mathematics

Article

Fractional Derivatives and Integrals: What Are They Needed For?

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Received: 19 December 2019; Accepted: 22 January 2020; Published: 25 January 2020

Abstract: The question raised in the title of the article is not philosophical. We do not expect general answers of the form "to describe the reality surrounding us". The question should actually be formulated as a mathematical problem of applied mathematics, a task for new research. This question should be answered in mathematically rigorous statements about the interrelations between the properties of the operator's kernels and the types of phenomena. This article is devoted to a discussion of the question of what is fractional operator from the point of view of not pure mathematics, but applied mathematics. The imposed restrictions on the kernel of the fractional operator should actually be divided by types of phenomena, in addition to the principles of self-consistency of mathematical theory. In applications of fractional calculus, we have a fundamental question about conditions of kernels of fractional operator of non-integer orders that allow us to describe a particular type of phenomenon. It is necessary to obtain exact correspondences between sets of properties of kernel and type of phenomena. In this paper, we discuss the properties of kernels of fractional operators to distinguish the following types of phenomena: fading memory (forgetting) and power-law frequency dispersion, spatial non-locality and power-law spatial dispersion, distributed lag (time delay), distributed scaling (dilation), depreciation, and aging.

Keywords: fractional calculus; fractional derivative; translation operator; distributed lag; time delay; scaling; dilation; memory; depreciation; probability distribution

MSC: 26A33 Fractional derivatives and integrals; 34A08 Fractional differential equations; 60E05 Distributions: general theory

1. Introduction

Why do we need fractional derivatives and integrals of non-integer order? We are not interested in the answer from the standpoint of philosophy or methodology of science. We are primarily interested in the answer from the point of view of applied mathematics, theoretical physics, economic theory, and other applied sciences. For application of fractional calculus [1–7], we want to have an answer in the form of exact mathematical statements that is formulated in precise and strict form. To get such an answer, it is required to formulate the question in mathematical form. The question should actually be formulated as a mathematical problem of applied mathematics, as a task for new research.

We also do not plan to delve into the "linguistic" question of which operators might be called fractional and which are not. The first author has already formulated their point of view on this issue in articles [8–12]. There are also many important contributions to this discussion (for example, see [13–16]). In this article, we do not plan to continue the discussion directly in this direction. We want to direct our discussion in a different direction. However, we will make an important remark for this paper. Please note that the proposed principle "No nonlocality. No fractional derivative" [11] cannot be turned into the principle "No memory. No fractional derivative". This is due to the fact that nonlocality in time cannot be reduced only to memory (about the concept of memory, see for example in articles [17,18]). It should also be noted here that the operators that describe the delay, lag, and scaling continuously distributed over time cannot be attributed to fractional operators if the distribution is described by probability density functions. These operators are integer order operators with distributed delay, lag, and scaling.

The fractional calculus, which is the theory of integrals and derivatives of fractional order, describes a wide variety of different types of operators with non-integer order [1–7]. Fractional calculus allows us to describe various phenomena and effects in natural and social sciences. For example, we should note the non-locality of power-law type, spatial dispersion of power type, fading memory, frequency dispersion of power type, intrinsic dissipation, the openness of systems (interaction with environment), fractional relaxation-oscillation, fractional viscoelasticity, fractional diffusion-waves, long-range interactions of power-law type, and many others [19,20].

In applied mathematics, it is important to have a tool that allows you to adequately select the type of fractional operators for the type of phenomena under consideration. It is necessary to have clear mathematical criteria for associating fractional operators of non-integer orders and those types of phenomena that they can describe. Differential and integral operators of non-integer orders are a powerful tool for modeling and description of processes that characterized by fading memory and spatial nonlocality. However, not all operators of non-integer orders can describe the effects of memory (or non-locality).

We should emphasize that not all fractional derivatives and integrals can be used for modeling the processes with memory. For example, the Kober and Erdelyi–Kober operators as well as the Caputo–Fabrizio integral and derivatives cannot be applied to describe phenomena with memory or spatial nonlocality. These operators can be applied only to describe processes with continuously distributed scaling (dilation) and lag (delay), respectively [21,22]. We also can state that these operators can be interpreted as derivatives and integrals of integer orders with scaling or lag, distributions of which are described by some probability density functions [21,22].

In application of the differential and integral operators with non-integer orders, a fundamental question arises about the correct subject interpretation of the different types of operators. Interpretation is not in the form of a description of one of the particular manifestations of real processes, but by one or another type of phenomena. We should clearly understand what type of effects and phenomena a given fractional operator of non-integer order can describe.

It is necessary to understand what types of fractional operators, what types of phenomena can be described in principle. The most important role in this description of phenomena must be understood by what types of fractional derivatives and integrals of non-integer order, in principle, what types (classes) of phenomena can describe.

In applications of fractional calculus, we can distinguish the following types (classes) of phenomena by some properties of kernels:

- fading memory (forgetting) and power-law frequency dispersion;
- spatial non-locality and power-law spatial dispersion;
- distributed lag (time delay);
- dictributed scaling (dilation);
- depreciation and aging.

These types of phenomena can be described by fractional operators of non-integer orders with some types of operator kernels. For these types of phenomena, we should have mathematical conditions on the operator kernels, which uniquely identify one of types of these phenomena.

Let us give some examples of the correspondence between the some fractional derivatives (or integrals) and the type of phenomena, which can be described by these operators in Table 1.

Examples of these type of phenomena in physics are described in Handbook of Fractional Calculus with Application [19,20].

Table 1. Examples of the correspondence between the some fractional derivatives (or integrals) and the type of phenomena.

№	Type of Phenomena:	Example of Fractional Operators:
1	Memory and Non-Locality in Time	Caputo and Riemann–Liouville
2	Spatial Non-Locality and Spatial Dispersion	Riess and Liouville
3	Distributed Time Delay and Lag	Caputo–Fabrizio
4	Distributed Dilation and Scaling	Kober and Erdelyi–Kober, Gorenflo–Luchko–Mainardi
5	Distributed Depreciation and Aging	Prabhakar and Kilbas–Saigo–Saxena

In this paper, we proposed the properties of operator kernels and corresponding types of phenomena. In fractional calculus, we do not have a list of correspondence between mathematical properties of the operators kernels and types of effects and phenomena. Mathematically rigorous conditions on the kernels of fractional differential and integral operators are necessary to distinguish between different types of phenomena and processes.

First of all, we must clearly distinguish between types of fractional operators and types of phenomena. This should not be just a list of examples of specific manifestations in the different sciences. In fractional calculus, we should have correspondence between the types of phenomena and the types of properties of operator kernels. In this article, we will explain in more detail the proposed approach to the interpretation of fractional derivatives and integrals.

2. Formulation of Mathematical Problem

This article does not claim to be a general consideration of fractional derivatives and integrals. To simplify the discussion of fractional operators, we will consider operators with respect to one variable t, which will be interpreted as time. A discussion of the problem of the relationship between the types of phenomena and the types of fractional operators will be constructed on the example of the following operator

$$\Big(D_{(K)}f\Big)(t) = \int_{t_0}^{t} K(t,\tau)\Big(\mathcal{D}_{\tau}^{(n)} f(\tau)\Big)d\tau, \tag{1}$$

where $\Big(\mathcal{D}_{\tau}^{(n)} f\Big)(\tau)$ is differential operator of the integer order n, where $n = 0, 1, 2, \ldots$, and $K(t,\tau)$ is a kernel of the operator. For example, we can consider the standard derivative of the integer order n, i.e.,

$$\Big(\mathcal{D}_{\tau}^{(n)} f\Big)(\tau) = f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}. \tag{2}$$

In general, the kernel $K(t,\tau)$ depends on the order n and initial point t_0, i.e., we should use $K_{n,t_0}(t,\tau)$. To simplify the notation, we will use $K(t,\tau)$, assuming that n and t_0 are already fixed. Expression (1) has a sense, if the integral (1) exists. In general, the function $f^{(n)}(\tau)$ does not have to be continuous function and the kernel $K(t,\tau)$ can have an integrable singularity of some kind.

