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New Results for Kneser Solutions of Third-Order Nonlinear Neutral Differential Equations

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Abstract: In this paper, we consider a certain class of third-order nonlinear delay differential equations $\left(r\left(w''\right)^{\alpha}\right)'(v) + q(v)\, x^{\beta}(\varsigma(v)) = 0$, for $v \geq v_0$, where $w(v) = x(v) + p(v)\, x(\vartheta(v))$. We obtain new criteria for oscillation of all solutions of this nonlinear equation. Our results complement and improve some previous results in the literature. An example is considered to illustrate our main results.

Keywords: oscillation criteria; thrid-order; delay differential equations

1. Introduction

The continuous development in various sciences is accompanied by the continued emergence of new models of difference and differential equations that describe this development. Studying the qualitative properties of differential equations helps to understand and analyze many life phenomena and problems; see [1]. Recently, the study of the oscillatory properties of differential equations has evolved significantly; see [2–10]. However, third-order differential equations attract less attention compared to first and second-order equations; see [11–20].

In this paper, we consider the third-order neutral nonlinear differential equation of the form

$$\left(r\left(w''\right)^{\alpha}\right)'(v) + q(v)\, x^{\beta}(\varsigma(v)) = 0, \quad \text{for } v \geq v_0, \tag{1}$$

where $w(v) = x(v) + p(v)\, x(\vartheta(v))$, α and β are ratios of odd positive integers. In this work, we assume the following conditions:

(I$_1$) $r \in C([v_0,\infty),(0,\infty))$
$$\int_{v_0}^{\infty} r^{-1/\alpha}(s)\, \mathrm{d}s = \infty;$$

(I$_2$) $p, q \in C([v_0,\infty),[0,\infty))$, $p(v) \leq p_0 < \infty$, q does not vanish identically;

(I$_3$) $\vartheta, \varsigma \in C^1([v_0,\infty),\mathbb{R})$, $\vartheta(v) < v$, $\varsigma(v) < v$, $\vartheta'(v) \geq \vartheta_0 > 0$, $\vartheta \circ \varsigma = \varsigma \circ \vartheta$ and $\lim_{v\to\infty} \vartheta(v) = \lim_{v\to\infty} \varsigma(v) = \infty$.

A solution of (1) means $x \in C([v_0, \infty))$ with $v_* \geq v_0$, which satisfies the properties $w \in C^2([v_*, \infty))$, $r(w'')^\alpha \in C^1([v_*, \infty))$ and satisfies (1) on $[v_*, \infty)$. We consider the nontrivial solutions of (1) which exist on some half-line $[v_*, \infty)$ and satisfy the condition $\sup\{|x(v)| : v_1 \leq v < \infty\} > 0$ for any $v_1 \geq v_*$.

Definition 1. *The* ***class*** S_1 *is a set of all solutions* x *of Equation (1) such that their corresponding function* w *satisfies*

$$\textbf{Case } (\mathbf{i}): \quad w(v) > 0, w'(v) > 0, w''(v) > 0;$$

and the ***class*** S_2 *is a set of all solutions of Equation (1) such that their corresponding function* w *satisfies*

$$\textbf{Case } (\mathbf{ii}): \quad w(v) > 0, w'(v) < 0, w''(v) > 0.$$

Definition 2. *If the nontrivial solution* x *is neither positive nor negative eventually, then* x *is called an* ***oscillatory*** *solution. Otherwise, it is a* ***non-oscillatory*** *solution.*

When studying the oscillating properties of neutral differential equations with odd-order, most of the previous studies have been concerned with creating a sufficient condition to ensure that the solutions are oscillatory or tend to zero; see [11–20]. For example, Baculikova and Dzurina [11,12], Candan [13], Dzurina et al. [15], Li et al. [18] and Su et al. [19] studied the oscillatory properties of (1) in the case where $\alpha = \beta$ and $0 \leq p(v) \leq p_0 < 1$. Elabbasy et al. [16] studied the oscillatory behavior of general differential equation

$$\left(r_2\left(\left(r_1(w')^\alpha\right)'\right)^\beta\right)'(v) + q(v) f(x(\varsigma(v))) = 0, \text{ for } v \geq v_0,$$

For an odd-order, Karpuz at al. [17] and Xing at al. [20] established several oscillation theorems for equation

$$\left(r_2\left(w^{(n-1)}\right)^\alpha\right)'(v) + q(v) x^\alpha(\varsigma(v)) = 0, \text{ for } v \geq v_0.$$

As an improvement and completion of the previous studies, Dzurina et al. [14], established standards to ensure that all solutions of linear equation

$$\left(r_2(r_1 w')'\right)'(v) + q(v) x(\varsigma(v)) = 0,$$

by comparison with first-order delay equations.

