI-C, consistent Akaike information criterion; BI-C, Bayesian information criterion; HQI-C, Hannan–Quinn information criterion; BE, beta exponential; TGE, transmuted generalized exponential; OIPE, odd inverse Pareto exponential; GLLE, generalized odd log-logistic exponential; APEE, alpha power exponentiated exponential; MOE, Marshall–Olkin exponential; TE, transmuted exponential.

| Model | -L | AI-C | CAI-C | BI-C | HQI-C | Estimates |
|---|---|---|---|---|---|---|
| IPLE | 180.351 | 366.702 | 367.559 | 371.099 | 368.164 | $\hat{\alpha} = 1.52792$ (0.77685)<br>$\hat{\beta} = 3.15494$ (1.42388)<br>$\hat{\lambda} = 2.65845 \times 10^7$ $(2.08706 \times 10^8)$ |
| BE | 181.626 | 369.252 | 370.109 | 373.649 | 370.71 | $\hat{\lambda} = 0.03202$ (0.000038)<br>$\hat{a} = 510.847$ (0.08904)<br>$\hat{b} = 0.39293$ (4.87011) |
| TGE | 181.851 | 369.702 | 370.559 | 374.099 | 371.159 | $\hat{\alpha} = 35.9735$ (20.016)<br>$\hat{\lambda} = 0.74507$ (0.29061)<br>$\hat{\theta} = 0.01357$ (0.00255) |
| OIPE | 182.661 | 371.321 | 372.178 | 375.718 | 372.779 | $\hat{\alpha} = 2004.08$ $(19,007.7)$<br>$\hat{\lambda} = 0.01671$ (0.00262)<br>$\hat{\beta} = 0.02631$ (0.25009) |
| EE | 182.724 | 369.447 | 369.861 | 372.379 | 370.419 | $\hat{\alpha} = 56.611$ (0.01696)<br>$\hat{\lambda} = 2.65845$ (0.00253) |
| APEE | 182.876 | 371.752 | 372.609 | 376.149 | 373.21 | $\hat{\alpha} = 1.90787 \times 10^{11}$ $(2.27727 \times 10^{12})$<br>$\hat{a} = 0.01729$ (0.00261)<br>$\hat{c} = 2.41738$ (1.91861) |
| GLLE | 183.422 | 370.845 | 371.258 | 373.776 | 371.816 | $\hat{\alpha} = 4.6185$ (0.68532)<br>$\hat{\lambda} = 0.00268$ (0.00013) |
| LE | 183.422 | 370.845 | 371.258 | 373.776 | 371.816 | $\hat{\alpha} = 0.00268$ (0.68532)<br>$\hat{\lambda} = 2.65845$ (0.00013) |
| APE | 183.787 | 371.574 | 371.987 | 374.505 | 372.545 | $\hat{\alpha} = 1.2042 \times 10^{12}$ $(3.72879 \times 10^{12})$<br>$\hat{a} = 0.01407$(0.00088) |
| MOE | 187.319 | 378.638 | 379.052 | 381.57 | 379.61 | $\hat{\alpha} = 340.079$ (319.528)<br>$\hat{a} = 0.02237$ (0.00349) |
| W | 194.425 | 392.85 | 393.264 | 395.782 | 393.822 | $\hat{a} = 2.46021$ (0.27047)<br>$\hat{b} = 309.814$ (23.7393) |
| TE | 202.028 | 408.056 | 408.47 | 410.988 | 409.028 | $\hat{\beta} = 0.00528$ (0.00099)<br>$\hat{\lambda} = -1.0000$ (0.78162) |
| E | 211.894 | 425.787 | 425.921 | 427.253 | 426.273 | $\hat{\lambda} = 0.00362$ (0.00064) |

**Table 7.** The Anderson–Darling (AD), Cramér–von Mises (CVM), K-S, and $p$-value of the IPLE model and other fitted models.

| Model | AD | CVM | K-S | $p$-Value |
|---|---|---|---|---|
| IPLE | 0.32689 | 0.03905 | 0.08389 | 0.97794 |
| BE | 0.49015 | 0.05744 | 0.10498 | 0.87228 |
| TGE | 0.47873 | 0.05966 | 0.11582 | 0.78399 |
| OIPE | 0.56721 | 0.07179 | 0.12591 | 0.69073 |
| EE | 0.56576 | 0.07133 | 0.12591 | 0.69072 |
| APEE | 0.57419 | 0.07242 | 0.12691 | 0.68125 |
| GLLE | 0.60687 | 0.070002 | 0.11695 | 0.77389 |