Remark 1. *In general, we can consider other differential operators* $\left(\mathcal{D}^{(n)}_{\tau} f\right)(\tau)$ *of the integer order n instead of the standard derivative* $f^{(n)}(\tau)$*. For example, we can consider the operators (1), where differential operator* $\left(\mathcal{D}^{(n)}_{\tau} f\right)(\tau)$ *is defined in the form.*

$$\left(\mathcal{D}^{(n)}_{\tau} f\right)(\tau) = \prod_{k=0}^{n}\left(1+\gamma+k+\beta^{-1}\tau\frac{d}{d\tau}\right), \tag{3}$$

which is used in the Gorenflo-Luchko-Mainardi (GLM) operator [23–25] with some parameters $\gamma \in \mathbb{R}$ *and* $\beta > 0$ *of the kernel* $K(t,\tau)$ *(for details see Equations (1) and (12) in [24], and Equations (4) and (39) in [25]). This operator is also known as the left-sided Caputo-type modification of the Erdelyi–Kober fractional derivative (see Equation (12) in [24] (p. 362)). Please note that the GLM operator was introduced for the first time in [23] in connection with the scale-invariant solutions of the time-fractional diffusion-wave equation (see Equation (58) on [23] (p. 188)). The special form (3) is needed in order to make this operator a left-inverse operator to the Erdelyi–Kober integral operator (see Equation (13) in [24] (p. 362)). Emphasize that the main property of any generalized (fractional) derivative is to be a left-inverse operator to the corresponding generalized (fractional) integral operator. Please note that the Kober and Erdelyi–Kober operators [1,4], as well as the Caputo–Fabrizio operators [26–28], cannot be applied for modeling processes with fading memory or spatial nonlocality. These operators can be used only to describe continuously distributed scaling (dilation) and lag (delay), respectively (sections below). Therefore we also can state that these operators are interpreted as derivatives and integrals of integer orders with scaling or lag, distributions of which are described by probability density functions.*

Remark 2. *We can also consider instead of* $\left(\mathcal{D}^{(n)}_{\tau} f\right)(\tau)$ *a fractional differential (or integral) operator* $\left(\mathcal{D}^{(\alpha)}_{\tau} f\right)(\tau)$ *of another type than the ones defined by the kernel* $K(t,\tau)$*. For example, we can use the Caputo fractional derivative,* $\left(\mathcal{D}^{(\alpha)}_{\tau} f\right)(\tau) = \left(D^{\alpha}_{C,0+} f\right)(\tau)$*, and the kernel is the probability density function of the gamma distriburion (for details, see the Section 7 of the article [29] and the papers [30–32]). Such a choice is necessary to describe the simultaneous presence of two such phenomena as distributed lag and fading memory.*

Let us give a formulation of a mathematical problem of applied mathematics, as task for new research in fractional calculus that will be illustrated in this paper below.

Mathematical problem of fractional calculus in application: What conditions must the kernel $K(t,\tau)$ of operator (1) have in order to describe one or another type of phenomena? It is necessary to obtain exact correspondences between sets of properties of kernel and type of phenomena.

In this paper, we describe the conditions on the kernel $K(t,\tau)$, which allow us to use operator of the form (1) to describe the following types of phenomena:

(Type I): Continuously Distributed Scaling (Dilation);

(Type II): Continuously Distributed Lag (Delay).

We also give some comments to the phenomena:

(Type III): Continuously Distributed Fading Memory;

(Type IV): Distributed Depreciation and Aging.

Let us give these conditions for phenomena of Types I and II in the form of the following statements. The conditions on the kernel $K(t,\tau)$ for phenomenon of Types III and IV are discussed in the separate sections of this paper.

Statement 1.

Let us assume that the kernel $K(t,\tau)$ of the operator (1) with $t_0 = 0$ satisfies the following conditions

$$K(\lambda t, \lambda\tau) = \lambda^{-1} K(t,\tau), \tag{4}$$

for all $\lambda > 0$, and the condition of non-negativity, and the normalization condition

$$K(1,x) \geq 0, \int_0^1 K(1,x)\, dx = K < \infty \tag{5}$$

for all $x \in (0,1)$, where K is a finite positive constant. In this case, operator (1) can be represented (by using the change of variable $\tau \to x = \tau/t$) in the form

$$\Big(D_{(K)} f\Big)(t) = K \int_0^1 \rho_1(x)\; S_x\Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big) dx \tag{6}$$

with a numerical factor K, where $\rho_1(x) = K(1,x)/K$ is the probability density function that satisfies the condition of non-negativity and the normalization condition

$$\rho_1(x) \geq 0, \int_0^1 \rho_1(x)\, dx = 1, \tag{7}$$

and S_x is the scaling (dilation) operator

$$\begin{gathered} S_x f(t) = f(t{\cdot}x), \\ S_x\Big(\mathcal{D}_z^{(n)} f(z)\Big) = \Big(\mathcal{D}_z^{(n)} f(z)\Big)_{z=t\cdot x}, \\ S_x f^{(n)}(t) = \left(\frac{d^n f(z)}{dz^n}\right)_{z=t\cdot x}. \end{gathered} \tag{8}$$

Then operator (1) describes the continuously distributed scaling (dilations). In physics and economics, the dilation is the change of scale of objects and processes.

Remark 3. *Please note that using property (4) also allows us to write the operator (1) as the Mellin-type convolution*

$$\Big(D_{(K)} f\Big)(t) = \int_0^t K\Big(\frac{t}{\tau}, 1\Big)\Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big)\frac{d\tau}{\tau}, \tag{9}$$

which differs from the Mellin convolution by the upper limit of t instead of infinity. Using the kernel

$$K_H(x,1) = \begin{cases} K(x,1) & x > 1, \\ 0 & x \leq 1. \end{cases} \tag{10}$$

The operator (7) can be represented in the form

$$\Big(D_{(K)} f\Big)(t) = K_H *_{\mathcal{M}} f^{(n)} = \int_0^\infty K_H\Big(\frac{t}{\tau}, 1\Big)\Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big)\frac{d\tau}{\tau}, \tag{11}$$

where $*_{\mathcal{M}}$ is the Mellin convolution [33,34]. This representation allows us to propose a generalization the operator (9) by using the of the Mellin convolution in the definition of these generalized operators [29].

Remark 4. *Operators (1) and (2) with kernel, which satisfies the conditions (4) and (5), cannot be considered to be fractional derivative of non-integer order for positive integer values of n. The correct interpretation of these operators is integer order derivatives with the continuously distributed scaling (dilation). Please note that as a basis for the definition of these operators, which actually are integer order operators, one can use expression (6) with conditions (7) instead of Equation (1) with conditions (4) and (5).*

To have fractional generalization of these operators there are two ways: (A) we can use a fractional differential (or integral) operator $\Big(\mathcal{D}_\tau^{(\alpha)} f\Big)(\tau)$ instead of $\Big(\mathcal{D}_\tau^{(n)} f\Big)(\tau)$; (B) we can also use the kernel $\rho_1(x)$, which is not satisfied the normalization condition (7). In the work [29], we proposed a fractional generalization of this type of operators by the way (A) to describe processes with fading memory and distributed scaling.

Remark 5. *In our opinion, the Kochubei's approach to general fractional calculus [35–37], which is based on the Laplace convolution, can be applied to formulate new general fractional calculus, which will be based on the Mellin convolution. Moreover, the general fractional operators (9) and (11) can be used to formulate a generalization of the Luchko operational calculus [24,38], where the Mellin convolution will be used instead of the Laplace convolution.*

Statement 2.