The main objective of this paper is to obtain new criteria for oscillation of all solution of nonlinear Equation (1). Our results complement and improve the results in [11–19] which only ensure that non-oscillating solutions tend to zero.

Next, we state the following lemmas, which will be useful in the proof of our results.

Lemma 1. *Assume that* $c_1, c_2 \in [0, \infty)$ *and* $\gamma > 0$*. Then*

$$(c_1 + c_2)^\gamma \leq \mu\left(c_1^\gamma + c_2^\gamma\right), \tag{2}$$

where

$$\mu := \begin{cases} 1 & \text{if } \gamma \leq 1 \\ 2^{\gamma-1} & \text{if } \gamma > 1. \end{cases}$$

Lemma 2. *Let $u, g \in C([v_0,\infty),\mathbb{R})$, $u(v) = g(v) + ag(v-b)$ for $v \geq v_0 + \max\{0,c\}$, where $a \neq 1$, b are constants. Suppose that there exists a constant $l \in \mathbb{R}$ such that $\lim_{v\to\infty} u(v) = l$.*

$$
\begin{aligned}
(\mathbf{H}_1): &\quad \text{If } \liminf_{v\to\infty} g(v) = g_* \in \mathbb{R}, \text{ then } g_* = l/(1+a);\\
(\mathbf{H}_2): &\quad \text{If } \limsup_{v\to\infty} g(v) = g^* \in \mathbb{R}, \text{ then } g^* = l/(1+a).
\end{aligned}
$$

Lemma 3. *Let $x \in C^n([v_0,\infty),(0,\infty))$. Assume that $x^{(n)}(v)$ is of fixed sign and not identically zero on $[v_0,\infty)$ and that there exists a $v_1 \geq v_0$ such that $x^{(n-1)}(v)\,x^{(n)}(v) \leq 0$ for all $v \geq v_1$. If $\lim_{v\to\infty} x(v) \neq 0$, then for every $\mu \in (0,1)$ there exists $v_\mu \geq v_1$ such that*

$$
x(v) \geq \frac{\mu}{(n-1)!} v^{n-1} \left| x^{(n-1)}(v) \right| \text{ for } v \geq v_\mu.
$$

2. Criteria for Nonexistence of Decreasing Solutions

Through this paper, we will be using the following notation:

$$
\begin{aligned}
£w(v) &:= r\left(w''\right)^{\alpha}(v),\\
\widetilde{q}(v) &:= \min\{q(v), q(\vartheta(v))\}
\end{aligned}
$$

and

$$
\eta(v,u) := \int_u^v \frac{1}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s \text{ and } \widetilde{\eta}(v,u) = \int_u^v \left(\int_u^s \frac{1}{r^{\frac{1}{\alpha}}(\zeta)} \mathrm{d}\zeta \right) \mathrm{d}s,
$$

where $v \in [v_0,\infty)$.

Lemma 4. *Assume that $x \in S_2$. Then*

$$
w(u) \geq \widetilde{\eta}(\varpi,u)\, £^{1/\alpha} w(\varpi), \tag{3}
$$

for $u \leq \varpi$, and

$$
\left(£w(v) + \frac{(p_0)^{\beta}}{\vartheta_0} £w(\vartheta(v)) \right)' \leq -\frac{1}{\mu} \widetilde{q}(v)\, w^{\beta}(\varsigma(v)). \tag{4}
$$

Proof. Let x be an eventually positive solution of (1). Then, we can assume that $x(v) > 0$, $x(\vartheta(v)) > 0$ and $x(\varsigma(v)) > 0$ for $v \geq v_1$, where v_1 is sufficiently large. From Lemma 1, (1) and ($\mathbf{I}_2$), we obtain

$$
w^{\beta}(v) \leq \mu \left(x^{\beta}(v) + p_0^{\beta} x^{\beta}(\vartheta(v)) \right). \tag{5}
$$

Since $£w(v)$ is non-increasing, we have

$$
-w'(u) \geq \int_u^{\varpi} \frac{1}{r^{1/\alpha}(s)} £^{1/\alpha} w(s)\,\mathrm{d}s \geq £^{1/\alpha} w(\varpi) \int_u^{\varpi} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s, \text{ for } u \leq \varpi. \tag{6}
$$

