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# Stability Estimates for Finite-Dimensional Distributions of Time-Inhomogeneous Markov Chains

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**Abstract:** This paper is devoted to the study of the stability of finite-dimensional distribution of time-inhomogeneous, discrete-time Markov chains on a general state space. The main result of the paper provides an estimate for the absolute difference of finite-dimensional distributions of a given time-inhomogeneous Markov chain and its perturbed version. By perturbation, we mean here small changes in the transition probabilities. Stability estimates are obtained using the coupling method.

**Keywords:** finite-dimensional distributions; Markov chains; coupling method; stability

## 1. Introduction

The stability of Markov chains is an important topic that has attracted the interest of researchers for multiple recent decades. A noticeable contribution to the stability theory of Markov chains has been done by Tuominen, Thorisson, Tweedie (see [1,2]), and others.

There are many various problems related to the stability of Markov chains and many various methods of investigation, respectively. We can refer the reader to the book [3] that is devoted to the stability of time-homogeneous Markov chains. Its results are established using the tools of functional analysis and extending methods introduced by D. Revuz, one of the founders of the modern Markov chains theory (see [4]). It turns out, however, that such methods can not be easily extended to the time-inhomogeneous case and also they do not always allow a clear probabilistic interpretation. Other approaches that are used nowadays are splitting and coupling methods. The splitting method was used in the classical book [5], in order to derive the whole modern theory of Markov chains. The coupling method, which has some similarities to the splitting, but different in nature, was first introduced by Doeblin in [6] in 1938 and later described in the famous monographs [7,8]. This method is purely probabilistic and can be used to study both time-homogeneous and time-inhomogeneous Markov chains, and allows to get various stability results (see [9,10]).

For example, in the papers [11–13], the coupling method is used to obtain the ergodic properties of a Markov chain and the stability estimate of the form $||\lambda P^n(x,\cdot) - \mu P^n(x,\cdot)||$ for the transition probabilities of the same chain that starts from various initial distributions. In the paper [14], the coupling method is used to get the results in the time-inhomogeneous situation.

In this paper, we study the stability of the discrete-time, perturbed Markov chain on a general state space. Original and perturbed chains could be time-inhomogeneous. The main result of this paper is to obtain a stability estimate for finite-dimensional distributions of the form

$$|\mathbb{P}_x\{X_1^{(1)} \in B_1, X_2^{(1)} \in B_2, \dots, X_n^{(1)} \in B_n\} - \mathbb{P}_x\{X_1^{(2)} \in B_1, X_2^{(2)} \in B_2, \dots, X_n^{(2)} \in B_n\}|.$$

We construct a special coupling process for two different time-inhomogeneous Markov chains. Similar coupling processes are used in papers [15–17] to get various stability estimates under different conditions for time-homogeneous and time-inhomogeneous Markov chains.

Stability properties of Markov chains, including finite-dimensional stability for discrete-space chains, is also studied in papers [18–20] using another variation of the coupling method called "maximal coupling". Those stability results are applied to the analysis of the impact of the stress factor in the "widow pension" actuarial model in papers [21,22].

This paper is organized as follows. Section 2 includes the main notations and assumptions, such as minorization and uniform proximity conditions. Section 3 has a description of the coupling space specially constructed for the coupling of two different time-inhomogeneous Markov chains. Section 4 includes auxiliary lemmas that serve as the tools for the main results. Section 5 has the main theorem of the paper. Section 6 is devoted to the case when the uniform proximity condition does not hold. Section 7 includes a summary of the main results, comparison with other classical results in a similar area, and possible directions of the research.

## 2. Notation and Main Assumptions

Consider a pair of time-inhomogeneous, independent, discrete-time Markov chains $X_n^{(1)}$ and $X_n^{(2)}$, defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with the values in the general state space $(E, \mathcal{E})$, where $E$ is some set and $\mathcal{E}$ is a -$\sigma$-field.

For a given transition kernel $K\colon E \times \mathcal{E} \to [0, \infty)$ and probability measure $\mu(dx)$, we will define a measure $\mu K(dx)$ as follows:

$$\mu K(A) = \int_E \mu(dx) K(x, A).$$

The one step transition probabilities will be denoted as follows:

$$P_{it}(x, A) = \mathcal{P}\{X_{t+1}^{(i)} \in A | X_t^{(i)} = x\},$$

where $t \in \mathbb{N}_0$, and $\mathbb{N}_0$ is the set of non-negative integers, $i \in \{1, 2\}$.

Throughout the paper, it will be assumed that the transition probabilities of the chains $X^{(1)}$ and $X^{(2)}$ can be represented in the following way:

$$P_{it}(x, A) = Q_t(x, A) + (1 - Q_t(x, E)) R_{it}(x, A). \tag{1}$$

Here, $Q_t$ is a substochastic kernel, such that $0 \le Q_t(x, E) \le 1$ for each $x \in E$, and $R_{it}$ are transition kernels. In representation (1), $Q_t$ plays a role of a "common part" of two chains and $R_{it}$ is the "distinguishing parts". Note that such representation is always possible for any two transition probabilities (for example, by setting $Q_t(x, A) = 0$, $R_{it} = P_{it}$). However, later we shall impose a condition that requires $Q_t(x, E)$ to be separated from 0.

Next condition plays a crucial role in the coupling construction used in the present paper.

$(M)$ For every $t \ge 0$, there exists a probability measure $\nu_t$ defined on the state space $(E, \mathcal{E})$, and constant $\alpha_t \in (0, 1)$ such that, for all $x \in E$, $A \in \mathcal{E}$ and $i \in \{1, 2\}$

$$P_{it}(x, A) \ge \alpha_t \nu_t(A). \tag{2}$$

In addition, we assume that $\alpha := \inf_{t \ge 0} \alpha_t \in (0, 1)$.