Let us assume that the kernel $K(t,\tau)$ of the operator (1) with $t_0 = -\infty$ satisfies the following condition

$$K(t,\tau) = K(t-\tau) \tag{12}$$

for all $t > \tau$, the condition of non-negativity and the normability (or the normalization) condition

$$K(x) \geq 0, \int_0^\infty K(x)\, dx = K < \infty \tag{13}$$

for all $x \in (0,\infty)$, where K is a finite positive constant. In this case, operator (1) can be represented (by using the change of variable $\tau \to x = t - \tau$) in the form

$$\Big(D_{(K)} f\Big)(t) = K \int_0^\infty \rho_2(x) T_x \Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big) dx \tag{14}$$

with a finite positive constant K, where $\rho_2(x) = K(x)/K$ is the probability density function that satisfies the condition of non-negativity and the normalization condition

$$\rho_2(x) \geq 0, \int_0^\infty \rho_2(x)\, dx = 1, \tag{15}$$

and T_x is the translation (shift, lag) operator

$$T_x f(t) = f(t-x),\ T_x\Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big) = \Big(\mathcal{D}_z^{(n)} f(z)\Big)_{z=t-x}. \tag{16}$$

Then operators (1) describe the continuously distributed lag (time delay).

Remark 6. *Given the above, we can state that the operator with kernel, which satisfies the conditions (12) and (13), cannot be interpreted as fractional derivative of non-integer order for positive integer values of n. The correct interpretation of this operator is integer order derivative with the continuously distributed lag [29]. As a basis for the definition of this operator, which is integer order operators, we can use expression (14) with conditions (15) instead of Equation (1) with conditions (12) and (13).*

To have a fractional generalization of this operator, there are two ways: (A) to use a fractional differential (or integral) operator $\Big(\mathcal{D}_\tau^{(\alpha)} f\Big)(\tau)$ instead of $\Big(\mathcal{D}_\tau^{(n)} f\Big)(\tau)$; (B) to use the kernel $K(t,\tau)$, for which the normalization condition (15) is violated. In the work [29], we proposed a fractional generalization of this type operators by the way (A) to describe processes with memory and distributed lag. The fractional derivatives and integrals of non-integer orders, in which lag (time delay) is described by continuous probability distributions, were proposed in [29] (pp. 148–154), and used in macroeconomic models [30–32]. An example of fractional operators with distributed lag is also suggested in the Section 7 of the paper [29] (pp. 148–154).

Remark 7. *Please note that general operators of type (1) with the kernel (12) and without the condition (8) were considered by Anatoly N. Kochubei in works [35–37]. These works suggested concept of a general fractional calculus by using the differential operator based on Laplace convolution. Kochubei proposed the mathematical conditions on kernel of general fractional derivative, which lead to the fact that this general operator has a right inverse operator (a kind of a general fractional integral).*

3. Continuously Distributed Scaling (Dilation): Erdelyi–Kober Operators

As a generalization of the Riemann–Liouville fractional integral was proposed by Herman Kober. The Kober fractional integral [4] (p. 106), of the order $\alpha > 0$ is defined as

$$\left(I^{\alpha}_{K;0+;\eta}f\right)(t) = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t \tau^{\eta}\, (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{17}$$

where $\eta \in \mathbb{R}$. If function $f(t) \in L_p(\mathbb{R}_+)$, with $1 \leq p < \infty$, and $\eta > (1-p)/p$, the operator (17) is bounded [1] (p. 323). For $\eta = 0$, operator (17) can be expressed through the Riemann-Liouville integration by the expression

$$\left(I^{\alpha}_{K;0+;1}f\right)(t) = t^{-\alpha}\, (I^{\alpha}_{RL,0+}f)(t). \tag{18}$$

Changing the variable of integration by $\tau \to x = \tau/t$, the Kober operator (17) takes the form

$$\left(I^{\alpha}_{K;0+;\eta}f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\eta}(1-x)^{\alpha-1}\, f(x\, t) dx. \tag{19}$$

Expression (19) allows us to use the probability density function (p.d.f.) of the beta distribution in the form

$$\rho_{\alpha;\beta}(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } x \in [0,1], \tag{20}$$

and $\rho_{\alpha;\beta}(x) = 0$ if $x \notin [0,1]$, where $B(\alpha,\ \beta)$ is the beta function. Using (20), the Kober fractional integral is represented by the equation

$$\left(I^{\alpha}_{K;0+;\eta}f\right)(t) = K_{EK} \int_0^1 \rho_{\eta+1;\alpha}(x)\ f(x{\cdot}t) dx \tag{21}$$

with the constant

$$K_{EK} = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)}. \tag{22}$$

We note that expression (21) contains $f(x{\cdot}t)$ instead of $f(x)$. Therefore the variable $x > 0$ can be interpreted as a random variable, which describes scaling (dilation) with the gamma distribution. Using the scaling operator S_x: $S_x f(t) = f(x{\cdot}t)$, the Kober fractional integral (17) is represent by the equation

$$\left(I^{\alpha}_{K;0+;\eta}f\right)(t) = K_{EK} \int_0^1 \rho_{\eta+1;\alpha}(x) (S_x f(t)) dx, \tag{23}$$

where K_{EK} is defined by Equation (22). Equation (23) leads to the interpretation of the Kober operator as an expected value, where $x > 0$ is a random variable that describes the scaling and has the beta distribution up to numerical factor (22).

As a result, expression (23) gives a possibility to state that the Kober operator (17) can be interpreted as a continuously distributed dilation operator, in which the scaling variable has the beta distribution up to a constant factor (22).

The proposed interpretation of the Kober operator (17) allows us to generalize this operator by using other the probability density function instead of the beta distribution (20) and other lower and upper limits of integral in Equation (23). For example, the generalized operator of continuously distributed scaling (dilation) is define [29] by the expression

$$\left(D_{(\rho;S)}f\right)(t) = \int_0^\infty \rho(x)\left(S_x\left(\mathcal{D}_t^{(n)}f\right)(t)\right)dx, \tag{24}$$

where $n = 0, 1, 2, \ldots$, and $\rho(x) \geq 0$ is the probability density function such that

$$\int_0^\infty \rho(x)dx = 1. \tag{25}$$

In Equation (24) it is assumed that the integral $\int_0^\infty \rho(x) \left| S_x\left(\mathcal{D}_t^{(n)}f\right)(t)\right| dx$ converges, where $\mathcal{D}_x^{(n)} f(x)$ and $\rho(x)$ are piecewise continuous or continuous functions on $\mathbb{R}$. Here we can consider $\mathcal{D}_t^{(n)} f(t) = f^{(n)}(t)$.

The Erdelyi–Kober type operator [4] (p. 105), is defined by the equation

$$\left(I^{\alpha}_{EK;0+;\sigma,\eta}f\right)(t) = \frac{\sigma\, t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t \tau^{\sigma(\eta+1)-1}\, (t^{\sigma} - \tau^{\sigma})^{\alpha-1} f(\tau)d\tau, \tag{26}$$

where α > 0 is the order of integration. To get the notation of the paper (see Equation (1) in p. 360, [24]), we should change the indexes: $\sigma \to \beta$, $\alpha \to \delta$, $\eta \to \gamma$. In the case σ = 1, operator (26) is represented in the form of the Kober operator (17). Operator (26) can be represented by the equation

$$\left(I^{\alpha}_{EK;0+;\sigma,\eta}f\right)(t) = K_{EK} \int_0^1 \rho_{EK}(x)(S_x f(t))dx \tag{27}$$

with the probability density function

$$\rho_{EK}(x) = \frac{\sigma}{B(\eta+1,\alpha)} x^{\sigma(\eta+1)-1}(1-x^{\sigma})^{\alpha-1}, \tag{28}$$

and the constant factor K_{EK} defined by Equation (22). For σ = 1, the function (28) described beta distribution (20).

As a result, the Erdelyi–Kober and Kober operators are operators of integer orders with continuously distributed scaling (dilation). We should note that the fractional generalizations of these operators, which can be applied to describe simultaneously action of distributed scaling and fading memory, were proposed in [29].