Integrating this inequality from u to ϖ, we get

$$
w(u) - w(\varpi) \geq £^{1/\alpha} w(\varpi) \int_u^{\varpi} \left(\int_u^{\sigma} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s \right) \mathrm{d}\sigma.
$$

Thus,

$$
w(u) \geq \widetilde{\eta}(\varpi,u)\, £^{1/\alpha} w(\varpi). \tag{7}
$$

Now, from (1) and ($\mathbf{I}_3$), we obtain

$$\left(\pounds w\left(\vartheta\left(v\right)\right)\right)'\frac{1}{\vartheta'\left(v\right)}+q\left(\vartheta\left(v\right)\right)x^{\beta}\left(\varsigma\left(\vartheta\left(v\right)\right)\right)=0. \tag{8}$$

Using (1), (5) and (8), we have

$$\begin{aligned} 0 &\geq \left(\pounds w\left(v\right)\right)'+q\left(v\right)x^{\beta}\left(\varsigma\left(v\right)\right)+p_0^{\beta}\left(\frac{1}{\vartheta_0}\left(\pounds w\left(\vartheta\left(v\right)\right)\right)'+q\left(\vartheta\left(v\right)\right)x^{\beta}\left(\varsigma\left(\vartheta\left(v\right)\right)\right)\right) \\ &\geq \left(\pounds w\left(v\right)\right)'+\frac{1}{\vartheta_0}p_0^{\beta}\left(\pounds w\left(\vartheta\left(v\right)\right)\right)'+\widetilde{q}\left(v\right)\left(x^{\beta}\left(\varsigma\left(v\right)\right)+p_0^{\beta}x^{\beta}\left(\varsigma\left(\vartheta\left(v\right)\right)\right)\right). \end{aligned}$$

Thus,

$$\left(\pounds w\left(v\right)+\frac{1}{\vartheta_0}p_0^{\beta}\pounds w\left(\vartheta\left(v\right)\right)\right)'+\frac{1}{\mu}\widetilde{q}\left(v\right)w^{\beta}\left(\varsigma\left(v\right)\right)\leq 0. \tag{9}$$

The proof of the lemma is complete. □

Theorem 1. *If there exists a function $\delta\in C\left(\left[v_0,\infty\right),\left(0,\infty\right)\right)$ such that $\vartheta\left(v\right)\leq\delta\left(v\right),\varsigma^{-1}\left(\delta\left(v\right)\right)<v$ and the delay differential equation*

$$\phi'\left(v\right)+\frac{1}{\mu}\left(\frac{\varsigma_0}{\varsigma_0+p_0^{\beta}}\right)^{\beta/\alpha}\widetilde{q}\left(v\right)\left(\widetilde{\eta}\left(\vartheta\left(v\right),\delta\left(v\right)\right)\right)^{\beta}\phi^{\beta/\alpha}\left(\varsigma^{-1}\left(\delta\left(v\right)\right)\right)=0 \tag{10}$$

is oscillatory, then S_2 is an empty set.

Proof. Assume the contrary that x is a positive solution of (1) and which satisfies case **(ii)**. Then, we assume that $x\left(v\right)>0$, $x\left(\varsigma\left(v\right)\right)>0$ and $x\left(\vartheta\left(v\right)\right)>0$ for $v\geq v_1$, where v_1 is sufficiently large. Thus, from (1), we get $\left(r\left(w''\right)^{\alpha}\right)'\left(v\right)\leq 0$ for $v\geq v_1$. Using Lemma 4, we get (3) and (4). Combining (4) and (3) with $\left[u=\vartheta\left(v\right)\right.$ and $\left.\varpi=\delta\left(v\right)\right]$, we find

$$\left(\pounds w\left(v\right)+\frac{1}{\varsigma_0}p_0^{\beta}\pounds w\left(\varsigma\left(v\right)\right)\right)'+\frac{1}{\mu}\widetilde{q}\left(v\right)\left(\widetilde{\eta}\left(\vartheta,\delta\right)\right)^{\beta}\pounds^{\beta/\alpha}w\left(\delta\left(v\right)\right)\leq 0. \tag{11}$$

Since $\pounds w\left(v\right)$ is non-increasing, we see that $\pounds w\left(v\right)\leq\pounds w\left(\varsigma\left(v\right)\right)$, and hence

$$\pounds w\left(v\right)+\frac{1}{\varsigma_0}p_0^{\beta}\pounds w\left(\varsigma\left(v\right)\right)\leq\left(1+\frac{1}{\varsigma_0}p_0^{\beta}\right)\pounds w\left(\varsigma\left(v\right)\right). \tag{12}$$