Assume that $\{B_k, k \ge 0\}$ is some collection of sets from $\mathcal{E}$, and let

$$B_n^{(t)} = \otimes_{j=1}^{n} B_{t+j}.$$

Denote $\mathcal{O} \subset \mathbb{N}_0$ as the set of indices such that $\nu(B_k) = 0$ if and only if $k \in \mathcal{O}$. Finally, denote a set

$$\mathcal{O}_n^{(t)} = \mathcal{O} \cap \{t, \ldots, t+n-1\},\ n \geq 1$$

and let

$$k_n^{(t)} = card(\mathcal{O}_n^{(t)}) \tag{3}$$

be a number of elements in the set $\mathcal{O}_n^{(t)}$. We need also a notation for the smallest non-zero value of $\nu(B_k)$:

$$b_n^{(t)} = \min_{k \notin \mathcal{O}_n^{(t)}} \nu(B_k) > 0.$$

Our goal is to give the upper bounds for the difference of probabilities for the chains $X_n^{(i)}$ not to leave sets $B_k$ for $t \leq k \leq t+n$.

In order to do this, we shall impose various conditions on $Q_t$ to ensure that chains $X^{(1)}$ and $X^{(2)}$ are close enough. The most obvious condition that will be used is a uniform proximity condition. It is formulated as follows:

$(U)$ The following upper bound holds:

$$\varepsilon_n^{(t)} := \sup_{x \in E, t \leq k \leq t+n} (1 - Q_t(x, B_k)) < 1. \tag{4}$$

Later, we shall relax condition (4) to allow $Q_t(x, B_k)$ to reach 0 for some $t$.

In addition, we need a condition related to the properties of the sets $B_n$, $n \geq 0$. Thus, we shall introduce the following domination condition.

$(D)$ There exists a real-valued sequence $S_n \geq 0$, $n \geq 0$ with a finite sum $m = \sum\limits_{k \geq 0} S_k < \infty$, such that

$$\int_{B_t} \nu(dx) \int_{B_n^{(t)}} Q_t(x, dy_1) Q_{t+1}(y_1, dy_2) \ldots Q_{t+n}(y_n, B_{t+n+1}) \leq S_n. \tag{5}$$

This condition can be referred as a requirement for the expectation of time, spent by a chain in the sets $B_n$, to be finite.

## 3. Coupling of Two Independent Time-Inhomogeneous Markov Chains with Different Transition Probabilities

Our goal is to construct a coupling for chains $X^{(1)}$ and $X^{(2)}$ using the minorization condition $(M)$. Let us introduce the following "non-coupling" operator:

$$T_t(x_1, x_2; A, B) = \frac{P_{1t}(x_1, A) - \alpha_t \nu_t(A)}{1 - \alpha_t} \frac{P_{2t}(x_2, B) - \alpha_t \nu_t(B)}{1 - \alpha_t}. \tag{6}$$

Please note that $T_t$ is a stochastic operator in the sense that

$$T_t(x_1, x_2; E, E) = 1.$$

In addition, it is obvious that

$$T_t(x_1, x_2; A, E) = \frac{P_{1t}(x_1, A) - \alpha_t \nu_t(A)}{1 - \alpha_t},\ T_t(x_1, x_2; E, A) = \frac{P_{2t}(x_2, A) - \alpha_t \nu_t(A)}{1 - \alpha_t}.$$

To construct a coupling, let us consider the Markov chain $Z_n = (Z_{1n}, Z_{2n}, d_n),\ n \geq 0$ with the state space $(E \times E \times \{0,1,2\})$. For the chain $Z_n$, we define transition probabilities as follows:

$$\begin{aligned}
&\mathbb{P}_t(x,y,0;A,B,\{0\}) = (1-\alpha_t)T_t(x,y,A,B),\\
&\mathbb{P}_t(x,y,0;A,B,\{1\}) = \alpha_t\nu_t(A\cap B),\\
&\mathbb{P}_t(x,y,0;A,B,\{2\}) = 0,\\
&\mathbb{P}_t(x,x,1;A,B,\{0\}) = (1-Q_t(x,E))R_{1t}(x,A)R_{2t}(x,B),\\
&\mathbb{P}_t(x,x,1;A,B,\{1\}) = 0,\\
&\mathbb{P}_t(x,x,1;A,B,\{2\}) = Q_t(x,A\cap B),\\
&\mathbb{P}_t(x,y,2;A,B,\{i\}) = \mathbb{P}_t(x,y,1;A,B,\{i\}),
\end{aligned} \tag{7}$$

where $Q_t$, $R_{1t}$, and $R_{2t}$ are defined in (1), and $T_t$ is defined in (6).

We can give the following interpretation of the chain defined by formulas (7). The chain can be in one of two principal "modes", coupled mode, when $d \in \{1,2\}$ or decoupled mode, when $d = 0$. If the chain is coupled at the moment $t$ and has a value $x \in E$ (in the coupled mode $Z_n^{(1)} = Z_n^{(2)}$), we flip the coin with the probability of success $Q_t(x,E)$. In the case of success, the chain remains coupled, and we render the next state with a probability proportional to $Q_t(x,\cdot)$. Otherwise, with probability $1 - Q_t(x,E)$, the chain is moving to the decoupled mode and is splitting into two values $y$ and $z$ rendered with probabilities $R_{1t}(x,dy)$ and $R_{2t}(x,dz)$, respectively.

In the decoupled mode, the chain is two-dimensional and is moving according to two independent trajectories, each governed by $(P_{it}(x,\cdot) - \alpha_t\nu_t(\cdot))/(1-\alpha_t)$. We can interpret this motion as follows. If the chain is decoupled, we flip a coin with the probability of success $\alpha_t$. In the case of success, the chain is coupling on the step $t+1$, and the next value is rendered with a probability $\nu_t(\cdot)$ so that, in this case, $d_{t+1} = 1$. Otherwise, with probability $1-\alpha_t$, the chain remains decoupled and renders values of the next step according to the $T_t$. Thus, the chain trajectory could be decomposed into cycles of coupling–decoupling, which is a key idea of proofs of the results in this paper.