As a result, we can state that the operators (1) with kernels (4) and (5), the operators (6) with different probability density functions (7), and operators (23), (24), (27) can be applied to describe continuously distributed scale phenomena in economics, physics, and other sciences.

4. Continuously Distributed Delay (Lag): Caputo–Fabrizio Operator

The Caputo–Fabrizio operator is proposed in [26–28]. The Caputo–Fabrizio operator $D^{(\alpha)}_{CF}$ of the non-integer order $\alpha \in (0, 1)$ is defined (see Equation (2.2) of [26] (p. 74)) by the equation

$$\left(D^{(\alpha)}_{CF}f\right)(t) = \frac{m(\alpha)}{1-\alpha} \int_{t_0}^{t} exp\left\{-\frac{\alpha}{1-\alpha}(t-\tau)\right\} f^{(1)}(\tau)d\tau, \tag{29}$$

where $f^{(1)}(\tau) = df(\tau)/d\tau$ is the standard derivative of first order, $m(\alpha)$ is a "normalization" function. For $n > 1$, the Caputo–Fabrizio operator of the order $\alpha + n \in (n, n+1)$ is defined (see, Equation (2.8) of [26] (p. 76)) by the expression

$$\left(D_{CF}^{(\alpha+n)} f\right)(t) = \left(D_{CF}^{(\alpha)} f^{(n)}\right)(t), \tag{30}$$

where $\alpha \in (0, 1)$ and $f^{(n)}(\tau) = d^n f(\tau)/d\tau^n$ are the standard derivatives of integer order $n \in \mathbb{N}$. The Caputo–Fabrizio operator of the order $\alpha \in (n, n+1)$ is defined (see, Equation (2.8) of [26] (p.76)) by the expression

$$\left(D_{CF}^{(\alpha)} f\right)(t) = \frac{m(\alpha-n)}{n-\alpha+1} \int\limits_{t_0}^{t} exp\left\{-\frac{\alpha-n}{n-\alpha+1}(t-\tau)\right\} f^{(n+1)}(\tau) d\tau, \tag{31}$$

where $n = [\alpha]$. The Caputo–Fabrizio operators (31) of order $\alpha \in (n, n+1)$ with $t_0 = -\infty$ can be represented in the form

$$\left(D_{CF}^{(\alpha)} f\right)(t) = \frac{\lambda\, m(\alpha-n)}{\alpha-n} \int_{-\infty}^{t} exp\{-\,\lambda\,(t-\tau)\}\, f^{(n+1)}(\tau) d\tau, \tag{32}$$

where

$$\lambda = \frac{\alpha-n}{n-\alpha+1} \tag{33}$$

Changing the variable $\tau \to x = t - \tau$ of integration in (32), Equation (32) takes the form

$$\left(D_{CF}^{(\alpha)} f\right)(t) = \frac{\lambda\, m(\alpha-n)}{\alpha-n} \int_{0}^{\infty} exp\{-\,\lambda x)\}\, f^{(n+1)}(t-x) dx. \tag{34}$$

Equation (34) can be represented by expression (14) in the form

$$\left(D_{CF}^{(\alpha)} f\right)(t) = K_{CF} \int_{0}^{\infty} \rho(x) \left(T_x f^{(n+1)}(t)\right) dx, \tag{35}$$

where the positive constant K_{CF} is

$$K_{CF} = \frac{m(\alpha-n)}{\alpha-n}, \tag{36}$$

and $\rho(x)$ is the probability density function of the exponential distribution

$$\rho(x) = \lambda\, exp(-\,\lambda\, x), \tag{37}$$

for $x > 0$ and $\rho(x) = 0$ for $x \leq 0$, where $\lambda > 0$ is the parameter that is often called the rate parameter or the speed of response [39] (p. 27). It is also used the parameter $T = 1/\lambda$ as time-constant of exponentially distributed lag. This parameter T is interpreted as the length of the time delay [39] (p.27). The kernel (37) is actively used in economics to describe processes with distributed lag [39] (p. 26). We should note that distribution (37) describes the time between events in a Poisson point process, which is the continuous analogue of the geometric distribution. It is well-known that this distribution has the key property of being memoryless.

In the work [22], it is proved that the Caputo–Fabrizio operator of the order $\beta = n - 1/(\lambda + 1)$, coincides with derivative of integer order with exponentially distributed lag, where λ is the rate parameter (33) of the distribution (37), and $n = [\beta] + 1$. Therefore, the Caputo–Fabrizio operator can be interpreted as an integer order derivative with the exponentially distributed time delay.

The existence of the time delay is based on the fact that the processes have a finite speed, and the change of the input does not lead to instant changes of output. In physical sciences it is well-known that the finite speed of the process does not mean that there is memory in the process. Therefore continuously distributed lag cannot be considered to be a dependence of the state of as process on its history. The time delay cannot be interpreted as a memory.

As a result, the Caputo–Fabrizio operators cannot be applied to modeling memory or spatial nonlocality in processes, but this operator describes continuously (exponentially) distributed time delay.

The proposed interpretation of the Caputo–Fabrizio operator (35) allows us to generalize this operator [29] by using other the probability density function instead of the exponential distribution (37). For example, the generalized operator of continuously distributed scaling (dilation) is define [29] by the expression

$$\left(D_{(\rho;T)} f\right)(t) = \int_0^\infty \rho(x) \left(T_x f^{(n)}(t)\right) dx, \tag{38}$$

where $n = 0, 1, 2, \ldots$, and $\rho(x) \geq 0$ is the probability density function such that

$$\int_0^\infty \rho(x) dx = 1. \tag{39}$$

In Equation (38) it is assumed that the integral $\int_0^\infty \rho(x) \left| \left(T_x f^{(n)}(t)\right) \right| dx$ converges, where $f^{(n)}(\mathrm{x})$ and $\rho(x)$ are piecewise continuous or continuous functions on $\mathbb{R}$.

The fractional generalization of the Caputo–Fabrizio operator was proposed in [29] to take into account various distributions of delay time and power-law fading memory in one operator.

5. Continuously Distributed Fading Memory

To describe memory (the fading memory), we can use operators (1), for which the normability condition is not satisfied.

For example, the operator (1) with $t_0 = -\infty$ and the kernel

$$K(t, \tau) \; = \frac{1}{\Gamma(n-\alpha)} (t-\tau)^{n-\alpha-1} \tag{40}$$

is the left-sided Caputo fractional derivative of the order $\alpha \geq 0$ (see Equation (2.4.15) [4] (p. 92) for $a = -\infty$) that is defined by the equation

$$\left(D^{\alpha}_{C+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{41}$$

where $\Gamma(\alpha)$ is the gamma function, and $f^{(n)}(\tau)$ is the derivative of the integer order $n = [\alpha] + 1$ for non-integer values of α (and $n = \; \alpha$ for integer values of α). Changing the variable $\tau \to x = t - \tau$ operator (1) with the kernel (40) and $t_0 = -\infty$ can be represented in the form

$$\left(D_{(K)} f\right)(t) = \int_0^\infty K_c(x) \left(T_x f^{(n)}(t)\right) dx, \tag{42}$$

where the kernel

$$K_c(x) = \frac{x^{n-\alpha-1}}{\Gamma(n-\alpha)} \tag{43}$$

cannot be interpreted as a probability density function since the normalization condition is violated

$$\int_0^\infty K_c(x) \; \mathrm{dx} = \left(\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)} \right)_0^\infty = \infty \tag{44}$$

for non-integer values of α.

Let us describe some basic principles and properties of the kernel that should be taken into account to describe memory.

Principle of violation of normability. Processes with memory cannot be described by operators (1) if the operator kernel can be considered to be a probability density function. In other words, the memory function cannot be probability density function.

