mathematics

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pth Moment Stability of a Stationary Solution for a Reaction Diffusion System with Distributed Delays

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Received: 1 December 2019; Accepted: 30 January 2020; Published: 6 February 2020

Abstract: In this paper, the Sobolev embedding theorem, Holder inequality, the Lebesgue contrl convergence theorem, the operator norm estimation technique, and critical point theory are employed to prove the existence of nontrivial stationary solution for p-Laplacian diffusion system with distributed delays. Furthermore, by giving the definition of pth moment stability, the authors use the Lyapunovfunctional method and Kamke function to derive the stability of nontrivialstationary solution. Moreover, a numerical example illuminates the effectiveness of the proposed methods. Finally, an interesting further thought is put forward, which is conducive to the in-depth study of the problem.

Keywords: large-scale variational methods; Lyapunovfunctional; pth moment stability

1. Introduction

It is well known that, in practical engineering, electrons inevitably diffuse in the inhomogeneous electromagnetic field. In addition, hence, the stability analysis of the reaction–diffusion system has become a hot topic [1–14]. In recent decades, many authors, such as Linshan Wang, Qiankun Song and Jinde Cao, have studied the stability of Laplacian reaction–diffusion neural networks with time delay, and achieved fruitful results in Laplacian diffusion systems [7–14]. On the other hand, p-Laplacian diffusion systems have also been widely studied.

In 2018, the asymptotic behavior of a p-Laplacian reaction–diffusion dynamic system is studied in [15]:

$$\frac{\partial u}{\partial t} - a(lu)\Delta_p u = f(u) + h(t). \tag{1}$$

However, in practical engineering, the time delay is unavoidable, which may lead to chaos and instability of the system [16–24]. Thus, in this paper, we are to investigate the delayed p-Laplacian reaction–diffusion dynamic system. In addition, similar research has already begun. In [24], the p-Laplacian diffusion was firstly introduced in a time-delay dynamical system. Inspired by some methods of [24], the authors in [25] employed an impulsive differential inequality lemma to further study the time-delay neural networks with pulse perturbation. In [16], Ruofeng Rao and Shouming Zhong employed the Ekeland variational principle and Lyapunov stability theory to derive a globally exponential pth moment stability criterion for a Markovian jumping T–S fuzzy diffusion system with time-delays:

$$\begin{cases} \dfrac{du_i(t,x)}{dt} = \mathcal{D}_i div\Big(|\nabla u_i|^{p-2}\nabla u_i\Big) - b_i(u_i) + \sum\limits_{j=1}^{J}\rho_j(\omega(t))\Big(c_{ij}(r(t))f_i(u_i(t,x)) \\ \qquad\qquad + d_{ij}(r(t))g_i(u_i(t-\tau_i(t),x))\Big) + I_i, \quad i\in\mathcal{N}, t>0,\, x\in\Omega, \\ u_i(t,x) = 0, \quad i\in\mathcal{N}, t\geqslant 0, x\in\partial\Omega. \end{cases} \tag{2}$$

On the other hand, fuzzy logic theory has shown to be an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. Among various kinds of fuzzy methods, Takagi–Sugeno (T–S) fuzzy models provide a successful method to describe certain complex nonlinear systems using some local linear subsystems [26–31].

Motivated by some ideas and methods in [16,32–40,46], we are to investigate the stability of a class of p-Laplacian diffusion T–S fuzzy system via variational methods that are different from those of existing literature related to reaction–diffusion fuzzy systems.

2. Preliminaries

Consider the following p-Laplacian system with time delays:

Fuzzy rule j: **IF** $\omega_1(t)$ is μ_{j1} and $\cdots$ $\omega_m(t)$ is μ_{jm} **THEN**

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} = a_i(t,u_1,\cdots,u_n)\Delta_p u_i(t,x) - b_i(u_i(t,x)) + c_{ij}f_i(u_i(t,x)) \\ + d_{ij}\displaystyle\int_{t-\tau(t)}^{t} f_i(u_i(s,x))ds + J_i,\ i\in\mathcal{N},\ t\geqslant 0,\ x\in\Omega, \\ u_i(\theta,x) = \phi_i(\theta,x), \quad (\theta,x)\in[-\tau,0]\times\Omega,\ i\in\mathcal{N}, \\ u_i(t,x) = 0, \quad (t,x)\in[-\tau,+\infty)\times\partial\Omega,\ i\in\mathcal{N}, \end{cases} \tag{3}$$

where $\mathcal{N} \triangleq \{1,2,\cdots,n\}$ is a finite index set, and $\Omega \in R^n$ is a bounded set with a smooth boundary $\partial\Omega$. $\{\mu_{jk}(j=1,2,\cdots,N;\ k=1,2,\cdots,m)\}$ is a fuzzy set, $\omega_k(t)$ represents a premise variable, m is the number of premise variables, and N is the number of **IF-THEN** rules. $a_i(t,x,u_1,\cdots,u_n)$ represents a diffusion operator. Time delay $\tau(t)$ satisfies $\tau(t)\in(0,\tau]$, where $\tau>0$ is a constant. b_i represents the behavior function, dependent on t and x. c_{ij} and d_{ij} denote the strength of state links between neurons, f_i denotes the activation function of the neurons at i, and J_i denotes the external input of the neurons at i. Assume in this paper that J_i is a bounded quantity.