Denote by $\mathbb{P}^{(t)}_{xyd}$ a probability on a canonical probability space for the chain $Z$ with transition probabilities $\mathbb{P}_t$, starting at the moment $t$ from the state $Z_t = (x,y,d),\ x,y \in E,\ d \in \{0,1,2\}$. Symbol $\mathbb{E}^{(t)}_{xyd}$ denotes corresponding expectation. Let us also define $\mathbb{P}^{(t)}_{\nu 1}(\cdot) := \int_E \nu(dx)\mathbb{P}^{(t)}_{xx1}(\cdot)$ and corresponding expectation $\mathbb{E}^{(t)}_{\nu 1}$. Introduce a special notation for the first decoupling after the moment $t$:

$$\tau^{(t)} = \inf\{n \geq t : d_n = 0\},$$

and, in order to simplify further calculations, we shall introduce a special notation for some of the probabilities related to the coupling. Namely, for all $i \in \{1,2\},\ t \geq 0, n \geq 1,\ x,y \in B_t$ and $z,w,v \in E$, we put

$$u^{(t)}_{i,n}(x) := \mathbb{P}^{(t)}_{xx1}\{d_{t+n} = 1,\ Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n-1}\},\ n \geq 2, \tag{8}$$

$$q^{(t)}_n(dy) := \mathbb{P}^{(t)}_{\nu 1}\{d_{t+k} = 2,\ Z^{(1)}_t = Z^{(2)}_t \in B_t,\ Z^{(1)}_{t+k} = Z^{(2)}_{t+k} \in B_{t+k},\ Z^{(1)}_{t+n} = Z^{(2)}_{t+n} \in dy,\ k = \overline{1,n}\}, \tag{9}$$

$$h^{(t)}_{i,n}(x,y;dz,dw) := \mathbb{P}^{(t)}_{xy0}\{d_{t+k} = 0,\ Z^{(i)}_{t+k} \in B_{t+k},\ Z^{(1)}_{t+n} \in dz,\ Z^{(2)}_{t+n} \in dw,\ k = \overline{1,n}\}, \tag{10}$$

$$R_n(z,dw,dv) := R_{1n}(z,dw)R_{2n}(z,dv), \tag{11}$$

$$p^{(t)}_{i,n}(x) = \mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n}\}. \tag{12}$$

We put $u^{(t)}_{i,0}(x) = u^{(t)}_{i,1}(x) = 0$ and $q^{(t)}_0(dy) = 0$.

## 4. Auxiliary Lemmas

In this section, we shall introduce auxiliary lemmas that play an important role in getting the proofs of the main result.

**Lemma 1.** *For all $i \in \{1,2\}$, $x \in B_t$, $t \geq 0$, $n \geq 2$, the following inequality holds true:*

$$\sum_{k=0}^{n} u_{i,k}^{(t)}(x) q_{n-k}^{(t+k)}(E) \leq p_{i,n}^{(t)}(x) \leq 1,$$

*where $p_{i,n}^{(t)}(x)$ is defined in (12).*

**Proof.** Using definitions (7), (8) and denoting $D := \{0,1,2\}$, we can write:

$$u_{1,k}^{(t)}(x) = \mathbb{P}_{xx1}^{(t)}\{d_{t+k} = 1,\ Z_{t+j}^{(1)} \in B_{t+j},\ j = \overline{1,k-1}\} = \int_{B_{t+1}\times E\times D} \mathbb{P}_t(x,x,1;du_1,dv_1,dz_1)\times$$

$$\times \int_{B_{t+2}\times E\times D} \mathbb{P}_{t+1}(u_1,v_1,z_1;du_2,dv_2,dz_2)\ldots \int_{E\times E\times\{1\}} \mathbb{P}_{t+k-1}(u_{t+k-1},v_{t+k-1},0;du_{t+k},dv_{t+k},1) =$$

$$= \int_{B_{t+1}\times E\times D} \mathbb{P}_t(x,x,1;du_1,dv_1,dz_1)\ldots\mathbb{P}_{t+k-2}(u_{t+k-2},v_{t+k-2},z_{t+k-2};B_{t+k-1},E,0)\alpha_{t+k-1}\nu_{t+k-1}(E).$$

Thus, we established the equality

$$u_{1,k}^{(t)}(x) = \int_{B_{t+1}\times E\times D} \mathbb{P}_t(x,x,1;du_1,dv_1,dz_1)\ldots\mathbb{P}_{t+k-2}(u_{t+k-2},v_{t+k-2},z_{t+k-2};B_{t+k-1},E,0)\alpha_{t+k-1}. \quad (13)$$

Obviously, the similar equality holds true for $u_{2,k}^{(t)}$. Now, using definitions (7) and (9), we can write:

$$q_{n-k}^{(t+k)}(E) = \int_{B_{t+k}} \nu_{t+k-1}(dx) \int_{B_{t+k+1}} Q_{t+k+1}(x,dx_1)\ldots\int_{B_{t+n}} Q_{t+n-1}(x_{n-k-1},dx_{n-k}). \quad (14)$$

Combining (13) and (14), we get that

$$u_{1,k}^{(t)}(x)q_{n-k}^{(t+k)}(E) = \int_{B_{t+1}\times E\times D} \mathbb{P}_t(x,x,1;du_1,dv_1,dz_1)\ldots\mathbb{P}_{t+k-2}(u_{t+k-2},v_{t+k-2},z_{t+k-2};B_{t+k-1},E,0)\times$$

$$\times\alpha_{t+k-1}\int_{B_{t+k}} \nu_{t+k-1}(dx) \int_{B_{t+k+1}} Q_{t+k+1}(x,dx_1)\ldots\int_{B_{t+n}} Q_{t+n-1}(x_{n-k-1},dx_{n-k}).$$