The requirement of violation of the normability conditions is not enough for a comprehensive description of fading memory. We should have conditions for the kernel of operator (1), which allow us to use this operator to described memory.

Principle of causality. The main condition that must be satisfied for all types of memory is the fulfillment of the causality principle. It is obvious that the operators that describe memory phenomena should satisfy the causality principle. In mathematical form, the causality principle can be realized by the Kramers–Kronig relations [18].

In addition to these relations, we can state that the right-sided fractional derivatives (for example the Riemann–Liouville, Liouville, and Caputo-type) cannot be used to processes with the memory. The right-sided fractional integrals and derivatives are defined for $\tau > t$, where t is the present time moment. Therefore these operators describe dependencies of processes on the future states. The left-sided fractional operators describe the past states of the process.

Principle of memory fading. The important property of memory is the memory fading. The principle of memory fading was first proposed by Ludwig Boltzmann, and then it was significantly developed by Vito Volterra. This principle states that the increasing of the time interval leads to a decrease in the contribution of impact to the response. The exact mathematical formulation of this principle is given in [40–44], it is more complicated than that required for us in this paper, which is restricted by the operators (1). Therefore we will use a simplified formulation of the principle of memory fading [17].

Let us consider two functions $f(\tau)$ and $y(t)$, which are interpreted as the impact and response variables respectively, and we will assume that these functions are connected by the equation

$$y(t) = \int_0^t K(t,\tau)\Big(\mathcal{D}_\tau^{(n)} f(\tau)\Big) d\tau. \tag{45}$$

Let us assume that $\mathcal{D}_\tau^{(n)} f(\tau)$ is different from zero on a finite time interval $\tau \in [0,T]$, and which is zero outside this interval ($\mathcal{D}_\tau^{(n)} f(\tau) = 0$ for $t > T$). This means that we consider $H(T–\tau)\mathcal{D}_\tau^{(n)} f(\tau)$ instead of $\mathcal{D}_\tau^{(n)} f(\tau)$ in Equation (45) with times $t \in [T,\infty)$. Then Equation (45) gives

$$y(t) = \int_0^T K(t,\tau)\left(\mathcal{D}_\tau^{(n)} f(\tau)\right) d\tau \; for \; t < T. \tag{46}$$

We see that for $t > T$ there is no impact, but the response is different from zero ($y(t) \neq 0$ for $t > T$). This means that the memory about the impact, which acts on time interval $[0,T]$, is stored in the process. Therefore, we can state that this process saves the history of changes of the impact. Using the mean value theorem, there is a value $\xi \in [0,T]$ and Equation (46) can be written as

$$y(t) = K(t,\xi)\left(\mathcal{D}_\tau^{(n)} f(\tau)\right)_{\tau=\xi} T. \tag{47}$$

As a result, we can see that the behavior of the response $y(t)$ is determined by the behavior of the kernel $K(t,\tau)$ with fixed constant time $\tau = \xi$. The behavior of the kernel $K(t,\tau)$ at infinite increase of t ($t \to \infty$) and fixed τ determines the dynamics of the process with memory (See Table 2).

Table 2. Examples of the correspondence between the type of memory and type of operator kernels.

Type of Memory	Type of Kernel	Fading (Dissipation)
Memory of Insignificant Events	$\left\lvert \lim_{t\to\infty} K(t,\tau) \right\rvert = 0$	Fading Memory
Memory of Significant Events	$0 < \left\lvert \lim_{t\to\infty} K(t,\tau) \right\rvert < \infty$	Non-Fading Memory
Memory of Crises and Shocks	$\left\lvert \lim_{t\to\infty} K(t,\tau) \right\rvert = \infty$	Non-Fading Memory

Let us assume that there is the limit

$$\lim_{t\to\infty} K(t,\tau) = K_\infty(\tau) = K_\infty, \tag{48}$$

for all τ, when $\tau < t$. In this case, we can consider the three basic type of behavior of $K(t,\xi)$ at infinity $t \to \infty$.

First Type ($K_\infty = 0$): Memory of Insignificant Events (IE-memory). If the kernel tends to zero $(K(t,\tau) \to 0)$ at $t \to \infty$, then the process completely forgets about the impact that acts in the past. Then the process that is described by Equation (47) is reversible (is repeated) in a sense. We can say that the memory effects did not lead to irreversible changes of the process, since the memory about the impact has not been preserved forever. Therefore this type of memory can be called "the memory with complete forgetting" (or the memory of insignificant events). As a result, the mathematical characteristic of processes with fading memory can be described by the operator kernels that satisfy the following Principle of Memory Fading memory: Memory, which is described by the operator (45), is fading if the kernel satisfies the condition

$$\lim_{t\to\infty} K(t,\tau) = 0 \tag{49}$$

for all fixed values of τ. The memory will be called the memory with power-law fading if there is a parameter $\alpha > 0$ such that the limit $\lim_{t\to\infty} t^{-\alpha}\, K(t,\tau)$ is a finite constant for fixed τ. For example, the kernel (40) of the left-sided Caputo fractional derivative describes the power-law memory fading.

Second Type ($0 < |K_\infty| < \infty$): Memory of Significant Events (SE-memory). If the kernel $K(t,\tau)$ tends to a finite limit at $t \to \infty$, the impact leads to the irreversible consequences in the sense that the memory of the impact is preserved forever. Therefore this type of memory can be called "the memory with remembering forever" (or memory of significant events).

Third Type ($K_\infty = \infty$): Memory of Crises and Shocks (CS-memory). Unbounded increase of the kernel $K(t,\tau)$ at $t \to \infty$ (with fixed τ) characterizes an unstable process with memory. This kernel cannot be used to describe stable processes. However, this type of kernels can be used in the various models, which take into account the processes with crises and shocks (for example in economy), when we can expect a manifestation of instability phenomena. The behavior of processes with memory at time t is determined by the behavior of the operator kernel (memory function) in the previous time instants $\tau < t$. Therefore, an unbounded increase in the memory function at infinity $(t \to \infty)$ does not lead us to the rejection of consideration of such operator kernels. For example, in this type of memory one can assume that the operator kernel $K(t,\tau)$ is bounded for all $\tau < t$ for a fixed $t < \infty$. Therefore this type of memory can be called "the memory of crises and shocks".

Non-Monotony of Decrease. In general, the memory fading assumes a set of stronger restrictions on the operator kernels. For example, it is assumed that the fading memory is described by operator kernels, which tends to zero monotonically with increasing the time variable. This assumption means that it is less probable to expect of strengthening of the memory with respect to the more distant events. We should note that in economics the agents may remember sharp and significant changes of the variables despite the fact that these changes were more distant past compared to weaker changes in the near past. For this reason, in economics we can use operator kernels without property of monotonic decrease.

Principle of memory reversibility. In paper [17,18], we describe some general restrictions that can be imposed on the structure and properties of memory. For example we consider the principle of memory reversibility (the principle of memory recovery). The principle of memory reversibility is connected with the principle of duality of accelerator with memory and multiplier with memory, which is proposed in [45]. Mathematically this principle is based on the main property of any fractional derivative to be a left-inverse operator to the corresponding fractional integral operator.