**Table 7.** *Cont.*

| Model | AD | CVM | K-S | *p*-Value |
|---|---|---|---|---|
| LE | 0.60687 | 0.07000 | 0.11695 | 0.77389 |
| APE | 0.94303 | 0.13285 | 0.13769 | 0.57886 |
| MOE | 0.96658 | 0.11006 | 0.14402 | 0.52043 |
| W | 2.84585 | 0.45267 | 0.23395 | 0.06022 |
| TE | 5.46613 | 1.06381 | 0.37487 | 0.00025 |
| E | 8.12694 | 1.71592 | 0.46956 | $1.48825 \times 10^{-6}$ |

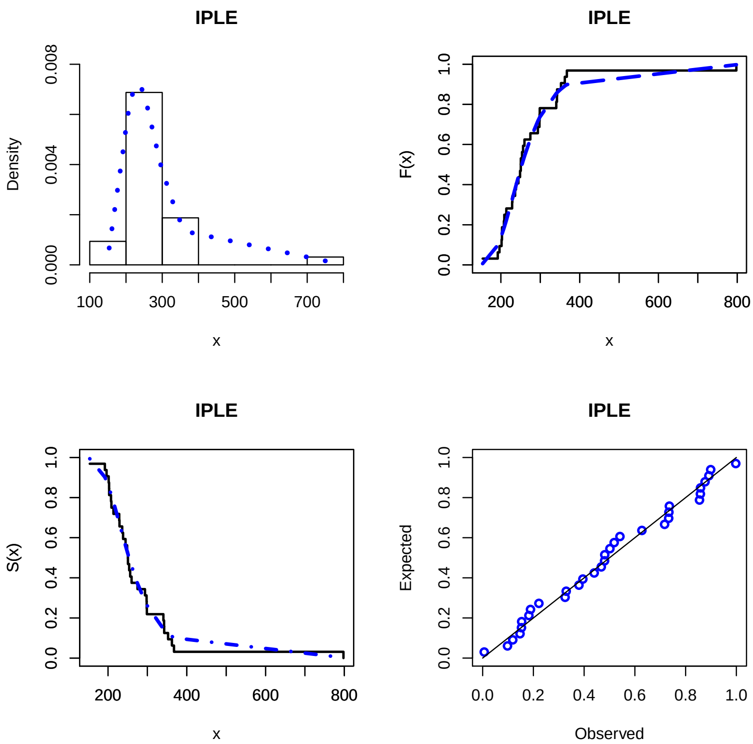


**Figure 4.** Histogram of the insurance dataset with the fitted IPLE PDF, CDF, survival function (SF), and P-P plot.

## 7. Conclusions

This paper proposes a new three parameter distribution, called the inverse power logistic exponential (IPLE) distribution, which can be used to model heavy-tailed data in insurance and other applied areas. The IPLE model generalizes the inverse Weibull, inverse logistic exponential, inverse Rayleigh, and inverse exponential distributions. The hazard rate function of the IPLE distribution can be decreasing, unimodal, increasing, and reversed-J-shaped, and J-shaped. Some of its mathematical properties were derived. The unknown parameters of the IPLE model were estimated by five classical estimators, called the maximum likelihood estimators, Anderson–Darling estimators, least-squares estimators, Cramér–von Mises estimators, and weighted least-squares estimators. The simulation results showed that all estimators perform very well in estimating the parameters of the IPLE distribution. Based on our study, the maximum likelihood method provided accurate estimates for the parameters of the IPLE distribution. The practical importance of the IPLE distribution was illustrated using real insurance data, showing its adequate fits and superiority as compared with other competing existing models. The proposed model may be used effectively in modeling data in several applied areas such as medicine, economics, reliability, life testing, and engineering, among others.

The work of this paper can be extended in some ways. For example, the IPLE parameters can be estimated under different censoring schemes using classical and Bayesian estimation. Exponentiated or transmuted versions of the IPLE model can be established, and a bivariate extension of it may also be studied. The application of the IPLE model may also be explored in other applied areas.