Using (7), we can see that, for any $u, v \in E$,

$$\begin{aligned} \alpha_{t+k-1}\textstyle\int_{B_{t+k}} \nu_{t+k-1}(dx) \int_{B_{t+k+1}} Q_{t+k+1}(x,dx_1)\ldots\int_{B_{t+n}} Q_{t+n-1}(x_{n-k-1},dx_{n-k}) = \\ = \textstyle\int_{B_{t+k}} \mathbb{P}_{t+k-1}(x,y,0;du_{t+k},du_{t+k},1)\times \\ \times \textstyle\int_{B_{t+k+1}} \mathbb{P}_{t+k}(u_{t+k},u_{t+k},1;du_{t+k+1},du_{t+k+1},2)\ldots\mathbb{P}_{t+n}(u_{t+n-1},u_{t+n-1},2;B_{t+n},B_{t+n},2). \end{aligned} \quad (15)$$

Finally, combining (13)–(15), we obtain the following equality:

$$u_{1,k}^{(t)}(x)q_{n-k}^{(t+k)}(E) = \int_{B_{t+1}\times E\times D} \mathbb{P}_t(x,x,1;du_1,dv_1,dz_1)\ldots\times$$

$$\int_{B_{t+k-1}\times E\times\{0\}} \mathbb{P}_{t+k-2}(u_{t+k-2},v_{t+k-2},z_{t+k-2};du_{t+k-1},dv_{t+k-1},dz_{t+k-1})\times$$

$$\times\int_{B_{t+k}\times E\times\{1\}} \mathbb{P}_{t+k-1}(u_{t+k-1},v_{t+k-1},0;du_{t+k},du_{t+k},dz_{t+k})\times$$

$$\times\int_{B_{t+k+1}} \mathbb{P}_{t+k}(u_{t+k},u_{t+k},1;du_{t+k+1},du_{t+k+1},2)\ldots P_{t+n}(u_{t+n-1},u_{t+n-1},2;B_{t+n},B_{t+n},2) =$$

$$= \mathbb{P}_{xx1}^{(t)}\{d_{t+k} = 1,\ d_{t+k+l} = 2,\ l = 1,n-k,\ Z_{t+j}^{(1)} \in B_{t+j},\ j = \overline{1,n}\}.$$

Taking into account the similar equality for $u^{(t)}_{2,k}$, we we can state that

$$u^{(t)}_{i,k}(x)q^{(t+k)}_{n-k}(E) = \mathbb{P}^{(t)}_{xx1}\{d_{t+k} = 1,\ d_{t+k+l} = 2,\ l = 1, n-k,\ Z^{(i)}_{t+j} \in B_{t+j},\ j = \overline{1,n}\}. \tag{16}$$

Furthermore, using (16), we can derive that

$$\sum_{k=0}^{n} u^{(t)}_{i,k}(x)q^{(t+k)}_{n-k}(E) = \sum_{k=0}^{n} \mathbb{P}^{(t)}_{xx1}\{d_{t+k} = 1,\ d_{t+k+l} = 2,\ l = 1, n-k,\ Z^{(i)}_{t+j} \in B_{t+j},\ j = \overline{1,n}\} \leq$$

$$\leq \mathbb{P}^{(t)}_{xx1}\{d_{t+n} = 2,\ Z^{(i)}_{t+j} \in B_{t+j},\ j = \overline{1,n}\} \leq p^{(t)}_{i,n}(x) \leq 1.$$

□

The next lemma gives an upper bound for a probability of a non-coupling during certain period. Here, $k^{(t)}_n$ is a cardinality of the set $\mathcal{O}^{(t)}_n$.

**Lemma 2.** *Assume that $b^{(t)}_n \alpha < 1$. Then, for all $i \in \{1,2\}$, $x, y \in B_t$, $t \geq 0$ and $n \geq 0$, the following inequality holds true:*

$$h^{(t)}_{i,n}(x, y; E, E) \leq (1 - b^{(t)}_n \alpha)^{-k^{(t)}_n} (1 - b^{(t)}_n \alpha)^n.$$

**Proof.** Taking the notation (7) for the transition probabilities and the notation (10) for $h^{(t)}_{i,n}$, we can derive the following relation:

$$h^{(t)}_{1,n}(x, y; E, E) = \prod_{k=0}^{n-1}(1 - \alpha_{t+k}) \int_{B_{t+1}\times E} T_t(x, y; dx_1, dy_1)\times$$

$$\int_{B_{t+2}\times E} T_{t+1}(x_1, y_1; dx_2, dy_2) \dots \int_{B_{t+n}\times E} T_{t+n-1}(x_{n-1}, y_{n-1}; dx_n, dy_n).$$

A similar formula holds true for $h^{(t)}_{2,n}(x, y; E, E)$. Thus, we can write:

$$h^{(t)}_{i,n}(x, y; E, E) = \prod_{k=0}^{n-1}(1 - \alpha_{t+k})\tilde{T}^{t,n}_i(x, y; E, E),$$

where operator $\tilde{T}_{1,t}(x, y; du, dv) = T_t(x, y; du \cap B_{t+1}, dv)$, $\tilde{T}_{2,t} = T_t(x, y; du, dv \cap B_{t+1})$, $T_t$ is defined in (6) and

$$\tilde{T}^{t,n}_i = \prod_{k=0}^{n-1} \tilde{T}_{i,t+k}.$$

Here, index $n$ stands for the number of terms in the product.