Remark 8. *We should note that there is an addition restriction on the kernel of the operator (1). In general, to have a self-consistent mathematical theory of the operators (1), the general fractional derivative (1) with $n = 1, 2, 3, \ldots$ should be a left-inverse operator to the corresponding general fractional integral operator (1) with $n = 0$. This requirement leads us to a relationship between the type of the operator kernels $K(t, \tau)$ and the order (and type) of the operators $\left(\mathcal{D}^{(n)}_{\tau} f(\tau)\right)$ of integer order $n = 1, 2, 3, \ldots$. For the kernel should depend on the order , i.e., $K(t, \tau) = K_n(t, \tau)$.*

Remark 9. *General fractional calculus was proposed by Anatoly N. Kochubei in [35–37] and based on the use of differential operators with Laplace convolution (the general Laplace-convolutional derivatives). The principle of memory reversibility means that the general operators should have right inverse (a kind of a fractional integral). We assume that the Kochubei approach to formulation of general fractional calculus, which is based on the Laplace convolution, can be applied to formulate new fractional calculus based on Mellin convolution. The general operators (the general Mellin -convolutional derivatives), which are based on Mellin convolution, and equations with these operators can be used to describe the scaling (dilation) phenomena in physics and economics.*

6. Properties of Kernels of Inverse Operators and Type of Phenomena

An addition restriction on the kernel of the operator (1) can be considered. The general operators (1) with $n = 1, 2, 3, \ldots$ can be considered to be the general fractional derivative. The general operators (1) with $n = 0$ can be considered to be general fractional integrals. In our opinion to have a self-consistent mathematical theory, the general fractional derivative (1) with $n = 1, 2, 3, \ldots$ should be a left-inverse operator to the corresponding general fractional integral operator (1) with $n = 0$. Therefore we proposed the following principle for fractional calculus: Any type of generalized (fractional) derivative should be a left-inverse operator to the corresponding type of generalized (fractional) integral operator. This principle can be considered to be a requirement of the existence of a generalization of the fundamental theorem of calculus, which is a theorem that links the concept of differentiating with the concept of integrating.

Obviously, this principle, this requirement lead us to a relationship between the type of the operator kernels $K_n(t, \tau)$ $n = 1, 2, 3, \ldots$, and the type of the kernel $K_0(t, \tau)$. Please note that this requirement also leads us to a relationship between the type of the operator kernels $K_n(t, \tau)$ and the order (and type) of the operators $\left(\mathcal{D}^{(n)}_{\tau} f(\tau)\right)$ of integer order $n = 1, 2, 3, \ldots$**First Question**: In connection with this principle, the natural question arises about the relationship between the properties of the kernels of fractional operators, considered to be the fractional integrals and as the fractional derivatives. In many cases, kernels belong to one type of functions. For example, the kernel of the left-sided Caputo fractional derivative (see Equation (2.4.15) in p. 92, [4]) has the form

$$K_n(t, \tau) = \frac{1}{\Gamma(n - \alpha)} (t - \tau)^{n-\alpha-1} \tag{50}$$

for $= 1, 2, 3, \ldots$. This fractional derivative is the left-inverse operator for the left-sided Riemann–Liouville fractional integral. The kernel of this integral is described by Equation (50) with $n = 0$ and negative α (see Equation (2.1.1) in [4] (p. 69)). The same situations we have for the Erdelyi–Kober operator and other types of fractional operators. However, this is not true in the general case.

Remark 10. *In paper [17,18], we describe some general restrictions that can be imposed on the structure and properties of memory. These restrictions are proposed as the principle of memory reversibility (the principle of memory recovery). Mathematically this principle is based on the property of any fractional derivative to be a left-inverse operator to the corresponding fractional integral operator.*

Statement 3.

The generalized (fractional) derivative (1) with $n = 1, 2, 3, \ldots$ must be the left inverse operator to the corresponding generalized (fractional) integral operator (1) with $n = 0$. However, the kernels $K_n(t, \tau)$ with $n = 1, 2, 3, \ldots$ of operator (1) and the kernel $K_0(t, \tau)$ of the fractional integral operator (1) with $n = 0$ can belong to different types of functions.

To prove this statement, we give an example of fractional operators of distributed orders.

In general, the parameter α that is the order of the fractional derivative or integral and describes the memory fading, can be distributed on an interval with some probability density function (the weight function). In the simplest case, we can use the continuous uniform distribution (CUD). The fractional integrals and derivatives of the uniform distributed order can be expressed thought the continual fractional integrals and derivatives, which were suggested by Adam M. Nakhushev [46,47]. The operators of non-integer orders, which are left inverse to the continual fractional integrals and derivatives, are proposed by Arsen V. Pskhu in [48,49]. Using the continual fractional integrals and derivatives, which were suggested by Nakhushev, we can define the integral and derivatives of uniform distributed order. These operators will be called the Nakhushev fractional integrals and derivatives. The corresponding inverse operators are proposed by Pskhu and therefore operators, which are inverse to fractional CUD fractional operators, will be called the Pskhu fractional integrals and derivatives.

In works of Pskhu [48,49] the notations $\mathrm{D}_{0+}^{[\alpha,\beta]}$ and $\mathrm{D}_{0+}^{-[\alpha,\beta]}$ are used for positive ($0 < \alpha < \beta$) and negative ($\alpha < \beta \leq 0$) values of α and β. In our opinion, this leads to confusion and misunderstanding in applications. Therefore we will use new notations, which allow us to see explicitly the integration and differentiation of the fractional orders.

The Nakhushev fractional integral can be defined (see Equation (5.1.7) of [49] (p. 136) and [48]) defined in the form

$$I_N^{[\alpha,\beta]} X(t) = \frac{1}{\beta - \alpha} \int_\alpha^\beta I_{RL,a+}^{\xi} X(t) d\xi = \int_0^t W(\alpha, \beta, t - \tau)\, X(\tau) d\tau, \tag{51}$$

where we use the function

$$W(\alpha, \beta, t) = \frac{1}{(\beta - \alpha)\, t} \int_\alpha^\beta \frac{t^\xi d\xi}{\Gamma(\xi)}. \tag{52}$$

Using Equation (5.1.26) of [49] (p. 143), the Nakhushev fractional derivative can be written in the form

$$D_N^{[\alpha,\beta]} X(t) = \left(\frac{d}{dx}\right)^n \int_0^t W(n - \alpha, n - \beta, t - \tau)\, X(\tau) d\tau, \tag{53}$$

where $\beta > \alpha > 0$. Please note that the Nakhushev fractional derivatives cannot be considered to be inverse operators for the Nakhushev fractional integration. The Pskhu fractional derivatives are inverse to the Nakhushev fractional integration and the Pskhu fractional integrals are inverse to the Nakhushev fractional derivatives.

The Pskhu fractional integral can be defined (see Equation (5.1.7) of [49] (p. 136), and [48]) by the expression

$$I_P^{[\alpha,\beta]} X(t) = (\alpha-\beta) \int_0^t (t-\tau)^{\beta-1} E_{\beta-\alpha}\left[(t-\tau)^{\beta-\alpha};\beta\right] X(\tau)d\tau, \tag{54}$$

where $\beta > \alpha > 0$. where $E_\alpha[z;\beta]$ is the Mittag–Leffler function that is defined by the expression

$$E_\alpha[z;\beta] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{55}$$

Using Equation (5.1.7) of [49] (p. 136), we can define the Pskhu fractional derivative as

$$D_P^{[\alpha,\beta]} X(t) = (\alpha-\beta) \left(\frac{d}{dx}\right)^n \int_0^t (t-\tau)^{-\alpha} E_{\beta-\alpha}^{n-1}\left[(t-\tau)^{\beta-\alpha};1-\alpha\right] X(\tau)d\tau, \tag{56}$$

where $\beta > \alpha > 0$ and the function $E_\alpha^\mu[z;\beta]$ is defined by the equation

$$E_\alpha^\mu[z;\beta] = \frac{\partial}{\partial\mu}(z^\mu \, E_\alpha[z;\beta+\mu])$$

As a result, we have that the Nakhushev fractional derivatives cannot be considered to be left-inverse operators for the Nakhushev fractional integrals [48,49]. Operators, which are left-inverse operator for the Nakhushev fractional derivatives and integrals, are the Pskhu fractional integrals and derivatives.

As a result, we proved that the kernels of the original and inverse operators can be of different types.

Second Question: If the kernels of generalized (fractional) derivative and the corresponding generalized (fractional) integral operator can be described by functions of different types, then the second natural question arises: Will these kernels describe the same types of phenomena? If the operator cores are different, then what is the difference in the phenomena described by these different types of cores? As a suggested answer on these questions, we can propose the following hypothesis.

