

Construction of Different Types Analytic Solutions for the Zhiber-Shabat Equation

Asif Yokus ^{1,†}, Hülya Durur ^{2,†}, Hijaz Ahmad ³ and Shao-Wen Yao ^{4,*}¹ Department of Actuary, Faculty of Science, Firat University, Elazig 23200, Turkey; asfyokus@yahoo.com² Department of Computer Engineering, Faculty of Engineering, Ardahan University, Ardahan 75000, Turkey; hulyadurur@ardahan.edu.tr³ Department of Basic Sciences, University of Engineering and Technology, Peshawar 25000, Pakistan; hijaz555@gmail.com⁴ School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China

* Correspondence: yaoshaowen@hpu.edu.cn

† These authors contributed equally to this work.

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Abstract: In this paper, a new solution process of $(1/G')$ -expansion and $(G'/G, 1/G)$ -expansion methods has been proposed for the analytic solution of the Zhiber-Shabat (Z-S) equation. Rather than the classical $(G'/G, 1/G)$ -expansion method, a solution function in different formats has been produced with the help of the proposed process. New complex rational, hyperbolic, rational and trigonometric types solutions of the Z-S equation have been constructed. By giving arbitrary values to the constants in the obtained solutions, it can help to add physical meaning to the traveling wave solutions, whereas traveling wave has an important place in applied sciences and illuminates many physical phenomena. 3D, 2D and contour graphs are displayed to show the stationary wave or the state of the wave at any moment with the values given to these constants. Conditions that guarantee the existence of traveling wave solutions are given. Comparison of $(G'/G, 1/G)$ -expansion method and $(1/G')$ -expansion method, which are important instruments in the analytical solution, has been made. In addition, the advantages and disadvantages of these two methods have been discussed. These methods are reliable and efficient methods to obtain analytic solutions of nonlinear evolution equations (NLEEs).

Keywords: $(1/G')$ -expansion method; the Zhiber-Shabat equation; $(G'/G, 1/G)$ -expansion method; traveling wave solutions; exact solutions

1. Introduction

The analysis of analytic solutions of nonlinear evolution equations (NLEEs) plays a significant role in the study of nonlinear physical phenomena. Various techniques have been tried to obtain analytic solutions, such as the sine-cosine method [1], extended sinh-Gordon equation expansion method [2,3], (G'/G) -expansion method [4,5], improved Bernoulli sub-equation function method [6], variational iteration algorithm-II [7–9], sub equation method [10], collocation method [11,12], $(1/G')$ -expansion method [13–15], first integral method [16], adomian decomposition methods [17–19], Hirota bilinear method [20], modified variational iteration algorithms [21–24], homotopy perturbation method [25], residual power series method [26], $(G'/G, 1/G)$ -expansion method [27] and so on [28–38].

In this study, our main purpose is to obtain the traveling wave solutions of the evolution equations in nonlinear dynamics. As it is known, scientific studies take place gradually. The first step is to examine a physical event, the second step is to model the event, the third step is to produce the solution of the model and the fourth step is to load the produced solution into physical meaning. In this

article, to produce the solution in the third stage and to prepare for the fourth stage. As it is known, in soliton theory, it will be much more valuable if solitons gain physical meaning. For example, today, the pandemic patients that affect the world may represent a stationary wave on a graph consisting of numerical data related to parameters such as number of patients and number of tests. Employees on this subject can relate to the solutions we will offer in this study. We consider the Zhiber-Shabat (Z-S) equation [39]

$$u_{xt} + pe^u + qe^{-u} + re^{-2u} = 0, \quad (1)$$

where p, r, q are arbitrary constants. When $r = 0, q \neq 0$, Equation (1) gives the well-known sinh-Gordon equation, while, $r \neq 0, q = 0$, gives the Dodd-Bullough-Mikhailov (DBM) equation. However, for $p = 0, q = -1, r = -1$, Equation (1) reduced to the Tzitzeica-Dodd-Bullough (TDB) equation, while for $r = q = 0$, gives the Liouville equation. These equations play an effective role in various scientific applications such as fluid dynamics, solid state physics, nonlinear optics and chemical kinetics. When the analytical solution of Equation (1) is found, the solutions of the sinh-Gordon, DBM, TDB and the Liouville equations can also be obtained.

Many researchers have investigated the Z-S equations and discussed its applications in different field of science and engineering. Some of these investigations are as follows: various types of solution for the Z-S equation are obtained [40] by using qualitative theory of polynomial differential system, while qualitative behavior and exact travelling wave solutions of the Z-S equation are obtained in [41]. Analytic solutions of the Z-S equation are obtained using the $(-\phi(\xi))$ -expansion method [42], exponential rational function method [43], while exact solutions of it are obtained using bifurcation theory and method of phase portraits analysis [44].