In view of a standard fuzzy inference method, the system (3) can be inferred as follows:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = a_i(t,u_1,\cdots,u_n)\Delta_p u_i - b_i(u_i) + \sum\limits_{j=1}^{N}\rho_j(\omega(t))\Big(c_{ij}f_i(u_i) \\ \qquad + d_{ij}\displaystyle\int_{t-\tau(t)}^{t} f_i(u_i(s,x))ds\Big) + J_i,\ \ t\geqslant 0,\ x\in\Omega,\ i\in\mathcal{N}, \\ u_i(\theta,x) = \phi_i(\theta,x), \quad (\theta,x)\in[-\tau,0]\times\Omega,\ i\in\mathcal{N}, \\ u_i(t,x) = 0, \quad (t,x)\in[-\tau,+\infty)\times\partial\Omega,\ i\in\mathcal{N}, \end{cases} \tag{4}$$

where $\omega(t) = (\omega_1(t),\cdots,\omega_m(t))$, and $\Upsilon_j(\omega(t))$ is the membership function corresponding to rule j. In addition, $\rho_j(\omega(t)) = \frac{\Upsilon_j(\omega(t))}{\sum_{i=1}^{N}\Upsilon_i(\omega(t))}$ with $\sum\limits_{j=1}^{N}\rho_j(\omega(t)) = 1$ and $\rho_j(\omega(t))\geqslant 0$.

In this paper, we assume

(H1) for any given $q \in (1, p)$,

$$\lim_{|r|\to\infty} \frac{b_i(r)}{r^{q-1}} = 0 = \lim_{|r|\to\infty} \frac{f_i(r)}{r^{q-1}};$$

(H2) for any given $i \in \mathcal{N}$, there is the corresponding positive number $\underline{a}_i$ such that

$$\underline{a}_i = \inf_{t,x,u} a_i(t, x, u_1, u_2, \cdots, u_n) > 0,$$

where $u = (u_1, u_2, \cdots, u_n)$;

(H3) for any given $i \in \mathcal{N}$, there is the corresponding positive number $\mathcal{F}_i$ such that

$$|f_i(s) - f_i(r)| \leqslant \mathcal{F}_i |s - r|, \quad \forall\, s, r \in R;$$

(H4) for any given $i \in \mathcal{N}$, there is the corresponding positive number $\bar{b}_i$ such that

$$\inf_{r\in R} b_i'(r) \geqslant \bar{b}_i > 0.$$

Definition 1. *The nontrivial solution $\xi^*(x) = (\xi_1^*(x), \cdots, \xi_n^*(x))^T$ of the system (4) is said to be pth moment stable if, for any given $\varepsilon > 0$ and any given initial value $\phi = (\phi_1, \cdots, \phi_n)^T$, there exists $\delta(\varepsilon, \phi) > 0$ for ϕ with $\sum_{i=1}^{n} \|\phi_i - \xi_i^*\|_\tau < \delta(\varepsilon, \phi)$ such that*

$$\sum_{i=1}^{n} \left(\int_\Omega |u_i(t, \phi, x) - \xi_i^*(x)|^p dx \right)^{\frac{1}{p}} < \varepsilon, \quad \forall\, t \geqslant t_0,$$

where $\|\phi_i - \xi_i^*\|_\tau = \sup_{-\tau\leqslant\theta\leqslant 0} \left(\int_\Omega |\phi_i(t, x) - \xi_i^*(x)|^p dx \right)^{\frac{1}{p}}$.

Here, the above definition mainly imitates the Definition 7.1 of [41] (Chapter 1).

3. Main Result

Theorem 1. *If the conditions (H1) and (H2) hold, there is a nontrivialstationary solution for fuzzy system (4). If, in addition, the conditions (H3) and (H4) and the following condition hold:*

$$\bar{b}_i > \mathcal{F}_i \sum_{j=1}^{N} (|c_{ij}| + \tau |d_{ij}|), \quad \forall\, i \in \mathcal{N}, \tag{5}$$

then, for any given $p > 1$, the nontrivialstationary solution is pth moment stable.

Proof. Denote $a_i(t) = a_i(t, u_1, \cdots, u_n)$. We shall complete the proof after two steps. □

Step 1. We claim that there is a nontrivialstationary solution for fuzzy system (4).

Indeed, for any given $i \in \mathcal{N}$, we consider the following functional:

$$I_i(\xi_i(x)) = \int_\Omega \left[\frac{1}{p} a_i(t) |\nabla \xi_i(x)|^p + B_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t)) \Big(c_{ij} + d_{ij}\tau(t) \Big) F_i(\xi_i(x)) - J_i \xi_i(x) \right] dx,$$

where $B_i(\xi_i) = \int_0^{\xi_i} b_i(r) dr$, and $F_i(\xi_i) = \int_0^{\xi_i} f(r) dr$.

Below, we prove $I_i \in C^1(W_0^{1,p}(\Omega), R)$.