Hypothesis of Duality: The kernels of the original and inverse operators of fractional calculus should describe dual types of phenomena.

This hypothesis is based on an attempt to answer the second question in the framework of economic interpretation, which is presented in the form of the principle of duality proposed in [45]. In this principle we describe duality of two basic economic concepts: the accelerator with memory and multiplier with memory (for details see [45]).

Remark 11. *We assume that the Kochubei approach to formulation of general fractional calculus, which is based on the Laplace convolution, can be applied to formulate new fractional calculus based on Mellin convolution. This allows us to describe duality of the economic concepts of the accelerator with scaling and multiplier with scaling.*

7. Memory with Lag: Distributed Lag Fractional Operators

In general, we can simultaneously take into account two different types of phenomena. For example, we can simultaneously take into account lagging and memory phenomena. For this, we proposed the distributed lag fractional calculus in [29]. Then this approach was applied to macroeconomic models.

To illustrate this approach, let us assume that the joint action of two phenomena: the lag with gamma distribution of delay time and the power-law fading memory. We will use the Caputo fractional derivatives to describe power-law memory. The continuously distributed delay time is described by the translation operator, where the delay time $\tau > 0$ is a random variable that is distributed on positive semiaxis. We can prove that the composition of these operators is represented as the Abel-type integral and integro-differential operators with the confluent hypergeometric Kummer function in the kernel.

The Caputo fractional derivative with gamma distributed lag is defined by the equation

$$\Big(D^{\lambda,a;\alpha}_{T;C;0+}f\Big)(t)=\int_0^t K^{\lambda,a}_{T}(\tau)\Big(D^{\alpha}_{C,0+}f\Big)(t-\tau)\,d\tau, \tag{57}$$

where the kernel $K^{\lambda,a}_{T}(\tau)$ is the probability density function of the gamma distribution

$$K^{\lambda,a}_{T}(\tau)=\begin{cases}\frac{\lambda^a\,\tau^{a-1}}{\Gamma(a)}\,exp(-\lambda\,\tau) & if\ \tau>0,\\ 0 & if\ \tau\le 0,\end{cases} \tag{58}$$

with the shape parameter $a>0$ and the rate parameter $\lambda>0$. If $a=1$, the function (58) describes the exponential distribution. Using the associative property of the Laplace convolution, the operators (57) can be represented [29] in the form

$$\Big(D^{\lambda,a;\alpha}_{T;C;0+}f\Big)(t)=\int_0^t K^{\lambda,a;n-\alpha}_{TRL}(\tau)f^{(n)}(t-\tau)\,d\tau, \tag{59}$$

where $n-1<\alpha\le n$, and the kernel $K^{\lambda,a;n-\alpha}_{TRL}(t)$ has the form

$$K^{\lambda,a;n-\alpha}_{TRL}(t)=\frac{\lambda^a\,\Gamma(a)}{\Gamma(a+n-\alpha)}\,t^{a+n-\alpha-1}F_{1,1}(a;a+n-\alpha;-\lambda t), \tag{60}$$

where $F_{1,1}(a;b;z)$ is the confluent hypergeometric Kummer function that is defined (see [4] (pp.29–30)) by the equation

$$F_{1,1}(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)dt=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(c)}{\Gamma(a)\Gamma(c+k)}\,\frac{z^k}{k!}, \tag{61}$$

where $a,\ z\in\mathbb{C}$, $Re(c)>Re(a)>0$ such that $c\neq 0,-1,-2,\ldots$. and series (61) is absolutely convergent for all $z\in\mathbb{C}$. It should be noted that the kernel (60) can be represented through the three parameter Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$, which is also called the Prabhakar function, by using the equation $F_{1,1}(a;c;z)=\Gamma(c)E^{a}_{1,c}(z)$. The Laplace transform of fractional operator (59) has the form

$$\Big(\mathcal{L}\Big(D^{\lambda,a;\alpha}_{T;C;0+}f\Big)(t)\Big)(s)=\frac{\lambda^a}{(s+\lambda)^a}\left(s^{\alpha}(\mathcal{L}Y)(s)-\sum_{j=0}^{n-1}s^{\alpha-j-1}f^{(j)}(0)\right), \tag{62}$$

where $n-1<\alpha\le n$.

As a result, the kernel $K^{\lambda,a;n-\alpha}_{TRL}(\tau)$ of the proposed special kind of the Abel-type fractional derivative describes the joint phenomenon of the power-law fading memory and the continuously distributed lag. Using Theorem 6.5 in [29] (pp. 145–146), and results of [31,32], we can describe the solution of the fractional differential equation

$$\Big(D^{\lambda,a;\alpha}_{T;C;0+}y\Big)(t)=\omega y(t)+F(t), \tag{63}$$

where the operator $D^{\lambda,a;\alpha}_{T;C;0+}$ is defined by Equation (59), $\alpha>0$ is the order of the operators, the parameters $a>0$ and $\lambda>0$ are the shape and rate parameters of the gamma distribution of delay time. The solution of Equation (63) can be represented in the form

$$y(t)=\sum_{j=0}^{n-1}S^{\alpha-j-1}_{\alpha,a}[\omega\,\lambda^{-a},\lambda|t]y^{(j)}(0)+\frac{1}{\omega}F(t)-\frac{1}{\omega}\int_0^t S^{\alpha}_{\alpha,a}\,[\omega\,\lambda^{-a},\lambda\big|t-\tau]\,F(\tau)d\tau, \tag{64}$$

with $n = [\alpha] + 1$, and $S^{\gamma}_{\alpha,\delta}\,[\mu,\lambda\big|t]$ is the special function that is defined by the expression

$$S^{\gamma}_{\alpha,\delta}\,[\mu,\lambda\big|t] = -\sum_{k=0}^{\infty} \frac{t^{\delta(k+1)-\alpha k-\gamma-1}}{\mu^{k+1}\Gamma(\delta(k+1)-\alpha k-\gamma)}\, F_{1,1}(\delta(k+1);\delta(k+1)-\alpha k-\gamma,-\lambda t), \tag{65}$$

where $F_{1,1}(a;b;z)$ is the confluent hypergeometric Kummer function (61).

In the connection with a possibility of composition of two or more kernels of operators that describe different phenomena, an important question arises about the following inverse mathematical problem. How we can identify and separate actions of two different type phenomena in it simultaneously action? In our opinion, the answer on this question is important to physics, mechanics, economics and other sciences.

8. Operator Kernel Behavior at Zero and Interpretation

In general, the type of behavior of the operator kernel (1) at $t \to 0$ can be important to different applications. We can assume the following type of behavior the kernel $K(t)$.

(1) The operator kernel tends to zero while the argument t tends to zero

$$\lim_{t\to 0+} K(t) = 0. \tag{66}$$

(2) The kernel $K(t)$ tends to finite nonzero constant while the argument t tends to zero

$$\lim_{t\to 0+} K(t) = K(0) = const. \tag{67}$$

(3) The kernel $K(t)$ tends to infinity as the argument t tends to zero

$$\lim_{t\to 0+} K(t) = \pm\infty.$$

A lot of kernels of the fractional integral and derivatives demonstrate only the third (or first) type of behavior at zero for non-integer orders. Let us describe some examples of the operator kernels that have this type of behavior.

The kernel of the Riemann–Liouville fractional integral has the form

$$K_{RLI}(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \tag{68}$$

where $\alpha > 0$ [4] (p.69). We see that

$$K_{RLI}(0) = \begin{cases} 0 & if \quad \alpha > 1 \\ 1 & if \quad \alpha = 1 \\ \infty & if \quad 0 < \alpha < 1 \end{cases}. \tag{69}$$

This means that kernel of the Riemann-Liouville fractional integral can demonstrate three type of behavior at zero ($t = 0$). However, the second type of behavior ($K_{PI}(0)$=const) cannot be realized for non-integer orders $\alpha > 0$.