Using (11) along with (12), we have that $\phi\left(v\right):=\pounds w\left(v\right)+\frac{1}{\varsigma_0}p_0^{\beta}\pounds w\left(\varsigma\left(v\right)\right)$ is a positive solution of the differential inequality

$$\phi'\left(v\right)+\frac{1}{\mu}\widetilde{q}\left(v\right)\left(\widetilde{\eta}\left(\vartheta,\delta\right)\right)^{\beta}\left(\frac{\varsigma_0}{\varsigma_0+p_0^{\beta}}\right)^{\beta/\alpha}\phi^{\beta/\alpha}\left(\varsigma^{-1}\left(\delta\left(v\right)\right)\right)\leq 0.$$

By Theorem 1 [21], the associated delay Equation (10) also has a positive solution, which is a contradiction. The proof is complete. □

Theorem 2. *Assume that $\beta \geq \alpha$. If there exists a function $\theta \in C([v_0,\infty),(0,\infty))$ such that $\theta(v) \leq v$, $\vartheta(v) \leq \varsigma(\theta(v))$ and*

$$\limsup_{v\to\infty} M^{\beta-\alpha}\eta^{\alpha}(\vartheta,\varsigma(\theta)) \int_{\theta(v)}^{v} \widetilde{q}(s)\,\mathrm{d}s > \mu\left(1+\frac{1}{\varsigma_0}p_0^{\beta}\right), \tag{13}$$

then S_2 is an empty set.

Proof. As in the proof of Theorem 1, we obtain (12). Using Lemma 4, we get (3) and (4). Integrating (4) from $\theta(v)$ to v, we get

$$0 < \pounds w(v) + \frac{1}{\varsigma_0}p_0^{\beta}\,\pounds w(\varsigma(v)) \leq \pounds w(\theta(v)) + \frac{1}{\varsigma_0}p_0^{\beta}\,\pounds w(\varsigma(\theta(v))) - \frac{1}{\mu}\int_{\theta(v)}^{v}\widetilde{q}(s)\,w^{\beta}(\vartheta(s))\,\mathrm{d}s,$$

which together with (12) gives

$$\left(1+\frac{1}{\varsigma_0}p_0^{\beta}\right)\pounds w(\varsigma(\theta(v))) \geq \frac{1}{\mu}w^{\beta}(\vartheta(v))\int_{\theta(v)}^{v}\widetilde{q}(s)\,\mathrm{d}s. \tag{14}$$

Since $w'(v) < 0$, there exists a constant $M > 0$ such that $w(v) \geq M$ for $v \geq v_2$, and hence (14) becomes

$$\left(1+\frac{1}{\varsigma_0}p_0^{\beta}\right)\pounds w(\varsigma(\theta(v))) \geq \frac{M^{\beta-\alpha}}{\mu}w^{\alpha}(\vartheta(v))\int_{\theta(v)}^{v}\widetilde{q}(s)\,\mathrm{d}s.$$

From (3) $[u = \vartheta(v)$ and $\varpi = \varsigma(\theta(v))]$, we find

$$\left(1+\frac{1}{\varsigma_0}p_0^{\beta}\right) \geq \frac{M^{\beta-\alpha}}{\mu}\eta^{\alpha}(\vartheta,\varsigma(\theta))\int_{\theta(v)}^{v}\widetilde{q}(s)\,\mathrm{d}s.$$

From above inequality, taking the lim sup on both sides, we obtain a contradiction to (13). The proof is complete. □

Corollary 1. *Assume that there exists a function $\delta \in C([v_0,\infty),(0,\infty))$ such that $\vartheta(v) \leq \delta(v)$, $\varsigma^{-1}(\delta(v)) < v$. Then S_2 is an empty set, if one of the statements is hold:*
($\mathbf{b}_1$) *$\alpha = \beta$ and*

$$\liminf_{v\to\infty}\int_{\vartheta^{-1}(\delta(v))}^{v}\widetilde{q}(s)\,\widetilde{\eta}(\varsigma(s),\delta(s))\,\mathrm{d}s > \frac{\vartheta_0+p_0^{\beta}}{\vartheta_0\mu e}; \tag{15}$$

($\mathbf{b}_2$) *$\alpha < \beta$, there exists a function $\xi(v) \in C^1([v_0,\infty))$ such that $\xi'(v) > 0$, $\lim_{v\to\infty}\xi(v) = \infty$,*

$$\limsup_{v\to\infty}\frac{\beta\xi'\left(\vartheta^{-1}(\delta(v))\right)\left(\vartheta^{-1}(\delta(v))\right)'}{\alpha\xi'(v)} < 1 \tag{16}$$

and

$$\liminf_{v\to\infty}\left[\frac{1}{\mu\xi'(v)}\left(\frac{\vartheta_0}{\vartheta_0+p_0^{\beta}}\right)^{\beta/\alpha}\widetilde{q}(v)\,\varsigma(\varsigma,\delta)\,e^{-\xi(v)}\right] > 0. \tag{17}$$