We have:

$$\prod_{k=0}^{n-1}(1 - \alpha_{t+k})\tilde{T}^{t,n}_i(x, y; E \times E) = \int_{B^{(t)}_n} (P_{it}(x, dy_1) - \alpha_t \nu_t(dy_1)) \dots (P_{it+n-1}(y_{n-1}, B_{t+n}) - \alpha_{t+n-1}\nu_{t+n-1}(B_{t+n}))$$

Let us consider the value $P_{it}(y, B_{t+1}) - \alpha_t \nu_t(B_{t+1})$. If $t + 1 \notin \mathcal{O}$, then $\nu(B_{t+1}) \geq b^{(t)}_n$, for all $n \geq 1$, so:

$$P_{it}(y, B_{t+1}) - \alpha_t \nu_t(B_{t+1}) \leq 1 - \alpha_t b^{(t)}_n \leq 1 - \alpha b^{(t)}_n.$$

If $t + 1 \in \mathcal{O}$, then

$$P_{it}(y, B_{t+1}) - \alpha_t \nu_t(B_{t+1}) \leq 1 = \frac{1 - \alpha b^{(t)}_n}{1 - \alpha b^{(t)}_n}.$$

Finally, we can derive

$$h_{i,n}^{(t)}(x,y;E,E) = \prod_{k=0}^{n-1}(1-\alpha_{t+k})\tilde{T}_i^{t,n}(x,y;E,E)$$

$$\leq \prod_{j=0}^{n-1}\left(\sup_{y\in E} P_{it+j}(y,B_{t+j+1}) - \alpha_{t+j}\nu_{t+j}(B_{t+j+1})\right) \leq$$

$$\leq \frac{(1-\alpha b_n^{(t)})^n}{(1-\alpha b_n^{(t)})^{k_n^{(t)}}} = (1-\alpha b_n^{(t)})^{n-k_n^{(t)}} = (1-b_n^{(t)}\alpha)^{-k_n^{(t)}}(1-b_n^{(t)}\alpha)^n.$$

□

Next, three lemmas play a key role in the subsequent proof of the main result of the paper.

**Lemma 3.** *Assume that $b_n^{(t)}\alpha < 1$. Then, for all $i \in \{1,2\}$, $t \geq 0$, $n \geq 1$ and $x \in B_t$, the following inequality holds true:*

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k},\ k=\overline{1,n}, d_{t+n}=0\} \leq \varepsilon_n^{(t)}\frac{1-(1-b_n^{(t)}\alpha)^{n-1}}{b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}},$$

*where $\varepsilon_n^{(t)}$ is defined in (4).*

**Proof.** In the case $n = 1$, the statement of the lemma is obvious. Thus, we shall consider the case $n \geq 2$. Using definitions (7)–(11) of the corresponding probabilities and considering the last coupling time $k \geq 0$ and the last decoupling time $1 \leq j \leq n-k$, we can decompose probability $\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, d_{t+n}=0,\ k=\overline{1,n}\}$, in the following way

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(1)} \in B_{t+k}, d_{t+n}=0,\ k=\overline{1,n}\} =$$

$$= \sum_{k=0}^{n-1}\sum_{j=1}^{n-k}\int_{B_{t+k}}\int_{B_{t+k+1}\times E} u_{1,k}^{(t)}(x)q_{j-1}^{(t+k)}(dy)(1-Q_{t+k+j}(y,E))\times$$

$$\times R_{t+k+j}(y,du,dv)h_{n-k-j}^{(t+k+j)}(u,v;E,E) \leq$$

$$\leq \varepsilon_n^{(t)}(1-b_n^{(t)}\alpha)^{-k_n^{(t)}}\sum_{k=0}^{n-1}\sum_{j=1}^{n-k}u_{1,k}^{(t)}(x)q_{j-1}^{(t+k)}(E)(1-b_n^{(t)}\alpha)^{n-k-j}.$$

In the last inequality, we used definition (4) of $\varepsilon_n^{(t)}$, Lemma 2, and the fact that $R_{it}(x,E)=1$. Now, we can change the order of summation in the previous inequality and derive that

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, d_{t+n}=0,\ k=\overline{1,n}\} \leq \varepsilon_n^{(t)}(1-b_n^{(t)}\alpha)^{-k_n^{(t)}}\sum_{j=1}^{n}(1-\alpha b_n^{(t)})^{j-1}\times$$

$$\times\sum_{k=0}^{n-j}u_{1,k}^{(t)}(x)q_{n-j-k}^{(t+k)}(E),$$

Using Lemma 1, we get $\sum\limits_{k=0}^{n-j}u_{1,k}^{(t)}(x)q_{n-j-k}^{(t+k)}(E) \leq 1$ and finally

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(1)} \in B_{t+k}, d_{t+n}=0,\ k=\overline{1,n}\} \leq \varepsilon_n^{(t)}(1-b_n^{(t)}\alpha)^{-k_n^{(t)}}\sum_{j=1}^{n}(1-\alpha b_n^{(t)})^{j-1} =$$

$$= \varepsilon_n^{(t)} \frac{1 - (1 - \alpha b_{n-1}^{(t)})^{n-1}}{b_n^{(t)} \alpha (1 - b_n^{(t)} \alpha)^{k_n^{(t)}}}.$$

Similarly, we can obtain inequality for $i = 2$. $\square$

**Lemma 4.** *Assume that $b_n^{(t)} \alpha < 1$. Then, for all $i \in \{1, 2\}$, $x \in B_t$, $t \geq 0$ and $n \geq 2$ such that $t + n \notin \mathcal{O}$, the following inequality holds true:*

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(i)} \in B_{t+k}, d_{t+n} = 1,\ k = \overline{1, n}\} \leq \varepsilon_n^{(t)} \alpha_{t+n-1} \nu_{t+n-1}(B_{t+n}) \frac{1 - (1 - b_n^{(t)} \alpha)^{n-2}}{b_n^{(t)} \alpha (1 - b_n^{(t)} \alpha)^{k_n^{(t)}}},$$

*where $\varepsilon_n^{(t)}$ is defined in (4).*

**Proof.** It follows from (7) that the transition into the state $\{d_{t+n} = 1\}$ is possible only from the state $\{d_{t+n-1} = 0\}$. Then, using Markov property, we can write:

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, d_{t+n} = 1,\ k = \overline{1, n}\} =$$

$$\int\limits_{B_{t+n-1} \times E} \mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, Z_{t+n-1} = (du, dv, 0),\ k = \overline{1, n-1}\} \times$$

$$\times \mathbb{P}_{uv0}^{(t+n-1)} \{d_{t+n} = 1, Z_{t+n}^{(1)} \in B_{t+n}\} =$$

$$= \alpha_{t+n-1} \nu_{t+n-1}(B_{t+n}) \mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, d_{t+n-1} = 0,\ k = \overline{1, n-1}\}.$$

Using Lemma 3, we get that

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, d_{t+n} = 1,\ k = \overline{1, n}\} \leq \alpha_{t+n-1} \nu_{t+n-1}(B_{t+n}) \varepsilon_n^{(t)} \frac{1 - (1 - b_n^{(t)} \alpha)^{n-2}}{b_n^{(t)} \alpha (1 - b_n^{(t)} \alpha)^{k_n^{(t)}}} =$$

The same reasoning and transformations hold for $i = 2$. $\square$

**Lemma 5.** *Assume domination condition $(D)$ holds true. Then, for all $i \in \{1, 2\}$, $x \in B_t$, $t \geq 0$ and $n > 2$ such that $t + n \notin \mathcal{O}$, the following inequality holds true:*

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(i)} \in B_{t+k}, \exists j\ d_{t+j} = 0,\ d_{t+n} = 2,\ k = \overline{1, n}\} \leq \varepsilon_n^{(t)} \frac{m}{b_n^{(t)} \alpha (1 - b_n^{(t)} \alpha)^{k_n^{(t)}}},$$

*where $m$ is a constant from the domination condition (5).*

**Proof.** The same ideas are used in the proof of this statement as in the proof of Lemma 4. Namely,

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, \exists j\ d_{t+j} = 0,\ d_{t+n} = 2,\ k = \overline{1, n}\} =$$

$$= \sum_{j=2}^{n-1} \int\limits_{B_{t+j}} \mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, Z_{t+j} \in (dy, dy, 1)\ k = \overline{1, j-1}\} \times$$

$$\times \mathbb{P}_{yy1} \{Z_{t+l}^{(1)} \in B_{t+l},\ d_{t+l} = 2,\ l = \overline{j+1, n}\}.$$

Using Lemma 4 and domination condition (5), we can derive:

$$\mathbb{P}_{xx1}^{(t)} \{Z_{t+k}^{(1)} \in B_{t+k}, \exists j\ d_{t+j} = 0,\ d_{t+n} = 2,\ k = \overline{1, n}\} \leq$$

$$\sum_{j=2}^{n-1} \varepsilon_n^{(t)} \alpha_{t+j-1} \nu_{t+j-1}(B_{t+j}) \frac{1-(1-b_n^{(t)}\alpha)^{j-2}}{b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}} q_{n-j}^{(t+j)} \le$$

$$\le \sum_{j=2}^{n-1} \varepsilon_n^{(t)} \alpha_{t+j-1} \nu_{t+j-1}(B_{t+j}) \frac{1-(1-b_n^{(t)}\alpha)^{j-2}}{b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}} S_{n-j} \le$$

$$\le \frac{\varepsilon_n^{(t)}}{b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}} \sum_{j=2}^{n-1} (1-(1-b_n^{(t)}\alpha)^{j-2}) S_{n-j} \le \varepsilon_n^{(t)} \frac{\sum\limits_{j=2}^{n-1} S_{n-j}}{b_n^{(t)}(1-b_n^{(t)}\alpha)^{k_n^{(t)}}} \le$$

$$\le \varepsilon_n^{(t)} \frac{m}{b_n^{(t)}\alpha(1-\alpha)^{k_n^{(t)}}}.$$

□

## 5. Main Result

Now, we can state the main result of the paper.

**Theorem 1.** *Assume $X_n^{(1)}$ and $X_n^{(2)}$ are two time-inhomogeneous, irreducible, non-periodic Markov chains defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, admitting representation (1), together with minorization condition $(M)$, uniform proximity condition $(U)$, and domination condition $(D)$.*

*Then, the following inequality holds true for all $n \ge 2$:*

$$|\mathcal{P}\{X_{t+k}^{(1)} \in B_{t+k},\ k=\overline{1,n} | X_t^{(1)} = x\} - \mathcal{P}\{X_{t+k}^{(2)} \in B_{t+k},\ k=\overline{1,n} | X_t^{(2)} = x\}| \le$$
$$\frac{\varepsilon_n^{(t)}}{b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}} \left(1-(1-\alpha b_n^{(t)})^{n-1} + \alpha_{t+n-1}\nu_{t+n-1}(B_{t+n})(1-(1-\alpha b_n^{(t)})^{n-2}) + m\right). \quad (17)$$

**Proof.** First, we will note that

$$\mathcal{P}\{X_{t+k}^{(i)} \in B_{t+k},\ k=\overline{1,n} | X_t^{(i)} = x\} = \mathbb{P}_{xx1}^{(t)}\{Z_{t=k}^{(i)} \in B_{t+k},\ k=\overline{1,n}\} = p_{i,n}^{(t)}(x).$$

Secondly, we can note that, on the set $\{\tau^{(t)} > n\}$ values, $Z_{t+k}^{(1)}$ and $Z_{t+k}^{(2)}$ coincide if $Z_t = (x,x,1)$, and $k = \overline{1,n}$. Thus, we can write:

$$|p_{1,n}^{(t)}(x) - p_{2,n}^{(t)}(x)| = |\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(1)} \in B_{t+k}, k=\overline{1,n},\ \tau^{(t)} \le n\} -$$
$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(2)} \in B_{t+k}, k=\overline{1,n},\ \tau^{(t)} \le n\}| \le$$
$$\max_{i\in\{1,2\}} \left\{\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, k=\overline{1,n},\ \tau^{(t)} \le n\}\right\}.$$

Next, we can decompose a probability $\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, k=\overline{1,n}, \tau^{(t)} \le n\}$ by the first after $t$ decoupling ($d_k = 0$). We note that $\{\exists k \in \{1,\ldots,n\},\ d_{t+k} = 0\} \subset \{\tau^{(t)} \le n\}$.