Indeed, let

$$I_i(\xi_i(x)) = \Lambda_i(\xi_i(x)) + \Gamma_i(\xi_i(x))$$

with

$$\Gamma_i(\xi_i(x)) = \int_\Omega \left[B_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t))\Big(c_{ij} + d_{ij}\tau(t)\Big)F_i(\xi_i(x)) - J_i\xi_i(x)\right]dx$$

and

$$\Lambda_i(\xi_i(x)) = \int_\Omega \frac{1}{p} a_i(t)|\nabla\xi_i(x)|^p dx.$$

For convenience, we denote

$$g(x,\xi_i) = b_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t))\Big(c_{ij} + d_{ij}\tau(t)\Big)f_i(\xi_i(x)) - J_i$$

and $G(x,\xi_i) = \int_0^{\xi_i} g(x,t)dt = B_i(\xi_i(x)) - \sum\limits_{j=1}^{N} \rho_j(\omega(t))\Big(c_{ij} + d_{ij}\tau(t)\Big)F_i(\xi_i(x)) - J_i\xi_i(x)$.

Combining the continuity hypothesis and (H1) results in

$$|g(x,\xi_i)| \leqslant C_0 + M_0|\xi_i|^{q-1},$$

where both $C_0 > 0$ and $M_0 > 0$ are constants.

Let $v \in W_0^{1,p}(\Omega)$ with $\|v\| < r$, for the functional $\Gamma_i(\xi_i) = \int_\Omega G(x,\xi_i(x))dx$, we can see it from the differential mean value theorem that the Gateaux differential of $\Gamma_i(\xi_i)$ is

$$D\Gamma_i(\xi_i, v) = \lim_{s\to 0}\int_\Omega g(x,\xi_i + \theta s v)v dx,$$

where $\theta \in [0,1]$. Now, we try to employ the Lebesgue control convergence theorem to deal with the limit. For $|s| \leqslant 1$, the Young inequality derives

$$\begin{aligned}
&|g(x,\xi_i+\theta s v)v| \\
\leqslant&\Big(C_0 + M_0|\xi_i+\theta s v|^{q-1}\Big)|v| \\
\leqslant&\frac{q-1}{q}\Big(C_0 + M_0|\xi_i+\theta s v|^{q-1}\Big)^{\frac{q}{q-1}} + \frac{1}{q}|v|^q \\
\leqslant&2^{\frac{1}{q-1}}\frac{q-1}{q}\Big(C_0^{\frac{q}{q-1}} + M_0^{\frac{q}{q-1}}|\xi_i+\theta s v|^q\Big) + \frac{1}{q}|v|^q \\
\leqslant&2^{\frac{1}{q-1}}\frac{q-1}{q}\Big(C_0^{\frac{q}{q-1}} + M_0^{\frac{q}{q-1}}2^{q-1}(|\xi_i|^q+|v|^q)\Big) + \frac{1}{q}|v|^q,
\end{aligned}$$

in which the function $2^{\frac{1}{q-1}}\frac{q-1}{q}\Big(C_0^{\frac{q}{q-1}} + M_0^{\frac{q}{q-1}}2^{q-1}(|\xi_i|^q+|v|^q)\Big) + \frac{1}{q}|v|^q$ is a Lebesgue Integrable function due to Sobolev embedding theorem [42]. In addition, then the Lebesgue control convergence theorem yields

$$D\Gamma_i(\xi_i, v) = \int_\Omega \lim_{s\to 0} g(x,\xi_i+\theta s v)v dx = \int_\Omega g(x,\xi_i)v dx.$$

It is not difficult to prove that $D\Gamma_i(\xi_i, v)$ is linear on v. In fact,

$$D\Gamma_i(\xi_i, k_1u + k_2v) = \int_\Omega g(x,\xi_i)(k_1u+k_2v)dx = k_1D\Gamma_i(\xi_i,u) + k_2D\Gamma_i(\xi_i,v).$$

On the other hand, it follows from $|g(x,\xi_i)| \leqslant C_0 + M_0|\xi_i|^{q-1}$ that the operator $\mathfrak{S} : \xi_i \to g(x,\xi_i)$ is the bounded continuous operator of $L^q(\Omega) \to L^{\frac{q}{q-1}}(\Omega)$. In addition, the Sobolev embedding theorem yields

$$|D\Gamma_i(\xi_i,v)| \leqslant \|g(x,\xi_i)\|_{L^{\frac{q}{q-1}}}\|v\|_{L^q} \leqslant C\|g(x,\xi_i)\|_{L^{\frac{q}{q-1}}}\|v\|,$$

where $\|v\| = \left(\int_{\Omega} |\nabla v|^p dx \right)^{\frac{1}{p}}$. Hence, $\|D\Gamma_i(\xi_i)\| \leqslant C\|g(x,\xi_i)\|_{L^{\frac{q}{q-1}}}$, which implies that

$$D\Gamma_i(\xi_i, v) = \langle D\Gamma_i(\xi_i), v\rangle, \quad \forall\, \xi_i, v \in W_0^{1,p}(\Omega).$$

Below, we shall show that the Gateaux derivative $D\Gamma_i(\xi_i)$ is continuous on ξ_i. In fact,

$$|\langle D\Gamma_i(\xi_i) - D\Gamma(u), v\rangle| \leqslant C\|g(x,\xi_i) - g(x,u)\|_{L^{\frac{q}{q-1}}}\|v\|,$$

which implies the norm of the operator

$$\|D\Gamma_i(\xi_i) - D\Gamma(u)\| \leqslant C\|g(x,\xi_i) - g(x,u)\|_{L^{\frac{q}{q-1}}}.$$