In the current work, we are interested in constructing exact solutions of the Zhiber-Shabat (Z-S) equation using $(1/G')$ -expansion method and $(G'/G, 1/G)$ -expansion method. The solutions of the equation have not been studied with either method. In this study, both to provide the literature with the solution produced by these methods and to discuss the advantages and disadvantages of the methods.

2. $(1/G')$ -Expansion Method

Consider a general form of NLEEs as

$$S\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0. \quad (2)$$

Let $u = u(x, t) = u(\xi), \xi = x - wt, w \neq 0$, where w is a constant speed of the wave. After using transformation, it can be converted into the following nonlinear ODE for $u(\xi)$:

$$Q(u, u', u'', u''', \dots) = 0. \quad (3)$$

The solution of Equation (3) is assumed to have the form

$$u(\xi) = a_0 + \sum_{i=1}^m a_i \left(\frac{1}{G'}\right)^i, \quad (4)$$

where $a_i, i = 0, 1, \dots, m$ are constants and $G = G(\xi)$ provides the following second order IODE

$$G'' + \lambda G' + \mu = 0, \quad (5)$$

where, λ and μ are constants to be determined after,

$$\frac{1}{G'} = \frac{\lambda}{-\mu + \lambda A(\cosh(\xi\lambda) - \sinh(\xi\lambda))}, \quad (6)$$

where A is an integral constant. If the desired derivatives of the Equation (4) are calculated and substituting in the Equation (3), a polynomial with the argument $(1/G')$ is attained. An algebraic equation system is created by equalizing the coefficients of this polynomial to zero. These equations are solved with the help of the package program and put into place in the default Equation (3) solution function. Finally, the solutions of Equation (1) are obtained.

3. $(G'/G, 1/G)$ -Expansion Method

Consider the following general form of NLEEs

$$Z(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, \dots) = 0. \quad (7)$$

If $u(x, t) = u(\xi)$, $\xi = x - wt$ where w is a constant, when transmutation is applied to Equation (7), it becomes a NLODE and this equation may be written as:

$$z(u', u'', u''', \dots) = 0. \quad (8)$$

Complexity can be reduced by integrating Equation (8). By the nature of this method, $G(\xi)$ is a quadratic function ODE solution,

$$G''(\xi) + \lambda G(\xi) = \mu. \quad (9)$$

Furthermore, to provide operational aesthetics as $\phi = \phi(\xi) = G'/G$ and $\psi = \psi(\xi) = \frac{1}{G(\xi)}$. We may write derivatives of functions defined here;

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \quad (10)$$

We can offer the behavior of solution function Equation (9) according to the state of λ , taking into account the equations given by the Equation (10).

i: If $\lambda < 0$

$$G(\xi) = c_1 \sinh(\sqrt{-\lambda}\xi) + c_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}, \quad (11)$$

whereas c_1 and c_2 are reel numbers. Considering Equation (11);

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2} (\phi^2 - 2\mu\psi + \lambda), \quad \sigma = c_1^2 - c_2^2, \quad (12)$$

Equation (12) is written.

ii: If $\lambda > 0$

$$G(\xi) = c_1 \sin(\sqrt{\lambda}\xi) + c_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}, \quad (13)$$

here c_1 and c_2 are reel numbers. Considering Equation (13), there is following equation;

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2} (\phi^2 - 2\mu\psi + \lambda), \quad \sigma = c_1^2 + c_2^2, \quad (14)$$

iii: If $\lambda = 0$

$$G(\xi) = \frac{\mu}{2}\xi^2 + c_1\xi + c_2, \quad (15)$$

here c_1 and c_2 are reel numbers. Considering Equation (15), there is following equation;

$$\psi^2 = \frac{1}{c_1^2 - 2\mu c_2} (\phi^2 - 2\mu\psi). \quad (16)$$

In terms of ϕ and ψ polynomials, solution of Equation (8) is ;

$$U(\xi) = a_0 + \sum_{i=1}^n (a_i \phi^i + b_i \psi^i). \quad (17)$$

In this study, we reorganized the solution function in classical $(G'/G, 1/G)$ -expansion method as Equation (17) with the logic of solution functions of (G'/G) and $(1/G')$ -expansion methods. This logic is considered together with the classical $(G'/G, 1/G)$ -expansion method and the method can be developed in future studies and different solutions can be offered.