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## References

1. Gupta, R.D.; Kundu, D. Generalized exponential distribution: Different method of estimations. *J. Stat. Comput. Simul.* **2001**, *69*, 315–337. [CrossRef]
2. Jones, M.C. Families of distributions arising from distributions of order statistics. *Test* **2004**, *13*, 1–43. [CrossRef]
3. Barreto-Souza, W.; Santos, A.H.; Cordeiro, G.M. The beta generalized exponential distribution. *J. Stat. Comput. Simul.* **2010**, *80*, 159–172. [CrossRef]
4. Khan, M.S.; King, R.; Hudson, I.L. Transmuted generalized exponential distribution: A generalization of the exponential distribution with applications to survival data. *Commun. Stat. Simul. Comput.* **2017**, *46*, 4377–4398. [CrossRef]
5. Pinho, L.G.B.; Cordeiro, G.M.; Nobre, J.S. The Harris extended exponential distribution. *Commun. Stat. Theory Methods* **2015**, *44*, 3486–3502. [CrossRef]
6. Afify, A.Z.; Cordeiro, G.M.; Yousof, H.M.; Alzaatreh, A.; Nofal, Z.M. The Kumaraswamy transmuted-G family of distributions: Properties and applications. *J. Data Sci.* **2016**, *14*, 245–270.
7. Lemonte, A.J.; Cordeiro, G.M.; Moreno-Arenas, G. A new useful three-parameter extension of the exponential distribution. *Statistics* **2016**, *50*, 312–337. [CrossRef]
8. Rasekhi, M.; Alizadeh, M.; Altun, E.; Hamedani, G.G.; Afify, A.Z.; Ahmad, M. The modified exponential distribution with applications. *Pak. J. Stat.* **2017**, *33*, 383–398.
9. Mahdavi, A.; Kundu, D. A new method for generating distributions with an application to exponential distribution. *Commun. Stat. Theory Methods* **2017**, *46*, 6543–6557. [CrossRef]
10. Nassar, M.; Afify, A.Z.; Shakhatreh, M. Estimation methods of alpha power exponential distribution with applications to engineering and medical data. *Pak. J. Stat. Oper. Res.* **2020**, *16*, 149–166. [CrossRef]
11. Afify, A.Z.; Zayed, M.; Ahsanullah, M. The extended exponential distribution and its applications. *J. Stat. Theory Appl.* **2018**, *17*, 213–229. [CrossRef]
12. Mansoor, M.; Tahir, M.H.; Cordeiro, G.M.; Provost, S.B.; Alzaatreh, A. The Marshall–Olkin logistic-exponential distribution. *Commun. Stat. Theory Methods* **2019**, *48*, 220–234. [CrossRef]
13. Afify, A.Z.; Suzuki, A.K.; Zhang, C.; Nassar, M. On three-parameter exponential distribution: Properties, bayesian and non-bayesian estimation based on complete and censored samples. *Commun. Stat. Simul. Comput.* **2019**. [CrossRef]
14. Nassar, M.; Kumar, D.; Dey, S.; Cordeiro, G.M.; Afify, A.Z. The Marshall–Olkin alpha power family of distributions with applications. *J. Comput. Appl. Math.* **2019**, *351*, 41–53. [CrossRef]
15. Afify, A.Z.; Mohamed, O.A. A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications. *Mathematics* **2020**, *8*, 135. [CrossRef]
16. Aldahlan, M.A.; Afify, A.Z. A new three-parameter exponential distribution with applications in reliability and engineering. *J. Nonlinear Sci. Appl.* **2020**, *13*, 258–269. [CrossRef]
17. Al-Babtain, A.A.; Shakhatreh, M.K.; Nassar, M.; Afify, A.Z. A New Modified Kies Family: Properties, Estimation Under Complete and Type-II Censored Samples, and Engineering Applications. *Mathematics* **2020**, *8*, 1345. [CrossRef]
18. Abbas, S.; Jahngeer, A.; Shahbaz, S.; Afify, A.Z.; Shahbaz, M. Topp-Leone moment exponential distribution: Properties and applications. *J. Natl. Sci. Found. Sri Lanka* **2020**, *48*. Available online: https://jnsfsl.sljol.info/articles/abstract/8836/ (access on 7 September 2020).
19. Afify, A.Z.; Gemeay, A.M.; Ibrahim, N.A. The heavy-tailed exponential distribution: Risk measures, estimation, and application to actuarial data. *Mathematics* **2020**, *8*, 1276. [CrossRef]