This means the operator $\mathfrak{T} : g(x,\xi_i) \to D\Gamma_i(\xi_i)$ is continuous on $L^{\frac{q}{q-1}} \to (W_0^{1,p})^*$. In addition, the Sobolev embedding theorem yields that the operator $\mathfrak{I} : \xi_i \to \xi_i$ is continuous on $W_0^{1,p} \to L^q(\Omega)$. In addition, hence, $D\Gamma_i = \mathfrak{T} \circ \mathfrak{S} \circ \mathfrak{I} : \xi_i \to D\Gamma_i(\xi_i)$ is the continuous operator of $W_0^{1,p} \to (W_0^{1,p})^*$. Thereby, Γ_i is Frechet differentiable at any $\xi_i \in W_0^{1,p}(\Omega)$. In addition, $\Gamma_i \in C^1(W_0^{1,p}(\Omega), R)$.

In addition, it follows from [43,44] and the condition of $a_i(t)$ that $\Lambda_i \in C^1(W_0^{1,p}(\Omega), R)$. Therefore, we have proved $I_i \in C^1(W_0^{1,p}(\Omega), R)$.

If, for any given $i \in \mathcal{N}$, the critical point of I_i exists, say $\xi^* = (\xi_1^*, \xi_2^*, \cdots, \xi_n^*)$, a nontrivial stationary solution for the fuzzy system, where $\xi_i^* = \xi_i^*(x)$. Below, we shall prove the existence of the critical point. In fact, the condition (H1) yields

$$|b_i(r)| + |f_i(r)| \leqslant C_1 + |r|^{q-1}, \quad \forall\, r \in R, \tag{6}$$

where $C_1 > 0$. By the arbitrariness of q, we select a suitable constant $q \in (1,p)$ such that

$$|F_i(r)| \leqslant C_1|r| + \frac{1}{q}|r|^q, \quad \forall\, r \in R. \tag{7}$$

Letting C_1 be big enough, we can prove

$$|B_i(r)| \leqslant C_1|r| + \frac{1}{q}|r|^q, \quad \forall\, r \in R. \tag{8}$$

In fact, combining the continuity of b_i and the hypothesis (H1) yields that there exists $\tilde{C}_1 > 0$ such that

$$|B_i(r)| \leqslant \tilde{C}_1|r| + \frac{1}{q}|r|^q, \quad \forall\, r \in R,$$

which deduces the inequality (8) due to the big C_1.

From the Sobolev embedding theorem, we know that there are $C_2, C_3 > 0$ such that

$$I_i(\xi_i) \geqslant \frac{1}{p}\underline{a}_i\|\xi_i\|^p - C_2\|\xi_i\|^q - C_3\|\xi_i\|, \tag{9}$$

where $\|\xi_i\| = \left(\int_{\Omega} |\nabla \xi_i|^p dx \right)^{\frac{1}{p}}$. In addition, (3.5) and $1 < q < p$ yield that the lower bound of I_i exists. We shall prove that I_i is coercive on $W_0^{1,p}(\Omega)$. Due to $\|\eta_k\| \to \infty$ and $I_i(\eta_k) \leqslant C$, (9) leads to a contradiction. Hence, I_i must be coercive on $W_0^{1,p}(\Omega)$. In addition, hence, there exists the constant

$$c_i = \inf_{v \in W_0^{1,p}(\Omega)} I_i(v),$$

from which there exists a minimization sequence $\{\eta_k\} \subset W_0^{1,p}(\Omega)$ such that $I_i(\eta_k) \to c_i$ whenever $k \to \infty$. In addition, we know from the coercivity and the lower bound of the functional I_i that c_i is the global minimum of I_i on the Sobolev space $W_0^{1,p}(\Omega)$. Moreover, if $\eta_k \to \xi_i^* \in W_0^{1,p}(\Omega)$, then $I_i(\xi_i^*) = c_i$ with $I_i'(\xi_i^*) = 0$, and ξ_i^* must be in the bounded subset of $W_0^{1,p}(\Omega)$. Thus, the minimization sequence $\{\eta_k\} \subset W_0^{1,p}(\Omega)$ satisfies $I_i(\eta_k) \to c_i$ and $I_i'(\eta_k) \to 0$. In the inequality (9), let $\xi_i = \eta_k$. Then, (9) yields that $\{\eta_k\}_{k=1}^{\infty}$ is bounded on $W_0^{1,p}(\Omega)$. Now, we claim that there exists $\xi_i^* \in W_0^{1,p}(\Omega)$ with $\eta_k \to \xi_i^*$. Indeed, we define the operators $\mathfrak{A}_i, \mathfrak{B}_i : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ as follows:

$$< \mathfrak{A}_i(v), w >= a_i(t) \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla w dx, \quad \forall\, v, w \in W_0^{1,p}(\Omega) \tag{10}$$

and

$$< \mathfrak{B}_i(v), w >= \int_{\Omega} \left[b_i(v) - \sum_{j=1}^{N} \rho_j(\omega(t)) \Big(c_{ij} f_i(v) + d_{ij} \tau(t) f_i(v) \Big) - J_i \right] w dx, \ \forall\, v, w \in W_0^{1,p}(\Omega). \tag{11}$$