Wherein, a_i ($i = 0, 1, \dots, m$) and b_i ($i = 1, \dots, m$) counts then are constants to be determined. m is a positive equilibrium term that may be attained by comparing the maximum order derivative and the maximum order nonlinear term in Equation (8). If Equation (17) is written in Equation (8) with Equations (10), (12), (14) or (16), a polynomial function associated with ϕ and ψ is written. Each term coefficient of $\phi^i \psi^j$ ($i = 0, 1, \dots, m$) ($j = 1, \dots, m$) of the attained polynomial functions are equated to zero and a system algebraic equations is attained for $a_i, b_i, w, \mu, c_1, c_2$ and λ ($i = 0, 1, \dots, m$). The required coefficients are obtained by solving the algebraic equation with the help of computer package programs. These coefficients found are written in Equation (17) and $u(\xi)$ solution function of Equation (8) is obtained and if $\xi = x - wt$ transmutation is employed in reverse order, we will attain analytic solution $u(x, t)$ of Equation (7).

4. Solutions of the (Z-S) Equation Using $(1/G')$ -Expansion Method

We consider Equation (1). Using transmutation $u(x, t) = u(\xi)$, $\xi = x - wt$, we obtain

$$-wu_{\xi\xi} + pe^u + qe^{-u} + re^{-2u} = 0, \quad (18)$$

where w is the wave speed. To implement this method, we use transmutation $u = \ln v$ and $v = V(\xi)$, Equation (18) becomes

$$-w(VV'' - (V')^2) + pV^3 + qV + r = 0. \quad (19)$$

In Equation (19), we find balancing term $m = 2$ and in Equation (4), the following situation is obtained:

$$V = a_0 + a_1 \left(\frac{1}{G'} \right) + a_2 \left(\frac{1}{G'} \right)^2, \quad a_2 \neq 0, \quad (20)$$

where a_0, a_1, a_2 unknown constants to be determined later. Replacing Equation (20) into Equation (19) and the coefficients of the algebraic Equation (1) are equal to zero, we can establish the following algebraic equation systems

$$\begin{aligned} \left(\frac{1}{G'[\xi]} \right)^0 &: r + qa_0 + pa_0^3 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^1 &: qa_1 - w\lambda^2 a_0 a_1 + 3pa_0^2 a_1 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^2 &: -3w\lambda\mu a_0 a_1 + 3pa_0 a_1^2 + qa_2 - 4w\lambda^2 a_0 a_2 + 3pa_0^2 a_2 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^3 &: -2w\mu^2 a_0 a_1 - w\lambda\mu a_1^2 + pa_1^3 - 10w\lambda\mu a_0 a_2 - w\lambda^2 a_1 a_2 + 6pa_0 a_1 a_2 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^4 &: -w\mu^2 a_1^2 - 6w\mu^2 a_0 a_2 - 5w\lambda\mu a_1 a_2 + 3pa_1^2 a_2 + 3pa_0 a_2^2 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^5 &: -4w\mu^2 a_1 a_2 - 2w\lambda\mu a_2^2 + 3pa_1 a_2^2 = 0, \\ \left(\frac{1}{G'[\xi]} \right)^6 &: -2w\mu^2 a_2^2 + pa_2^3 = 0. \end{aligned} \quad (21)$$

Case I.

$$\mu = \frac{p\lambda a_0 a_1}{2(q + 3pa_0^2)}, \quad a_2 = \frac{pa_0 a_1^2}{2(q + 3pa_0^2)}, \quad w = \frac{q + 3pa_0^2}{\lambda^2 a_0}, \quad r = -qa_0 - pa_0^3, \quad (22)$$

considering Equation (6), substituting Equation (22) into Equation (20), the following solution is attained

$$V = \left[a_0 + \frac{pa_0a_1^2}{2(q+3pa_0^2) \left(A \cosh[\lambda(x-wt)] - A \sinh[\lambda(x-wt)] - \frac{pa_0a_1}{2(q+3pa_0^2)} \right)^2} + \frac{a_1}{A \cosh[\lambda(x-wt)] - A \sinh[\lambda(x-wt)] - \frac{pa_0a_1}{2(q+3pa_0^2)}} \right]. \quad (23)$$

In addition, if Equation (23) is written instead of $u = \ln v$ transformation, the analytical solution of Equation (1) is as follows,

$$u_1(x, t) = \ln \left[a_0 + \frac{pa_0a_1^2}{2(q+3pa_0^2) \left(A \cosh[\lambda(x-wt)] - A \sinh[\lambda(x-wt)] - \frac{pa_0a_1}{2(q+3pa_0^2)} \right)^2} + \frac{a_1}{A \cosh[\lambda(x-wt)] - A \sinh[\lambda(x-wt)] - \frac{pa_0a_1}{2(q+3pa_0^2)}} \right]. \quad (24)$$

The hyperbolic traveling wave solution of Equation (24) produced from the $(1/G')$ -expansion method is as in Figure 1.