Now, we will analyze three possibilities: $d_{t+n} = 0$, $d_{t+n} = 1$ or $d_{t+n} = 2$. We can write

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, k=\overline{1,n}, \tau^{(t)} \le n\} =$$

$$= \mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, d_{t+n} = 0,\ k=\overline{1,n}\} + \mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, d_{t+n} = 1,\ k=\overline{1,n}\} +$$
$$+ \mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)} \in B_{t+k}, \exists j\ d_{t+j} = 0,\ d_{t+n} = 2,\ k=\overline{1,n}\}.$$

Consider all three terms separately. Estimate for the first term is given in Lemma 3, for the second term in Lemma 4, and for the third term in Lemma 5.

Combining the three estimates from Lemmas 3–5, we obtain inequality (17). □

## 6. The Case When Uniform Proximity Condition Is Violated

In this section, we will consider a case when uniform proximity condition (4) does not hold. We assume that there is a set $\mathbb{T}$ of non-negative integers, such that we can not expect $Q_t(x,E)$ to be small when $t \in \mathbb{T}$. In other words, we expect that proximity condition holds for all $t \notin \mathbb{T}$. Thus, we shall introduce another proximity condition:
(U2) Proximity condition:

$$\hat{\varepsilon} = \sup_{t \notin \mathbb{T}, x \in E} (1 - Q_t(x,E)) < 1. \tag{18}$$

We will also denote

$$\hat{\varepsilon}_n^{(t)} = \sup_{0 \le u \le t, u \notin \mathbb{T}, x \in E} (1 - Q_u(x,E)) \le \hat{\varepsilon} < 1, \tag{19}$$

and

$$\eta_n^{(t)} = \max\{k \in \{t, t+1, \ldots, t+n\} \cap \mathbb{T}\} - t,\ \eta_n^{(t)} \in \{0, \ldots, n\} \cup \{\infty\}. \tag{20}$$

In case $\{t, t+1, \ldots, t+n\} \cap \mathbb{T} = \varnothing$, we will put $\eta_n^{(t)} = \infty$. Let us introduce a special definition

$$\kappa_n^{(t)} = \begin{cases} k^{(t)}_{\eta_n^{(t)}}, \text{ if } \eta_n^{(t)} < \infty, \\ 0, \text{ if } \eta_n^{(t)} = \infty. \end{cases}$$

Now, we will derive analogues of Lemmas 3–5 under condition (18).

**Lemma 6.** *Assume that $b_n^{(t)}\alpha < 1$ and $\eta_n^{(t)} < n$. Then, for all $i \in \{1,2\}$, $t \ge 0$, $n \ge 2$ and $x \in B_t$, the following inequality holds true:*

$$\mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n}, d_{t+n} = 0\} \le$$
$$\le (1 - b_n^{(t)}\alpha)^{-k_n^{(t)}}(1 - b_n^{(t)}\alpha)^{n - \eta_n^{(t)}} + \hat{\varepsilon}_n^{(t)} \left( \frac{1 - (1 - b_n^{(t)}\alpha)^{n - \eta_n^{(t)} - 1}}{b_n^{(t)}\alpha (1 - b_n^{(t)}\alpha)^{k_n^{(t)}}} \right)^2,$$

*where $\hat{\varepsilon}_n^{(t)}$ is defined in (19) and $\eta_n^{(t)}$ is defined in (20).*

**Proof.** In this proof, we will write $\eta$ instead of $\eta_n^{(t)}$ in order to simplify derivations.
Let us decompose probability $\mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n}, d_{t+n} = 0\}$ by the time $\eta$:

$$\mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n}, d_{t+n} = 0\} =$$
$$\mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n},\ d_{t+\eta} \in \{0,1,2\},\ d_{t+n} = 0\} \le$$
$$\le \mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,n},\ d_{t+\eta} = 0,\ d_{t+n} = 0\} =$$
$$= \int \mathbb{P}^{(t)}_{xx1}\{Z^{(i)}_{t+k} \in B_{t+k},\ k = \overline{1,\eta},\ Z_{t+\eta} \in (du, dv, 0)\} \times$$
$$\times \mathbb{P}^{(t+\eta)}_{uv0}\{Z^{(i)}_{t+k} \in B_k,\ k = \overline{t+\eta+1, t+n},\ d_{t+n} = 0\} \le$$
$$\sup_{u,v \in B_\eta \times B_\eta} \mathbb{P}^{(t+\eta)}_{uv0}\{Z^{(i)}_k \in B_k,\ k = \overline{t+\eta+1, t+n},\ d_{t+n} = 0\}.$$

Let us now define: $t_0 = t + \eta$, $n_0 = n - \eta$. Then, we can decompose probability $\mathbb{P}^{(t+\eta)}_{uv0}\{Z^{(i)}_k \in B_k,\ k = \overline{t+\eta+1, t+n},\ d_{t+n} = 0\}$ by the moment of the first coupling (taking into account the case in which no coupling happens between $t_0$ and $t_0 + n_0$):

$$\mathbb{P}^{(t+\eta)}_{uv0}\{Z^{(i)}_k \in B_k,\ k = \overline{t+\eta+1, t+n},\ d_{t+n} = 0\} \le h^{(t_0)}_{i,n_0}(u,v;E,E) +$$

$$\alpha_{t_0+k-1}\sum_{k=1}^{n_0-1} h_{i,k-1}^{(t_0)}(u,v;E,E)\sup_x \mathbb{P}_{xx1}^{(t_0+k)}\{Z_{t_0+l}^{(t)}\in B_{t_0+l},\ l=\overline{k,n_0},\ d_{t_0+n_0}=0\}.$$

Now, using Lemma 3 to estimate $\mathbb{P}_{xx1}^{(t_0+k)}\{Z_{t_0+l}^{(t)}\in B_{t_0+l},\ l=\overline{k,n_0},\ d_{t_0+n_0}=0\}$ and the fact that: $\alpha_{t_0+k-1}\sum\limits_{k=1}^{n_0-1} h_{i,k}^{(t_0)}(u,v;E,E)\le (1-b_n^{(t)}\alpha)^{-k_n^{(t)}}\frac{1-(1-b_n^{(t)}\alpha)^{n_0-1}}{b_n^{(t)}\alpha}$, we get the statement of the lemma. □

The same exact decomposition can be used to prove other two lemmas.