It follows from (H1) that there are positive numbers C_4, C_5 such that

$$\left| b_i(r) - \sum_{j=1}^{N} \rho_j(\omega(t)) \Big(c_{ij} f_i(r) + d_{ij} \tau(t) f_i(r) \Big) - J_i \right| \leqslant C_4 + C_5 |r|^{q-1}, \ \forall\, r \in R.$$

From [4,5], the operators $\mathfrak{A}_i$ and $\mathfrak{A}_i^{-1}$ are continuous, and $\mathfrak{B}_i$ is compact operator. On one hand,

$$< I_i(\eta_k), v >=< \mathfrak{A}_i(\eta_k), v > + < \mathfrak{B}_i(\eta_k), v >, \quad \forall\, v \in W_0^{1,p}(\Omega), \tag{12}$$

and

$$\mathfrak{A}_i(\eta_k) + \mathfrak{B}_i(\eta_k) \to 0, \quad k \to \infty. \tag{13}$$

On the other hand, $\{\eta_k\}_{k=1}^{\infty}$ is bounded on the space $W_0^{1,p}(\Omega)$. Thus, it follows by the reflexivity of $W_0^{1,p}(\Omega)$ that there is a subsequence of $\{\eta_k\}_{k=1}^{\infty}$ that is weak convergent, say, $\{\eta_k\}_{k=1}^{\infty}$. Since $\mathfrak{B}$ is compact operator, there is a subsequence of $\{\eta_k\}_{k=1}^{\infty}$ such that $\{\mathfrak{B}_i(\eta_k)\}_{k=1}^{\infty}$ is convergent. In addition, $\{\mathfrak{A}_i(\eta_k)\}_{k=1}^{\infty}$ owns a convergent subsequence. Moreover, the continuity of $\mathfrak{A}^{-1}$ yields that $\{\eta_k\}_{k=1}^{\infty}$ owns a convergent subsequence, say, $\eta_k \to \xi_i^* \in W_0^{1,p}(\Omega)$. Hence, we have proved $I_i(\xi_i^*) = c_i$ and $I_i'(\xi_i^*) = 0$. By the arbitrariness of i, we have also proved that there is a nontrivial stationary solution $\xi^* = (\xi_1^*, \xi_2^*, \cdots, \xi_n^*)$ for the system (4).

Step 2. To prove that ξ_* is pth moment stable.

Consider the Lyapunov–Krasovskii functional $\mathbb{V}_i = \mathbb{V}_{i1} + \mathbb{V}_{i2}$, where

$$\mathbb{V}_{i1} = \int_{\Omega} |u_i(t,x) - \xi_i^*(x)|^p dx, \tag{14}$$

$$\mathbb{V}_{i2} = p\mathcal{F}_i \sum_{j=1}^{N} |d_{ij}| \int_{\Omega} \left(\int_{-\tau}^{0} dz \int_{t+z}^{t} |u_i(t,x) - \xi_i^*(x)|^{p-1} |u_i(s,x) - \xi_i^*(x)| ds \right) dx, \tag{15}$$

then the conditions (H2)–(H4), boundary value condition, variational method, and Young inequality deduce

$$\begin{aligned}
\frac{d\mathbb{V}_{i1}}{dt} =& p\int_\Omega |u_i(t,x)-\xi_i^*(x)|^{p-2}\Big(u_i(t,x)-\xi_i^*(x)\Big)\Big[a_i(t)\sum_{j=1}^n \frac{\partial}{\partial x_j}\Big((|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial x_j})-(|\nabla \xi_i^*|^{p-2}\frac{\partial \xi_i^*}{\partial x_j})\Big) \\
&-\Big(b_i(u_i)-b_i(\xi_i^*)\Big)+\sum_{j=1}^N \rho_j(\omega(t))\Big(c_{ij}(f_i(u_i)-f_i(\xi_i^*))+d_{ij}\int_{t-\tau(t)}^t (f_i(u_i(s,x))-f_i(\xi_i^*(x)))ds\Big)\Big]dx \\
\leqslant& -\bar{b}_i p\int_\Omega |u_i(t,x)-\xi_i^*(x)|^p dx + p\mathcal{F}_i\int_\Omega |u_i(t,x)-\xi_i^*(x)|^{p-2}|u_i(t,x)-\xi_i^*(x)|\Big[\sum_{j=1}^N\Big(|c_{ij}||u_i-\xi_i^*| \\
&+|d_{ij}|\int_{t-\tau}^t |u_i(s,x)-\xi_i^*(x)|ds\Big)\Big]dx \\
=& -\Big(\bar{b}_i p - p\mathcal{F}_i\sum_{j=1}^N |c_{ij}|\Big)\int_\Omega |u_i(t,x)-\xi_i^*(x)|^p dx \\
&+p\mathcal{F}_i\sum_{j=1}^N |d_{ij}|\int_\Omega\Big(|u_i(t,x)-\xi_i^*(x)|^{p-1}\int_{t-\tau}^t |u_i(s,x)-\xi_i^*(x)|ds\Big)dx.
\end{aligned} \tag{16}$$