Case II.

$$a_0 = \frac{pa_1^2 + \sqrt{p}\sqrt{pa_1^4 - 48qa_2^2}}{12pa_2}, \quad w = \frac{pa_2}{2\mu^2}, \quad \lambda = \frac{\mu a_1}{a_2},$$

$$r = \frac{-\frac{pa_1^6}{a_2} - \frac{\sqrt{p}a_1^4\sqrt{pa_1^4 - 48qa_2^2}}{a_2} - \frac{24qa_2\sqrt{pa_1^4 - 48qa_2^2}}{\sqrt{p}}}{432a_2^2}, \quad (25)$$

considering Equation (6), replacing Equation (25) into Equation (20), the following solution is attained

$$V = \frac{1}{12} \left[\frac{\sqrt{pa_1^4 - 48qa_2^2}}{\sqrt{p}a_2} + a_1^2 \left(\frac{\frac{12}{Ae^{a_1\left(\frac{pt}{2\mu} - \frac{x\mu}{a_2}\right)}a_1 - a_2} + \frac{1}{a_2}}{\left(-Ae^{a_1\left(\frac{pt}{2\mu} - \frac{x\mu}{a_2}\right)}a_1 + a_2 \right)^2} \right) \right]. \quad (26)$$

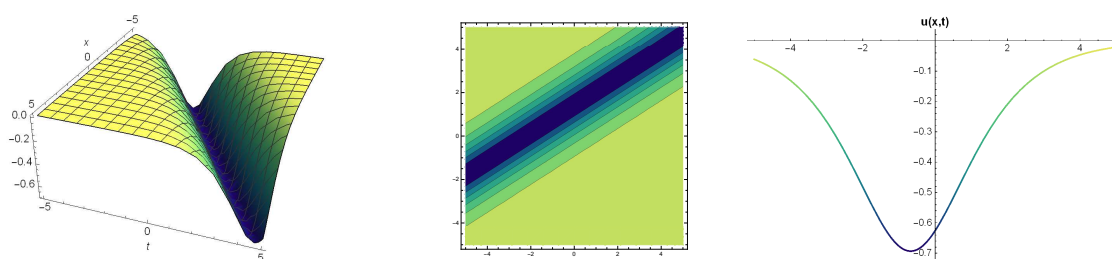


Figure 1. 3D, contour and 2D graphs respectively for $p = -1$, $\lambda = -0.8$, $A = -3$, $a_0 = -1$, $a_1 = -1$, $q = 2$ values of Equation (24).

In addition, if Equation (26) is written instead of $u = \ln v$ transformation, the analytical solution of Equation (1) is as follows,

$$u_2(x, t) = \ln \left[\frac{1}{12} \left[\frac{\sqrt{pa_1^4 - 48qa_2^2}}{\sqrt{pa_2}} + a_1^2 \left(\frac{12}{Ae^{a_1 \left(\frac{pt}{2\mu} - \frac{x\mu}{a_2} \right)} a_1 - a_2} + \frac{1}{a_2} \right) + \frac{12a_2}{\left(-Ae^{a_1 \left(\frac{pt}{2\mu} - \frac{x\mu}{a_2} \right)} a_1 + a_2 \right)^2} \right] \right]. \quad (27)$$

The analytic solution of Equation (27) produced from the $(1/G')$ -expansion method is as in Figure 2.

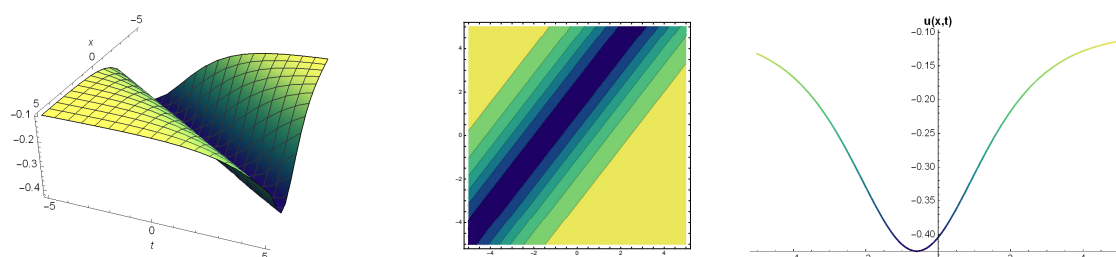


Figure 2. 3D, contour and 2D graphs respectively for $p = -1$, $\mu = -0.8$, $A = -3$, $a_2 = -1$, $a_1 = -1$, $q = 2$ values of Equation (27).