Proof. It is well-known from [22,23] that conditions (15)–(17) imply the oscillation of (10). □

3. Criteria for Nonexistence of Increasing Solutions

Theorem 3. *Assume that* $\vartheta(v) \leq \varsigma(v)$ *and* $\varsigma'(v) > 0$. *If there exist a function* $\sigma(v)$ *and* $v_1 \geq v_0$ *such that*

$$\limsup_{v\to\infty} \int_{v_1}^{v} \left[\frac{1}{\mu}\sigma(s)\,\widetilde{q}(s) - \frac{(\sigma'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma(s)\,\eta(\varsigma(s),s_1)\,\varsigma'(s))^{\alpha}} \left(1+\frac{\sigma_0^{\beta}}{\vartheta_0}\right)\right] \mathrm{d}s = \infty, \tag{18}$$

then S_1 *is an empty set.*

Proof. Let x be a positive solution of (1) and which satisfies case **(i)**. In view of case **(i)**, we can define a positive function by

$$\psi(v) = \sigma(v)\frac{\pounds w(v)}{w^{\alpha}(\varsigma(v))}. \tag{19}$$

Hence, by differentiating (19), we get

$$\psi'(v) = \sigma'(v)\frac{\pounds w(v)}{w^{\alpha}(\varsigma(v))} + \sigma(v)\frac{(\pounds w(v))'}{w^{\alpha}(\varsigma(v))} - \frac{\alpha\sigma(v)\,\pounds w(v)\,w^{\alpha-1}(\varsigma(v))\,w'(\varsigma(v))\,\varsigma'(v)}{w^{2\alpha}(\varsigma(v))}. \tag{20}$$

Substituting (19) into (20), we have

$$\psi'(v) = \sigma(v)\frac{(\pounds w(v))'}{w^{\alpha}(\varsigma(v))} + \frac{\sigma'(v)}{\sigma(v)}\psi(v) - \frac{\alpha\eta(\varsigma(v),v_1)\,\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\psi^{\frac{\alpha+1}{\alpha}}(v). \tag{21}$$

Now, define another positive function by

$$\varpi(v) = \sigma(v)\frac{\pounds w(\vartheta(v))}{w^{\alpha}(\varsigma(v))}. \tag{22}$$

By differentiating (22), we get

$$\varpi'(v) = \sigma'(v)\frac{\pounds w(\vartheta(v))}{w^{\alpha}(\varsigma(v))} + \sigma(v)\frac{(\pounds w(\vartheta(v)))'}{w^{\alpha}(\varsigma(v))} \tag{23}$$

$$-\frac{\alpha\sigma(v)\,\pounds w(\vartheta(v))\,w^{\alpha-1}(\varsigma(v))\,w'(\varsigma(v))\,\varsigma'(v)}{w^{2\alpha}(\varsigma(v))}. \tag{24}$$

Substituting (22) into (23) implies

$$\varpi'(v) = \sigma(v)\frac{(\pounds w(\vartheta(v)))'}{w^{\alpha}(\varsigma(v))} + \frac{\sigma'(v)}{\sigma(v)}\varpi(v) - \frac{\alpha\eta(\varsigma(v),v_1)\,\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\varpi^{\frac{\alpha+1}{\alpha}}(v). \tag{25}$$

We can write the inequalities (21) and (25) in the form

$$\begin{aligned}
\psi'(v) + \frac{\sigma_0^{\beta}}{\vartheta_0}\varpi'(v) \leq\; & \sigma(v)\frac{(\pounds w(v))' + \frac{\sigma_0^{\beta}}{\vartheta_0}(\pounds w(\vartheta(v)))'}{w^{\alpha}(\varsigma(v))} \\
& + \frac{\sigma'(v)}{\sigma(v)}\psi(v) - \frac{\alpha\eta(\varsigma(v),v_1)\,\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\psi^{\frac{\alpha+1}{\alpha}}(v) \\
& + \frac{\sigma_0^{\beta}}{\vartheta_0}\left(\frac{\sigma'(v)}{\sigma(v)}\varpi(v) - \frac{\alpha\eta(\varsigma(v),v_1)\,\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\varpi^{\frac{\alpha+1}{\alpha}}(v)\right).
\end{aligned} \tag{26}$$