(15) yields

$$\begin{aligned}
\frac{d\mathbb{V}_{i2}}{dt} =& p\mathcal{F}_i\sum_{j=1}^N |d_{ij}|\int_\Omega\Big(\int_{-\tau}^0 |u_i(t,x)-\xi_i^*(x)|^p ds - \int_{-\tau}^0 |u_i(t,x)-\xi_i^*(x)|^{p-1}|u_i(t+s,x)-\xi_i^*(x)|ds\Big)dx \\
=& p\mathcal{F}_i\sum_{j=1}^N |d_{ij}|\int_\Omega\Big(\tau|u_i(t,x)-\xi_i^*(x)|^p - \int_{-\tau}^0 |u_i(t,x)-\xi_i^*(x)|^{p-1}|u_i(t+s,x)-\xi_i^*(x)|ds\Big)dx \\
=& p\mathcal{F}_i\sum_{j=1}^N |d_{ij}|\int_\Omega\Big(\tau|u_i(t,x)-\xi_i^*(x)|^p - \int_{t-\tau}^t |u_i(t,x)-\xi_i^*(x)|^{p-1}|u_i(s,x)-\xi_i^*(x)|ds\Big)dx
\end{aligned} \tag{17}$$

Therefore,

$$\frac{d\mathbb{V}_i}{dt} \leqslant -\Big(\bar{b}_i p - p\mathcal{F}_i\sum_{j=1}^N(|c_{ij}|+\tau|d_{ij}|)\Big)\int_\Omega |u_i(t,x)-\xi_i^*(x)|^p dx \leqslant 0, \quad \forall\, t\geqslant 0,\, i\in\mathcal{N}.$$

Thus, ξ_* is pth moment stable due to the definition of $\mathbb{V}_i$ and [41] (Theorem 2.1).

4. Numerical Example

In fuzzy system (2.2), the number of **IF-THEN** rules is $N = 2$. Let $n = 2$, and for $i = 1,2$, $f_i(r) = i\sin r$, $b_i(r) = 5ir$, then (H1) holds for $a_i(t,x_1,x_2,u_1,u_2) = \frac{i+3}{10}[2+\sin(x_1x_2+u_1+u_2)]$. In addition, then $\underline{a}_i = \frac{i+3}{10} > 0$, i.e., the condition (H2) is fulfilled. Hence, Theorem 1 yields that there exists a nontrivial solution $\xi^* = (\xi_1^*(x_1,x_2),\xi_2^*(x_1,x_2))$. In addition, letting $\mathcal{F}_i = i$, the condition (H3) holds for $\bar{b}_i = 5i > 0$, and so the condition (H4) holds too. Let $\tau(t) = 2+\cos t$, then $\tau = 3$. Letting $c_{ij} = \frac{i+j}{30}, d_{ij} = \frac{ij}{30}$, the direct computation leads to

$$\bar{b}_i = 5i > i\sum_{j=1}^2 \frac{i+j+3ij}{30} = \mathcal{F}_i\sum_{j=1}^N(|c_{ij}|+\tau|d_{ij}|), \quad \forall\, i\in\{1,2\},$$

i.e.,

$$\bar{b}_1 = 5 > \frac{14}{30} = \sum_{j=1}^2 \frac{1+j+3j}{30} = \mathcal{F}_1\sum_{j=1}^N(|c_{1j}|+\tau|d_{1j}|)$$

and

$$\bar{b}_2 = 10 > \frac{5}{3} = 2\sum_{j=1}^2 \frac{2+j+6j}{30} = \mathcal{F}_2\sum_{j=1}^N(|c_{2j}|+\tau|d_{2j}|),$$

and so condition (5) is fulfilled. It follows from Theorem 1 that, for any given $p > 1$, the nontrivialstationary solution $\xi^* = (\xi_1^*(x_1,x_2),\xi_2^*(x_1,x_2))$ is pth moment stable.

Remark 1. *In this numerical example, $\tau = 3$ is the bigger upper bound of time delays, which shows the effectiveness of the proposed methods.*

5. Conclusions and Further Considerations

Mainly inspired by some methods and ideas of literature [16,27,40] related to p-Laplace, we employed the critical point theory, variational technique, and Lyapunov functional method to derive the existence theorem of pth moment stable non-trivial stationary solutions for a class of p-Laplacian reaction–diffusion delay systems. The theorem holds for all $p > 1$, and the methods used in this paper are different from those in previous related literature to some extent. For example, we proved $I_i \in C^1(W_0^{1,p}(\Omega), R)$ while it was not proved in related literature. A numerical example illustrates the effectiveness of the proposed methods. In [45] (Theorem 4.3), Ruofeng Rao and Shouming Zhong proposed a stability criterion for the delayed feedback financial system, in which the pulse effect occurs at a long time (see [45] (Remark 7) for details). How can the impulse control method be used to derive the stability criterion for the p-Laplacian diffusion system (4)? This is an interesting problem.