20. Alizadeh, M.; Afify, A.Z.; Eliwa, M.S.; Ali, S. The odd log-logistic Lindley-G family of distributions: Properties, Bayesian and non-Bayesian estimation with applications. *Comput. Stat.* **2020**, *35*, 281–308. [CrossRef]
21. Lan, Y.; Leemis, L.M. The logistic-exponential survival distribution. *Nav. Res. Logist.* **2008**, *55*, 252–264. [CrossRef]
22. Alkarni, S.H. Extended inverse lindley distribution: Properties and application. *Springer-Plus* **2015**, *4*, 670. [CrossRef] [PubMed]
23. Mead, M. Generalized inverse gamma distribution and its applications in reliability. *Commun. Stat. Theory Method* **2015**, *44*, 1426–1435. [CrossRef]
24. Sharma, V.K.; Singh, S.K.; Singh, U.S.; Agiwal, V. The inverse lindley distribution: A stress-strength reliability model with application to head and neck cancer data. *J. Ind. Prod. Eng.* **2015**, *32*, 162–173. [CrossRef]
25. Barco, K.V.P.; Mazucheli, J.; Janeiro, V. The inverse power lindley distribution. *Commun. Stat. Simul. Comput.* **2017**, *46*, 6308–6323. [CrossRef]
26. Nassar, M.; Afify, A.Z.; Dey, S.; Kumar, D. A new extension of weibull distribution: Properties and different methods of estimation. *J. Comput. Appl. Math.* **2018**, *336*, 439–457. [CrossRef]
27. Afify, A.Z.; Nassar, M.; Cordeiro, G.M.; Kumar, D. The Weibull Marshall–Olkin Lindley distribution: Properties and estimation. *J. Taibah Univ. Sci.* **2020**, *14*, 192–204. [CrossRef]
28. Sen, S.; Afify, A.Z.; Al-Mofleh, H.; Ahsanullah, M. The quasi xgamma-geometric distribution with application in medicine. *Filomat* **2019**, *33*, 5291–5330. [CrossRef]
29. Nassar, M.; Afify, A.Z.; Shakhatreh, M.; Dey, S. On a new extension of Weibull distribution: Properties, estimation, and applications to one and two causes of failures. *Qual. Reliab. Eng. Int.* **2020**, *36*, 2019–2043. [CrossRef]
30. Al-Mofleh, H.; Afify, A.Z.; Ibrahim, N.A. A new extended two-parameter distribution: Properties, estimation methods, and applications in medicine and geology. *Mathematics* **2020**, *8*, 1578. [CrossRef]
31. Nadarajah, S.; Kotz, S. The beta exponential distribution. *Reliab. Eng. Syst. Saf.* **2006**, *91*, 689–697. [CrossRef]
32. Marshall, A.W.; Olkin, I. A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika* **1997**, *84*, 641–652. [CrossRef]
33. Tian, Y.; Tian, M.; Zhu, Q. Transmuted linear exponential distribution: A new generalization of the linear exponential distribution. *Commun. Stat. Simul. Comput.* **2014**, *43*, 2661–2677. [CrossRef]
34. Akaike, H. Information theory and an extension of the maximum likelihood principle. In *International Symposium on Information Theory*, 2nd ed.; Armenian SSR: Tsahkadsor, Armenia, 1973; pp. 267–281.
35. Sugiura, N. Further analysts of the data by Akaike's information criterion and the finite corrections: Further analysts of the data by Akaike's. *Commun. Stat. Theory Methods* **1978**, *7*, 13–26. [CrossRef]
36. Hannan, E.J.; Quinn, B.G. The determination of the order of an autoregression. *J. R. Stat. Soc. Ser. B* **1979**, *41*, 190–195. [CrossRef]
37. Schwarz, G. Estimating the dimension of a model. *Ann. Stat.* **1978**, *6*, 461–464. [CrossRef]
38. Anderson, T.W.; Darling, D.A. Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. *Ann. Math. Stat.* **1952**, *23*, 193–212. [CrossRef]
39. Afify, A.Z.; Cordeiro, G.M.; Yousof, H.; Saboor, A.; Ortega, E.M.M. The Marshall–Olkin additive Weibull distribution with variable shapes for the hazard rate. *Hacettepe J. Math. Stat* **2018**, *47*, 365–381. [CrossRef]


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