5. Solutions of the (Z-S) Equation Using $(G'/G, 1/G)$ -Expansion Method

We consider Equation (1). Using transmutation $u(x, t) = u(\xi)$, $\xi = x - wt$, $w \neq 0$, we get

$$-wu_{\xi\xi} + pe^u + qe^{-u} + re^{-2u} = 0, \quad (28)$$

To apply this method, we use transmutation $u = \ln v$ and $v = V(\xi)$, Equation (28) becomes

$$-w(VV'' - (V')^2) + pV^3 + qV + r = 0. \quad (29)$$

In Equation (29), we find balancing term $m = 2$ and in Equation (10), the following situation is obtained

$$u(\xi) = a_0 + a_1\phi[\xi] + b_1\psi[\xi] + a_2\phi[\xi]^2 + b_2\psi[\xi]^2, \quad (30)$$

where a_0 , a_1 , a_2 , b_1 , b_2 constants to be determined are unknown. Replacing Equation (30) into Equation (29) and the coefficients of the algebraic Equation (1) are equal to zero, we can establish the following algebraic equation systems

$$\begin{aligned}
\text{Cons : } & r + qa_0 + pa_0^3 + w\lambda^2 a_1^2 - \frac{w\lambda^2 \mu^2 a_1^2}{\mu^2 + \lambda^2 \sigma} - 2w\lambda^2 a_0 a_2 + \frac{2w\lambda^2 \mu^2 a_0 a_2}{\mu^2 + \lambda^2 \sigma} - \frac{w\lambda^2 \mu a_0 b_1}{\mu^2 + \lambda^2 \sigma} \\
& + \frac{4w\lambda^3 \mu^3 a_2 b_1}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{4w\lambda^3 \mu a_2 b_1}{\mu^2 + \lambda^2 \sigma} - \frac{2w\lambda^3 \mu^2 b_1^2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{w\lambda^3 b_1^2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda^2 a_0 b_1^2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{2p\lambda^3 \mu b_1^3}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{q\lambda^2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{4w\lambda^3 \mu^2 a_0 b_2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{2w\lambda^3 a_0 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda^2 a_0^2 b_2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{8w\lambda^4 \mu^2 a_2 b_2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{2w\lambda^4 a_2 b_2}{\mu^2 + \lambda^2 \sigma} + \frac{6w\lambda^4 \mu b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{12p\lambda^3 \mu a_0 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} = 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\phi [\xi] : & qa_1 - 2w\lambda a_0 a_1 + 3pa_0^2 a_1 + 2w\lambda^2 a_1 a_2 - \frac{2w\lambda^2 \mu^2 a_1 a_2}{\mu^2 + \lambda^2 \sigma} - \frac{2w\lambda^2 \mu a_1 b_1}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{3p\lambda^2 a_1 b_1^2}{\mu^2 + \lambda^2 \sigma} - \frac{2w\lambda^3 \mu^2 a_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{6p\lambda^2 a_0 a_1 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{12p\lambda^3 \mu a_1 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} = 0,
\end{aligned}$$

$$\begin{aligned}
(\phi [\xi])^2 : & -\frac{w\lambda \mu^2 a_1^2}{\mu^2 + \lambda^2 \sigma} + 3pa_0 a_1^2 + qa_2 - 8w\lambda a_0 a_2 + \frac{2w\lambda \mu^2 a_0 a_2}{\mu^2 + \lambda^2 \sigma} + 3pa_0^2 a_2 + 2w\lambda^2 a_2^2 \\
& - \frac{2w\lambda^2 \mu^2 a_2^2}{\mu^2 + \lambda^2 \sigma} - \frac{w\lambda \mu a_0 b_1}{\mu^2 + \lambda^2 \sigma} + \frac{4w\lambda^2 \mu^3 a_2 b_1}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{11w\lambda^2 \mu a_2 b_1}{\mu^2 + \lambda^2 \sigma} - \frac{2w\lambda^2 \mu^2 b_1^2}{(\mu^2 + \lambda^2 \sigma)^2} \\
& + \frac{2w\lambda^2 b_1^2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda a_0 b_1^2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda^2 a_2 b_1^2}{\mu^2 + \lambda^2 \sigma} - \frac{2p\lambda^2 \mu b_1^3}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{q\lambda b_2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{4w\lambda^2 \mu^2 a_0 b_2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{8w\lambda^2 a_0 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda a_0^2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda^2 a_1^2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{16w\lambda^3 \mu^2 a_2 b_2}{(\mu^2 + \lambda^2 \sigma)^2} \\
& + \frac{4w\lambda^3 a_2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{6p\lambda^2 a_0 a_2 b_2}{\mu^2 + \lambda^2 \sigma} + \frac{14w\lambda^3 \mu b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{12p\lambda^2 \mu a_0 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{12p\lambda^3 \mu a_2 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} = 0,
\end{aligned}$$