Author Contributions: Methodology, X.W.; formal analysis, R.R.; software, S.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: All the authors thank the anonymous reviewers for their constructive suggestions and pertinent comments, which actually improve this paper. This work was supported by the National Basic Research Program of China (2010CB732501), the Scientific Research Fund of Science Technology Department of Sichuan Province (2012JY010), and the Sichuan Educational Committee Science Foundation (08ZB002, 12ZB349, 14ZA0274, 18ZA0082).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Wang, X.; Rao, R.; Zhong, S. New Stability Criterion for Takagi-Sugeno Fuzzy Cohen-Grossberg Neural Networks with Probabilistic Time-Varying Delays. *Math. Probl. Eng.* **2017**, *2017*, 3793157. [CrossRef]
2. Liang, X.; Wang, L.; Wang, R. Random Attractor of Reaction-Diffusion Hopfield Neural Networks Driven by Wiener Processes. *Math. Probl. Eng.* **2018**, *2018*, 2538658. [CrossRef]
3. Rao, R.; Zhong, S. Boundedness and robust analysis of dynamics of nonlinear diffusion high-order Markovian jump time-delay system. *Adv. Differ. Equ.* **2018**, *2018*, 434. [CrossRef]
4. Wang, L.; Xu, R.; Wang, Z. Synchronization analysis for stochastic reaction–diffusion Cohen-Grossberg neural networks with Neumann boundary conditions via periodically intermittent control. *Adv. Differ. Equ.* **2017**, *2017*, 141. [CrossRef]
5. Wang, Z.; Wang, L.; Xu, R. Adaptive Exponential Synchronization for Stochastic Competitive Neural Networks with Time-Varying Leakage Delays and Reaction-Diffusion Terms. *Math. Probl. Eng.* **2017**, *2017*, 6987436. [CrossRef]
6. Rao, R.; Zhong, S.; Pu, Z. On the role of diffusion factors in stability analysis for p-Laplace dynamical equations involved to BAM Cohen-Grossberg neural network. *Neurocomputing* **2017**, *223*, 54–62. [CrossRef]
7. Wei, T.; Lin, P.; Wang, Y.; Wang, L. Stability of stochastic impulsive reaction-diffusion neural networks with S-type distributed delays and its application to image encryption. *Neural Netw.* **2019**, *116*, 35–45. [CrossRef]
8. Song, Q.; Cao, J. Global exponential stability and existence of periodic solutions in BAM networks with delays and reaction-diffusion terms. *Chaos Solitons Fractals* **2005**, *23*, 421–430. [CrossRef]
9. Song, Q.; Zhao, Z.; Li, Y. Global exponential stability of BAM neural networks with distributed delays and reaction-diffusion terms. *Phys. Lett. A* **2005**, *335*, 213–225. [CrossRef]
10. Song, Q.; Cao, J. Global exponential robust stability of Cohen-Grossberg neural network with time-varying delays and reaction-diffusion terms. *J. Frankl. Inst.* **2006**, *343*, 705–719. [CrossRef]
11. Song, Q.; Wang, Z. Dynamical behaviors of fuzzy reaction-diffusion periodic cellular neural networks with variable coefficients and delays. *Appl. Math. Model.* **2009**, *33*, 3533–3545. [CrossRef]