The kernel of the Caputo and Riemann–Liouville fractional derivatives has the form

$$K_{CD}(t) = K_{RLD}(t) = \frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1}, \tag{70}$$

where $n = [\alpha] + 1$, and $n - 1 < \alpha < n$ for non-integer values of order α [4] (pp.70–91). We see that

$$K_{CD}(0) = \begin{cases} 0 & if \quad 0 < \alpha < n-1 \\ 1 & if \quad \alpha = n-1 \\ \infty & if \quad \alpha > n-1 \end{cases} . \tag{71}$$

This means that kernel of the Caputo and Riemann-Liouville fractional derivatives can demonstrate only one (singular) type of behavior at zero ($t = 0$) for non-integer orders. The other two cases ($K_{CD}(t) = 0$ and $K_{CD}(0) = 1$) are not implemented for the following reasons: (A) The case $\alpha = n - 1$ cannot be used for the Caputo derivative since we have $\alpha = n$ for integer values of α (see Equation (2.4.3) in [4] (p.91)). For this case, the Riemann–Liouville fractional derivative is standard derivative of integer order. (B) The case $0 < \alpha < n - 1$ cannot be used by definition the Caputo and Riemann-Liouville fractional derivatives that contains the condition $n - 1 < \alpha < n$ for non-integer values of order α. We have a similar situation for the Erdelyi–Kober and Kober operators.

As a result, we see that the power-law kernels of fractional derivatives have significantly less variability in the behavior properties at zero. Please note that the variety of properties of operator kernel at zero is important for applications of these operators in economics and physics, for example.

Let us note that some important phenomena are described only by the kernels with second type of behavior. For example, in economics this condition is used for the kernels that describe the depreciation of fixed assets (of capital), depreciation of equipment, obsolescence, aging, wear and tear [50] (p. 20). The kernel $K(t - \tau)$ characterizes the share of fixed assets put into operation at time τ and continuing to operate at time $t > \tau$. Obviously, in this case, the condition $K(0) = 1$ must be satisfied. For this, economics often use the exponential functions and the probability density function of the exponential distribution.

The kernels of the Riemann–Liouville, Caputo, Erdelyi–Kober fractional operators of non-integer order cannot be used to describe the depreciation or aging phenomena in economy. To describe these phenomena we can use the fractional operators with the Prabhakar function, the hypergeometric function, the Kummer (confluent hypergeometric) function in the kernels. In the framework of fractional calculus, these operators were proposed and described more than forty years ago in [51], (see also [52,53]) for the Prabhakar function, [54,55] the Kummer (confluent hypergeometric) function, and [56] (see also [1] (pp. 731–737)) for the hypergeometric function.

Please note that the operators with the Kummer (confluent hypergeometric) function in the kernels can be interpreted as the joint effect of two phenomena: the memory with power-law fading and the lag with gamma distribution of delay time. In the paper [29] (see Theorem 4.3 and Equation (4.48) p. 137; see also Equations (4.53) and (6.7)), we use the operators with the Kummer (confluent hypergeometric) function in the kernels that is Laplace convolution of the kernel of the Caputo fractional derivatives and probability density function of the gamma distribution that describes the distribution of the delay time $\tau > 0$.

The kernel of the Prabhakar fractional integral has the form

$$K_{PI}(t) = t^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega t^{\rho}] = t^{\mu-1} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{(\omega t^{\rho})^k}{k!}. \tag{72}$$

We can see that the kernel (72) can demonstrate three type of behavior at zero

$$K_{PI}(0) = \begin{cases} 0 & if \quad \mu > 1 \\ 1 & if \quad \mu = 1 \\ \infty & if \quad 0 < \mu < 1 \end{cases} \tag{73}$$

The kernel of the Kilbas–Saigo–Saxena fractional derivative [53] (that is also called the Prabhakar fractional derivative), which is proposed in [53] and it is left-inverse operator for the Prabhakar fractional integral, has the form

$$K_{PD}(t) = t^{n-\mu-1} E^{-\gamma}_{\rho,n-\mu}[\omega t^{\rho}], \tag{74}$$

where $n \geq [Re(\mu)] + 1$ with $Re(\mu) > 0$. We should emphasize that in kernel (74), we can use all positive integer values $n \geq [Re(\mu)] + 1$, where $Re(\mu) > 0$ since n is defined as $n = [\mu + \nu] + 1$ with $Re(\mu), Re(\nu) > 0$ in Theorem 9 in [53] (p. 47)).

Using expression (74), we get the following properties of kernel (74) in the initial point

$$K_{PD}(0) = \begin{cases} 0 & if \quad 0 < \mu < n-1 \\ 1 & if \quad \mu = n-1 \\ \infty & if \quad \mu > n-1 \end{cases} . \tag{75}$$

As a result, the kernel of the Kilbas–Saigo–Saxena fractional derivative can demonstrate three type of behavior at zero. Please note that this operator remains a fractional operator and under condition $\mu = n - 1$. This behavior significantly distinguishes this operator from other fractional derivatives, which usually have a singularity at zero.

Therefore to satisfy the initial conditions $K(0) = 1$ for the operator kernel, we can use the kernels with the Prabhakar function. These kernels allow us to use the fractional integrals and derivatives with the Prabhakar function in the kernel, which proposed in the works [51–53], to describe depreciation processes in economics. In addition, we can state that the kernel $K_{PI}(t)$ is the complete monotonic function for the case $\omega < 0$, $0 < \rho, \mu \leq 1$, $0 < \gamma \leq \mu/\rho$. The property of the complete monotonicity is important for the interpretation of operator kernels that describe standard depreciation phenomena. However, we can assume that the requirement of complete monotonicity for depreciation kernels is not necessary, when taking into account modernization of the equipment.

9. Conclusions

In this paper, we discussed an interpretation of fractional derivatives and integrals from the point of view of applied mathematics, theoretical physics, and economic theory. We state that it is important to connect all restrictions on the fractional operator kernels with types of phenomena, in addition to the self-consistency of mathematical theory. In applications of fractional calculus, we have a fundamental question about conditions of kernels of non-integer order operators that allow us to describe one or another type of phenomena. It is necessary to obtain exact correspondences between sets of properties of kernel and type of phenomena. In this paper, we describe some important properties of fractional operator kernels that can determine the characteristic features of certain types of phenomena. We consider the possible characteristic properties of kernels of fractional operators to distinguish the following types of phenomena: fading memory (forgetting) and power-law frequency dispersion; spatial non-locality and power-law spatial dispersion; distributed lag (time delay); distributed scaling (dilation); depreciation and aging.

Let us briefly describe possible directions for application of the proposed approach.

a) We should note the power-law kernels function can be used to consider an approximation of the generalized memory functions [57]. Using the generalized Taylor series in the Trujillo-Rivero-Bonilla form for the memory function, we proved [57] that the equations with memory functions can be represented through the Riemann–Liouville fractional integrals and the Caputo fractional derivatives of non-integer orders for wide class of the kernels. We can also note that the Abel-type fractional integral operator with Kummer function in the kernel (see Equation (37.1) in [1] (p. 731), and [32]) can be represented as an infinite series of the Riemann–Liouville fractional integrals.
b) We can have new types of phenomena in quantum theory, where we should take into account the intrinsic dissipation, the openness of systems, an interaction with environment [58–61].

c) We can expecte new types of phenomena in nonlinear, chaotic systems and for self-organization processes [62–65], where we should take into account the new types of attractors, patterns and effects.

At the same time, we emphasize that we have in mind not new regular applications of fractional calculus to the description of various particular phenomena in various science. We mean exact correspondence between the types of phenomena and the types of properties of fractional operator kernels.

Author Contributions: Contribution of V.E.T. is connected with fractional calculus. Contribution of S.S.T. is connected with probability theory. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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