Taking into account Lemma 1, (4) and (26), we obtain

$$\begin{aligned}\psi'(v)+\frac{\sigma_0^{\beta}}{\vartheta_0}\varpi'(v) \leq\; & -\sigma(v)\left(\frac{\widetilde{q}(v)}{\mu}\right)\\ & +\frac{\sigma'(v)}{\sigma(v)}\psi(v)-\frac{\alpha\eta(\varsigma(v),v_1)\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\psi^{\frac{\alpha+1}{\alpha}}(v)\\ & +\frac{\sigma_0^{\beta}}{\vartheta_0}\left(\frac{\sigma'(v)}{\sigma(v)}\varpi(v)-\frac{\alpha\eta(\varsigma(v),v_1)\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\varpi^{\frac{\alpha+1}{\alpha}}(v)\right).\end{aligned}$$

Applying the following inequality

$$Bu-Au^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^{\alpha}B^{\alpha+1}}{(\alpha+1)^{\alpha+1}A^{\alpha}},\quad A>0,$$

with

$$A=\frac{\alpha\eta(\varsigma(v),v_1)\varsigma'(v)}{\sigma^{\frac{1}{\alpha}}(v)}\text{ and }B=\frac{\sigma'(v)}{\sigma(v)},$$

we get

$$\begin{aligned}\psi'(v)+\frac{\sigma_0^{\beta}}{\vartheta_0}\varpi'(v) \leq\; & -\sigma(v)\frac{\widetilde{q}(v)}{\mu}+\frac{(\sigma'(v))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma(v)\eta(\varsigma(v),v_1)\varsigma'(v))^{\alpha}}\\ & +\frac{\frac{\sigma_0^{\beta}}{\vartheta_0}(\sigma'(v))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma(v)\eta(\varsigma(v),v_1)\varsigma'(v))^{\alpha}}.\end{aligned}$$

Integrating last inequality from v_1 to v, we arrive at

$$\int_{v_1}^{v}\left[\sigma(s)\frac{\widetilde{q}(s)}{\mu}-\frac{(\sigma'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma(s)\eta(\varsigma(s),s_1)\varsigma'(s))^{\alpha}}\left(1+\frac{\sigma_0^{\beta}}{\vartheta_0}\right)\right]\mathrm{d}s\leq\psi(v_2)+\frac{\sigma_0^{\beta}}{\vartheta_0}\varpi(v_2).$$

The proof is complete. □

Theorem 4. *Assume that there exist continuously differentiable functions $\sigma(v)$ and $\xi(v)$ and $\vartheta^{-1}(\delta(v))$ such that $\left(\vartheta^{-1}(\delta(v))\right)'>0, \xi'(v)>0$ and if (3) and one of the conditions (16), (17) or (15) holds, then Equation (1) is oscillatory.*

Theorem 5. *Assume that x is a positive solution of (1). If there exist $\theta\in C([v_0,\infty),(0,\infty))$ such that $\theta(v)<v$, $\varsigma(v)<\vartheta(\theta(v))$ and if conditions (3) and (13) hold, then Equation (1) is oscillatory.*

In this section we state and prove some results by considering

$$\varsigma(v)=v-\delta_0\text{ for }\delta_0\geq 0, p(v)=p_0\neq 1.$$

Lemma 5. *Let $x(v)$ be positive solution of Equation (1), eventually. Assume that $w(v)$ satisfies case* (**ii**). *If*

$$\int_{v_0}^{\infty}\int_{\phi}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty}q(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u\mathrm{d}\phi=\infty, \tag{27}$$

then

$$\lim_{v\to\infty} x(v) = 0. \tag{28}$$

Proof. Since $w(v)$ is a non-increasing positive function, there exists a constant $w_0 \geq 0$ such that $\lim_{v\to\infty} w(v) = w_0 \geq 0$. We claim that $w_0 = 0$. Otherwise, using Lemma 2, we conclude that $\lim_{v\to\infty} w(v) = w_0/(1+p_0) > 0$. Therefore, there exists a $v_2 \geq v_0$ such that, for all $v \geq v_2$

$$x(\varsigma(v)) > \frac{w_0}{2(1+p_0)} > 0. \tag{29}$$

From (1) and (29), we see that

$$(\pounds w((v)))' \leq -q(v)\left(\frac{w_0}{2(1+p_0)}\right)^{\beta}.$$

Integrating above inequality from v to ∞, we have

$$\pounds w((v)) \geq \left(\frac{w_0}{2(1+P_0)}\right)^{\beta} \int_v^{\infty} q(s)\,\mathrm{d}s.$$