**Lemma 7.** *Assume that $b_n^{(t)}\alpha<1$ and $\eta_n^{(t)}<n$. Then, for all $i\in\{1,2\}$, $x\in B_t$, $t\ge 0$ and $n>2$ such that $t+n\notin\mathcal{O}$, the following inequality holds true:*

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)}\in B_{t+k}, d_{t+n}=1,\ k=\overline{1,n}\}\le$$
$$\le (1-\alpha)^{-k_n^{(t)}}(1-b_n^{(t)}\alpha)^{n-\eta_n^{(t)}}+\hat{\varepsilon}_n^{(t)}\alpha_{t+n-1}\nu_{t+n-1}(B_{t+n})\left(\frac{1-(1-b_n^{(t)}\alpha)^{n-\eta_n^{(t)}-2}}{b_n^{(t)}\alpha(1-\alpha)^{k_n^{(t)}}}\right)^2,$$

*where $\hat{\varepsilon}_n^{(t)}$ is defined in (18).*

**Lemma 8.** *Assume the domination condition holds true and $\eta_n^{(t)}<n$. Then, for all $i\in\{1,2\}$, $x\in B_t$, $t\ge 0$ and $n\ge 2$ such that $t+n\notin\mathcal{O}$, the following inequality holds true:*

$$\mathbb{P}_{xx1}^{(t)}\{Z_{t+k}^{(i)}\in B_{t+k}, \exists j\ d_{t+j}=0,\ d_{t+n}=2,\ k=\overline{1,n}\}\le$$
$$\le (1-\alpha)^{-k_n^{(t)}}(1-b_n^{(t)}\alpha)^{n-\eta_n^{(t)}}+\hat{\varepsilon}_n^{(t)}\frac{m(1-b_n^{(t)}\alpha)^{n-\eta_n^{(t)}-1}}{(b_n^{(t)}\alpha(1-\alpha)^{k_n^{(t)}})^2},$$

*where $m$ is a constant from the domination condition (5).*

Now, we can state the stability theorem for this case.

**Theorem 2.** *Assume $X_n^{(1)}$ and $X_n^{(2)}$ are two time-inhomogeneous, irreducible, non-periodic Markov chains defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$, admitting representation (1), together with minorization condition $(M)$, proximity condition $(U2)$, domination condition $(D)$, and $\eta_n^{(t)}<n$.*

*Then, the following inequality holds true:*

$$|\mathcal{P}\{X_{t+k}^{(1)}\in B_{t+k},\ k=1,n|X_t^{(1)}=x\}-\mathcal{P}\{X_{t+k}^{(2)}\in B_{t+k},\ k=1,n|X_t^{(2)}=x\}|\le$$
$$3(1-\alpha)^{-k_n^{(t)}}(1-b_n^{(t)}\alpha)^{n-\eta_n^{(t)}}+\frac{\hat{\varepsilon}_n^{(t)}}{\left(b_n^{(t)}\alpha(1-b_n^{(t)}\alpha)^{k_n^{(t)}}\right)^2}\times$$
$$\times\left(\left(1-(1-\alpha b_n^{(t)})^{n-\eta_n^{(t)}-1}\right)^2+\alpha_{t+n-1}\nu_{t+n-1}(B_{t+n})\left((1-(1-\alpha b_n^{(t)})^{n-\eta_n^{(t)}-2})\right)^2+m(1-\alpha b_n^{(t)})^{n-\eta_n^{(t)}-1}\right).$$

**Proof.** The proof replicates the proof of Theorem 1. □

## 7. Conclusions

In this paper, we obtained a stability estimate for finite-dimensional distributions of a perturbed time-inhomogeneous Markov chain defined on a general state space. The stability estimate has the order of $\varepsilon m$ (see definition of $\varepsilon$ at (4) and the definition of $m$ at condition (D)). Estimates are obtained under the uniform proximity condition (4) and relaxed proximity condition (19). The results of the paper correlate with the similar results in the classical literature.

For example, the book [3] provides a series of stability estimates for time-homogeneous Markov chains where proximity estimates also have an order of $\varepsilon$, see ([3], Section V).

The principal example of an application for the results of the paper is a calculation of estimates for the value $P_x\{\tau > t\}$, where $\tau$ is a moment of reaching of the certain critical set $C$ by a perturbed time-inhomogeneous Markov chain $Y_t$. It is important that the results are obtained under relatively general conditions, which could be checked for practical applications. Theorems 1 and 2 provide an important improvement to the similar results of the papers [3] (comparing to this paper, our results are generalized to time-inhomogeneous case and we propose simpler conditions of stability) and [20] (unlike this paper, we consider general phase space, simpler and less restrictive conditions, and bounds that are easier to calculate).

The results of the paper admit the further improvements. For example, Doeblin's condition $(M)$ could be relaxed to the assumption $P_{it}(x, A) \geq \alpha_t \nu_t(A)$, $\forall x \in C$, where $C \in \mathcal{E}$ is a certain set. Such condition corresponds to the chains which are not of uniform-ergodic type (i.e., uniform mixing condition does not hold). Such results, as well as practical applications, are the subjects of a future study.

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