$$\begin{aligned}
(\phi [\xi])^3 : & -2wa_0 a_1 + pa_1^3 - 2w\lambda a_1 a_2 - \frac{2w\lambda \mu^2 a_1 a_2}{\mu^2 + \lambda^2 \sigma} + 6pa_0 a_1 a_2 - \frac{2w\lambda \mu a_1 b_1}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda a_1 b_1^2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{2w\lambda^2 \mu^2 a_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{4w\lambda^2 a_1 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{6p\lambda a_0 a_1 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{6p\lambda^2 a_1 a_2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{12p\lambda^2 \mu a_1 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} = 0,
\end{aligned}$$

$$\begin{aligned}
(\phi [\xi])^4 : & -wa_1^2 - 6wa_0 a_2 + 3pa_1^2 a_2 - \frac{2w\lambda \mu^2 a_2^2}{\mu^2 + \lambda^2 \sigma} + 3pa_0 a_2^2 - \frac{7w\lambda \mu a_2 b_1}{\mu^2 + \lambda^2 \sigma} + \frac{w\lambda b_1^2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{3p\lambda a_2 b_1^2}{\mu^2 + \lambda^2 \sigma} + \frac{6w\lambda a_0 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda a_1^2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{8w\lambda^2 \mu^2 a_2 b_2}{(\mu^2 + \lambda^2 \sigma)^2} + \frac{6w\lambda^2 a_2 b_2}{\mu^2 + \lambda^2 \sigma} \\
& - \frac{6p\lambda a_0 a_2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda^2 a_2^2 b_2}{\mu^2 + \lambda^2 \sigma} + \frac{8w\lambda^2 \mu b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} - \frac{12p\lambda^2 \mu a_2 b_1 b_2}{(\mu^2 + \lambda^2 \sigma)^2} = 0,
\end{aligned}$$

$$(\phi [\xi])^5 : -4wa_1 a_2 + 3pa_1 a_2^2 + \frac{4w\lambda a_1 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{6p\lambda a_1 a_2 b_2}{\mu^2 + \lambda^2 \sigma} = 0,$$

$$(\phi [\xi])^6 : -2wa_2^2 + pa_2^3 + \frac{4w\lambda a_2 b_2}{\mu^2 + \lambda^2 \sigma} - \frac{3p\lambda a_2^2 b_2}{\mu^2 + \lambda^2 \sigma} = 0,$$