It follows that

$$w''(v) \geq \left(\frac{w_0}{2(1+P_0)}\right)^{\frac{\beta}{\alpha}} \left(\frac{1}{r(v)}\int_v^{\infty} q(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}. \tag{30}$$

Integrating (30) from v to ∞, yields

$$-w'(v) \geq \left(\frac{w_0}{2(1+P_0)}\right)^{\frac{\beta}{\alpha}} \int_v^{\infty}\left(\frac{1}{r(u)}\int_v^{\infty} \mathrm{q}(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u.$$

Integrating again from v_2 to ∞, we obtain

$$w(v_2) \geq \left(\frac{w_0}{2(1+P_0)}\right)^{\frac{\beta}{\alpha}} \int_{v_2}^{\infty}\int_{\phi}^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty} \mathrm{q}(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u\mathrm{d}\phi,$$

which contradicts with (27). Therefore, $\lim_{v\to\infty} w(v) = 0$, and from the inequality $0 < x(v) \leq w(v)$, we have property (28). The proof is complete. □

Theorem 6. *Let condition (27) be satisfied and suppose that there exists a function $\varrho \in C(I,\mathbb{R})$ such that $\varrho(v) \leq \varsigma(v)$, $\varrho(v) < v$ and $\lim_{v\to\infty} \varrho(v) = \infty$. If the first-order delay differential equation*

$$y'(v) + \frac{q(v)}{(1+p_0)^{\beta}}\left(\int_{v_1}^{\varrho(v)}\int_{u_1}^{\phi} a^{-1/\gamma}(s)\,\mathrm{d}s\mathrm{d}u\right)^{\beta} y^{\frac{\beta}{\alpha}}(\varrho(v)) = 0$$

is oscillatory, then every solution $x(v)$ of Equation (1) is either oscillatory or satisfies (28).

Proof. Assume that $x(v)$ is positive solution of (1), eventually. This implies that there exists $v_1 \geq v_o$ such that either $(\mathbf{i})$ or $(\mathbf{ii})$ hold for all $v \geq v_1$.
For $(\mathbf{ii})$, by lemma 5, we see that (28) holds.

For (**i**), since $w'(v)$ is a non-decreasing positive function, there exists a constant c_0 such that $\lim_{v\to\infty} w'(v) = c_0 > 0$ (or $c_0 = \infty$). By Lemma 2, we have

$$\lim_{v\to\infty} x'(v) = c_0/(1+p_0) > 0,$$

which implies that $x(v)$ is a non-decreasing function and taking into account $\delta_0 \geq 0$, we get

$$w(v) = x(v) + p_0 x(v-\delta_0) \leq (1+p_0)\, x(v),$$

therefore

$$x(v) \geq \frac{1}{1+p_0} w(v),$$

for $\varrho(v) \leq \varsigma(v)$, and

$$x(\varsigma(v)) \geq x(\varrho(v)) \geq \frac{1}{1+p_0} w(\varrho(v)).$$

By substitution in (1), we have

$$(\pounds w(v))' + \frac{q(v)}{(1+p_0)^{\beta}} w^{\beta}(\varrho(v)) \leq 0. \tag{31}$$

Using (7) and (31), we get

$$(\pounds w(v))' + \frac{q(v)}{(1+p_0)^{\beta}} \left(\int_{v_2}^{\varrho(v)} \int_{u_1}^{\phi} a^{-1/\gamma}(s)\, \mathrm{d}s\mathrm{d}u \right)^{\beta} (\pounds w(\varrho(v)))^{\frac{\beta}{\alpha}} \leq 0.$$

Therefore, we have $y = \pounds w(v)$ is positive solution of a the first order delay equation

$$y'(v) + \frac{q(v)}{(1+p_0)^{\beta}} \left(\int_{v_1}^{\varrho(v)} \int_{u_1}^{\phi} a^{-1/\gamma}(s)\, \mathrm{d}s\mathrm{d}u \right)^{\beta} y^{\frac{\beta}{\alpha}}(\varrho(v)) \leq 0.$$

The proof is complete. □

Theorem 7. *If the first-order delay differential equation*

$$w'(v) + \frac{1}{\mu} \left(\frac{\vartheta_0}{\vartheta_0 + p_0^{\beta}} \right) \widetilde{q}(v) \frac{\lambda^{\beta} \varsigma^{2\beta}(v)}{2^{\beta} r^{\beta/\alpha}(\varsigma(v))} w^{\beta/\alpha}(\varsigma(v)) = 0 \tag{32}$$

is oscillatory, eventually. Then, every solution $x(v)$ of Equation (1) is either oscillatory or satisfies (28).