12. Song, Q.; Cao, J.; Zhao, Z. Periodic solutions and its exponential stability of reaction-diffusion recurrent neural networks with continuously distributed delays. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 65–80. [CrossRef]
13. Song, Q.; Cao, J. Dynamics of bidirectional associative memory networks with distributed delays and reaction-diffusion terms. *Nonlinear Anal. Real World Appl.* **2007**, *8*, 345–361. [CrossRef]
14. Yao, Q.; Wang, L.; Wang, Y. Periodic solutions to impulsive stochastic reaction-diffusion neural networks with delays. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *78*, 104865. [CrossRef]
15. Tomas, C.; Marta, H.; Pedro, M. Asymptotic behaviour of nonlocal p-Laplacian reaction–diffusion problems. *J. Math. Anal. Appl.* **2018**, *459*, 997–1015.
16. Rao, R.; Zhong, S. Existence of Exponential p-Stability Nonconstant Equilibrium of Markovian Jumping Nonlinear Diffusion Equations via Ekeland Variational Principle. *Adv. Math. Phys.* **2015**, *2015*, 812150. [CrossRef]
17. Li, X.; Yang, X.; Huang, T. Persistence of delayed cooperative models: Impulsive control method. *Appl. Math. Comput.* **2019**, *342*, 130–146. [CrossRef]
18. Li, X.; Shen, J.; Rakkiyappan, R. Persistent impulsive effects on stability of functional differential equations with finite or infinite delay. *Appl. Math. Comput.* **2018**, *329*, 14–22.
19. Rao, R. Global Stability of a Markovian Jumping Chaotic Financial System with Partially Unknown Transition Rates under Impulsive Control Involved in the Positive Interest Rate. *Mathematics* **2019**, *7*, 579. [CrossRef]
20. Li, X.; Yang, X.; Song, S. Lyapunov conditions for finite-time stability of time-varying time-delay systems. *Automatica* **2019**, *103*, 135–140. [CrossRef]
21. Rao, R.; Zhong, S.; Pu, Z. Fixed point and p-stability of T–S fuzzy impulsive reaction–diffusion dynamic neural networks with distributed delay via Laplacian semigroup. *Neurocomputing* **2019**, *335*, 170–184. [CrossRef]
22. Li, X.; Ding, Y. Razumikhin-type theorems for time-delay systems with Persistent impulses. *Syst. Control Lett.* **2017**, *107*, 22–27. [CrossRef]
23. Li, X.; Song, S.; Wu, J. Exponential Stability of Nonlinear Systems With Delayed Impulses and Applications. *IEEE Trans. Autom. Control* **2019**. [CrossRef]
24. Rao, R.; Zhong, S.; Wang, X. Stochastic stability criteria with LMI conditions for Markovian jumping impulsive BAM neural networks with mode-dependent time-varying delays and nonlinear reaction–diffusion. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 258–273. [CrossRef]
25. Wang, X.; Rao, R.; Zhong, S. LMI Approach to Stability Analysis of Cohen-Grossberg Neural Networks with p-Laplace Diffusion. *J. Appl. Math.* **2012**, *2012*, 523812. [CrossRef]
26. Tsai, Z.-R.; Hwang, J.-D. H_∞ Neural-Network-Based Discrete-Time Fuzzy Control of Continuous-Time Nonlinear Systems with Dither. *Math. Probl. Eng.* **2012**, *2012*, 314964. [CrossRef]
27. Stamov, G.T. Existence of almost periodic solutions for strong stable impulsive differential equations. *IMA J. Math. Control Inform.* **2001**, *18*, 153–160. [CrossRef]
28. Zhang, B.; Zhuang, J.; Liu, H.; Cao, J.; Xia, Y. Master-slave synchronization of a class of fractional-order Takagi-Sugeno fuzzy neural networks. *Adv. Differ. Equ.* **2018**, *2018*, 473. [CrossRef]
29. Bouzbida, M.; Hassine, L.; Chaari, A. Robust Kernel Clustering Algorithm for Nonlinear System Identification. *Math. Probl. Eng.* **2017**, *2017*, 2427309. [CrossRef]
30. Nirmala, V.J.; Saravanakumar, T.; Zhu, Q. Dissipative criteria for Takagi-Sugeno fuzzy Markovian jumping neural networks with impulsive perturbations using delay partitioning approach. *Adv. Differ. Equ.* **2019**, *2019*, 140. [CrossRef]
31. Lin, X.; Zhang, W.; Chen, B.-S. H_∞ Robust Tracking Control of Stochastic T–S Fuzzy Systems with Poisson Jumps. *Math. Probl. Eng.* **2018**, *2018*, 5257972. [CrossRef]
32. Wang, W.; Zhong, S.; Liu, F.; Cheng, J. Reachable set estimation for linear systems with time-varying delay and polytopic uncertainties. *J. Frankl. Inst.* **2019**, *356*, 7322–7346. [CrossRef]
33. Long, S.; Zhong, S.; Guan, H.; Zhang, D. Exponential stability analysis for a class of neutral singular Markovian jump systems with time-varying delays. *J. Frankl. Inst.* **2019**, *356*, 6015–6040. [CrossRef]
34. Yang, H.; Shu, L.; Wang, X.; Zhong, S. Synchronization of IT2 stochastic fuzzy complex dynamical networks with time-varying delay via fuzzy pinning control. *J. Frankl. Inst.* **2019**, *356*, 1484–1501. [CrossRef]
35. Zhang, R.; Zeng, D.; Zhong, S. Novel master-slave synchronization criteria of chaotic Lur e systems with time 7delays using sampled-data control. *J. Frankl. Inst.* **2017**, *354*, 4930–4954. [CrossRef]

36. Shan, Y.; She, K.; Zhong, S.; Cheng, J.; Zhao, C. Event-triggered passive control for Markovian jump discrete-time systems with incomplete transition probability and unreliable channels. *J. Frankl. Inst.* **2019**, *356*, 8093–8117. [CrossRef]
37. Yang, D. ; Li, X.; Shen, J.; Zhou, Z. State-dependent switching control of delayed switched systems with stable and unstable modes. *Math. Methods Appl. Sci.* **2018**, *41*, 6968–6983. [CrossRef]
38. Hu, J.; Sui, G.; Lv, X.; Li, X. Fixed-time control of delayed neural networks with impulsive perturbations. *Nonlinear Anal. Model. Control* **2018**, *23*, 904–920. [CrossRef]
39. Yang, D.; Li, X.; Qiu, J. Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback. *Nonlinear Anal. Hybrid Syst.* **2019**, *32*, 294–305. [CrossRef]
40. Rao, R.; He, Q. Non-zero Solution for the Quasi-linear. *Elliptic Equ. Chin. Quart. J. Math.* **2009**, *24*, 117–124.
41. Liao, X. *Theory Metrhods and Application of Stability*; Huazhong University of Science and Technology Press: Wuhan, China, 1999.
42. Wang, S. *Introduction of Sobolev Spaces and Partial Differential Equations*; Science Press: Beijing, China, 2019.
43. Drabek, P. *Solvability and Bifurcations of Nonlinear Equations*; Pitman Research Notes in Mathematics Series, 265; Longman: Harlow, UK, 1992.
44. Drabek, P.; Kufner, A.; Nicolosi, F. *Quasilinear Elliptic Equations with Degenerations and Singularities*; Walter de Gruyter: Berlin, Germany; New York, NY, USA, 1997.
45. Rao, R.; Zhong, S. Input-to-state stability and no-inputs stabilization of delayed feedback chaotic financial system involved in open and closed economy. *Discret. Contin. Dyn. Syst.* **2020**. [CrossRef]
46. Stamova, I.M., Stamov, G.T. Impulsive control on global asymptotic stability for a class of bidirectional associative memory neural networks with distributed delays. *Math. Comput. Model.* **2011**, *53*, 824–831. [CrossRef]