$$\begin{aligned}
\psi[\xi] : & -2w\lambda\mu a_1^2 + \frac{2w\lambda\mu^3 a_1^2}{\mu^2 + \lambda^2\sigma} + 4w\lambda\mu a_0 a_2 - \frac{4w\lambda\mu^3 a_0 a_2}{\mu^2 + \lambda^2\sigma} + qb_1 - w\lambda a_0 b_1 + \frac{2w\lambda\mu^2 a_0 b_1}{\mu^2 + \lambda^2\sigma} \\
& + 3pa_0^2 b_1 - 2w\lambda^2 a_2 b_1 - \frac{8w\lambda^2 \mu^4 a_2 b_1}{(\mu^2 + \lambda^2\sigma)^2} + \frac{10w\lambda^2 \mu^2 a_2 b_1}{\mu^2 + \lambda^2\sigma} + \frac{4w\lambda^2 \mu^3 b_1^2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{3w\lambda^2 \mu b_1^2}{\mu^2 + \lambda^2\sigma} \\
& + \frac{6p\lambda\mu a_0 b_1^2}{\mu^2 + \lambda^2\sigma} + \frac{4p\lambda^2 \mu^2 b_1^3}{(\mu^2 + \lambda^2\sigma)^2} - \frac{p\lambda^2 b_1^3}{\mu^2 + \lambda^2\sigma} + \frac{2q\lambda\mu b_2}{\mu^2 + \lambda^2\sigma} + \frac{8w\lambda^2 \mu^3 a_0 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{6w\lambda^2 \mu a_0 b_2}{\mu^2 + \lambda^2\sigma} \\
& + \frac{6p\lambda\mu a_0^2 b_2}{\mu^2 + \lambda^2\sigma} + \frac{16w\lambda^3 \mu^3 a_2 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{8w\lambda^3 \mu a_2 b_2}{\mu^2 + \lambda^2\sigma} - \frac{12w\lambda^3 \mu^2 b_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} + \frac{3w\lambda^3 b_1 b_2}{\mu^2 + \lambda^2\sigma} \\
& + \frac{24p\lambda^2 \mu^2 a_0 b_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{6p\lambda^2 a_0 b_1 b_2}{\mu^2 + \lambda^2\sigma} = 0, \\
\phi[\xi]\psi[\xi] : & 3w\mu a_0 a_1 - 4w\lambda\mu a_1 a_2 + \frac{4w\lambda\mu^3 a_1 a_2}{\mu^2 + \lambda^2\sigma} - w\lambda a_1 b_1 + \frac{4w\lambda\mu^2 a_1 b_1}{\mu^2 + \lambda^2\sigma} + 6pa_0 a_1 b_1 \\
& + \frac{6p\lambda\mu a_1 b_1^2}{\mu^2 + \lambda^2\sigma} + \frac{4w\lambda^2 \mu^3 a_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{w\lambda^2 \mu a_1 b_2}{\mu^2 + \lambda^2\sigma} + \frac{12p\lambda\mu a_0 a_1 b_2}{\mu^2 + \lambda^2\sigma} + \frac{24p\lambda^2 \mu^2 a_1 b_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{6p\lambda^2 a_1 b_1 b_2}{\mu^2 + \lambda^2\sigma} = 0, \\
\phi[\xi]^2\psi[\xi] : & w\mu a_1^2 + 10w\mu a_0 a_2 - 4w\lambda\mu a_2^2 + \frac{4w\lambda\mu^3 a_2^2}{\mu^2 + \lambda^2\sigma} - 2wa_0 b_1 + 3pa_1^2 b_1 - 5w\lambda a_2 b_1 \\
& + \frac{16w\lambda\mu^2 a_2 b_1}{\mu^2 + \lambda^2\sigma} + 6pa_0 a_2 b_1 - \frac{3w\lambda\mu b_1^2}{\mu^2 + \lambda^2\sigma} + \frac{6p\lambda\mu a_2 b_1^2}{\mu^2 + \lambda^2\sigma} - \frac{p\lambda b_1^3}{\mu^2 + \lambda^2\sigma} - \frac{14w\lambda\mu a_0 b_2}{\mu^2 + \lambda^2\sigma} \\
& + \frac{6p\lambda\mu a_1^2 b_2}{\mu^2 + \lambda^2\sigma} + \frac{16w\lambda^2 \mu^3 a_2 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{12w\lambda^2 \mu a_2 b_2}{\mu^2 + \lambda^2\sigma} + \frac{12p\lambda\mu a_0 a_2 b_2}{\mu^2 + \lambda^2\sigma} - \frac{16w\lambda^2 \mu^2 b_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} \\
& + \frac{7w\lambda^2 b_1 b_2}{\mu^2 + \lambda^2\sigma} - \frac{6p\lambda a_0 b_1 b_2}{\mu^2 + \lambda^2\sigma} + \frac{24p\lambda^2 \mu^2 a_2 b_1 b_2}{(\mu^2 + \lambda^2\sigma)^2} - \frac{6p\lambda^2 a_2 b_1 b_2}{\mu^2 + \lambda^2\sigma} = 0, \\
\phi[\xi]^3\psi[\xi] : & 5w\mu a_1 a_2 - 2wa_1 b_1 + 6pa_1 a_2 b_1 - \frac{9w\lambda\mu a_1 b_2}{\mu^2 + \lambda^2\sigma} + \frac{12p\lambda\mu a_1 a_2 b_2}{\mu^2 + \lambda^2\sigma} - \frac{6p\lambda a_1 b_1 b_2}{\mu^2 + \lambda^2\sigma} = 0, \\
\phi[\xi]^4\psi[\xi] : & 2w\mu a_2^2 - 4wa_2 b_1 + 3pa_2^2 b_1 - \frac{12w\lambda\mu a_2 b_2}{\mu^2 + \lambda^2\sigma} \\
& + \frac{6p\lambda\mu a_2^2 b_2}{\mu^2 + \lambda^2\sigma} + \frac{4w\lambda b_1 b_2}{\mu^2 + \lambda^2\sigma} - \frac{6p\lambda a_2 b_1 b_2}{\mu^2 + \lambda^2\sigma} = 0, \\
\psi[\xi]^4 : & -2w\mu^2 a_2 b_2 + 3w\mu b_1 b_2 + 3pb_1^2 b_2 - 2w\lambda b_2^2 + 3pa_0 b_2^2 = 0, \\
\phi[\xi]\psi[\xi]^4 : & 3pa_1 b_2^2 = 0, \\
\phi[\xi]^2\psi[\xi]^4 : & -2wb_2^2 + 3pa_2 b_2^2 = 0, \\
\psi[\xi]^5 : & 2w\mu b_2^2 + 3pb_1 b_2^2 = 0, \\
\psi[\xi]^6 : & pb_2^3 = 0,
\end{aligned}$$