Proof. As in the proof of Lemma 1, we get, from (1), (5) and (8), that (9) holds. Now, by using Lemma 3, we have

$$w(v) > \frac{\lambda}{2} v^2 w''(v). \tag{33}$$

Since $\frac{\mathrm{d}}{\mathrm{d}v} \pounds w(v) \leq 0$ and $\vartheta(v) \leq v$, we obtain $\pounds w(\vartheta(v)) \geq \pounds w(v)$, and so

$$\pounds w(v) + \frac{1}{\vartheta_0} p_0^{\beta} \pounds w(\vartheta(v)) \leq \left(1 + \frac{1}{\vartheta_0} p_0^{\beta} \right) \pounds w(v),$$

which with (9) gives

$$(\pounds w(v))' + \frac{1}{\mu}\left(\frac{\vartheta_0}{\vartheta_0 + p_0^{\beta}}\right)\widetilde{q}(v)\, w^{\beta}(\varsigma(v)) \leq 0.$$

Thus, from (33), we find

$$(\pounds w(v))' + \frac{1}{\mu}\left(\frac{\vartheta_0}{\vartheta_0 + p_0^{\beta}}\right)\widetilde{q}(v)\, \frac{\lambda^{\beta}}{2^{\beta}}\varsigma^{2\beta}(v)\left(w''(\varsigma(v))\right)^{\beta} \leq 0.$$

If we set $w := \pounds w(v) = r(w'')^{\alpha}$, then we have that $w > 0$ is a solution of delay inequality

$$w'(v) + \frac{1}{\mu}\left(\frac{\vartheta_0}{\vartheta_0 + p_0^{\beta}}\right)\widetilde{q}(v)\, \frac{\lambda^{\beta}\varsigma^{2\beta}(v)}{2^{\beta} r^{\beta/\alpha}(\varsigma(v))} w^{\beta/\alpha}(\varsigma(v)) \leq 0.$$

By Theorem 1 [21] the associated delay differential Equation (32) also has a positive solution. The proof is complete. □

Example 1. *Consider the third order delay differential equation*

$$\left[\left(\left[x(v) + px(\lambda v)\right]''\right)^{\alpha}\right]' + \frac{q_0}{v^{\alpha(n-1)+1}} x^{\alpha}(\gamma v) = 0, \qquad (34)$$

where $\gamma, \lambda \in (0,1)$. *Then* $\widetilde{q}(v) = \frac{q_0}{v^{2\alpha+1}}$, $\varsigma(v) = \gamma v$, $\vartheta(v) = \lambda v$, *set* $\sigma(v) = v^2$, $\zeta(v) = \frac{(\gamma+\lambda)v}{2}$.
It is easy to get $\eta(v,u) = (v-u)$, $\widetilde{\eta}(v,u) = \frac{(v-u)^2}{2}$ *and* $\vartheta^{-1}(v) = \frac{v}{\gamma}$.
By Theorem 3, (18) imply

$$q_0 > \frac{(2)^{\beta-1}(2\alpha)^{\alpha+1}}{\gamma^{2\alpha}(\alpha+1)^{\alpha+1}}\left(1 + \frac{\sigma_0^{\beta}}{\vartheta_0}\right),$$

also, by (15) with $\alpha = 1$, *we get*

$$\frac{q_0}{8}(\gamma - \lambda)^2 \ln\frac{2\gamma}{\lambda + \gamma} > \frac{\vartheta_0 + p_0}{\vartheta_0 e},$$

By Theorem 4 with $\alpha = 1$, *the Equation (34) is oscillatory if*

$$q_0 > \max\left\{\frac{1}{\gamma^2}\left(1 + \frac{\sigma_0}{\vartheta_0}\right), \frac{8(\vartheta_0 + p_0)}{(\gamma - \lambda)^2\left(\ln\frac{2\gamma}{\lambda+\gamma}\right)\vartheta_0 e}\right\}.$$

Remark 1. *The results in [11–19] only ensure that the non-oscillating solutions to Equation (34) tend to zero, so our method improves the previous results.*

Remark 2. *For interested researchers, there is a good problem which is finding new results for non existence of Kneser solutions for (1) without requiring*

$$\vartheta \circ \varsigma = \varsigma \circ \vartheta \text{ or } \left(\vartheta^{-1}(v)\right)' \geq \vartheta_0.$$

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