Our aim with the computer package program was reaching the solutions of system (31) and we attained the following situations.

If $\lambda < 0$,

Case I:

$$\begin{aligned}
a_0 &= \frac{4pw\lambda - \sqrt{-3p^3q + 4p^2w^2\lambda^2}}{3p^2}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = \frac{2w}{p}, \quad b_2 = 0, \quad \mu = 0, \\
r &= \frac{1}{27p^2} 2 \left(16w^3\lambda^3 + 3q\sqrt{p^2(-3pq + 4w^2\lambda^2)} + \frac{8w^2\lambda^2\sqrt{p^2(-3pq + 4w^2\lambda^2)}}{p} \right), \quad (32)
\end{aligned}$$

considering Equation (6), replacing Equation (32) into Equation (30), the following solution is attained

$$V = \frac{4pw\lambda - \sqrt{-3p^3q + 4p^2w^2\lambda^2}}{3p^2} + \frac{2w \left(c_2 \sqrt{-\lambda} \cosh \left[(-tw + x) \sqrt{-\lambda} \right] + c_1 \sqrt{-\lambda} \sinh \left[(-tw + x) \sqrt{-\lambda} \right] \right)^2}{p \left(c_1 \cosh \left[(-tw + x) \sqrt{-\lambda} \right] + c_2 \sinh \left[(-tw + x) \sqrt{-\lambda} \right] \right)^2}. \quad (33)$$

In addition, if Equation (33) is written instead of $u = \ln v$ transformation, the hyperbolic traveling wave solution of Equation (1) is as follows,

$$u_1(x, t) = \ln \left[\frac{4pw\lambda - \sqrt{-3p^3q + 4p^2w^2\lambda^2}}{3p^2} + \frac{2w \left(c_2 \sqrt{-\lambda} \cosh \left[(-tw + x) \sqrt{-\lambda} \right] + c_1 \sqrt{-\lambda} \sinh \left[(-tw + x) \sqrt{-\lambda} \right] \right)^2}{p \left(c_1 \cosh \left[(-tw + x) \sqrt{-\lambda} \right] + c_2 \sinh \left[(-tw + x) \sqrt{-\lambda} \right] \right)^2} \right]. \quad (34)$$

The hyperbolic traveling wave solution of Equation (34) produced from the $(G'/G, 1/G)$ -expansion method is as in Figure 3.

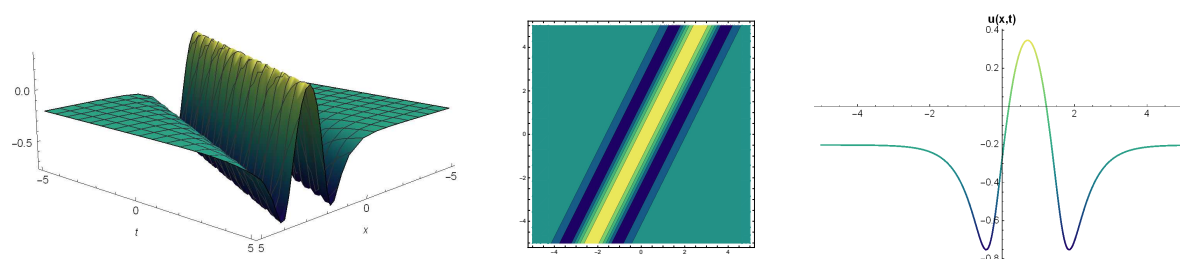


Figure 3. 3D, contour and 2D graphs respectively for $p = -0.5$, $\lambda = -1$, $c_2 = -1$, $c_1 = 5$, $q = -1$, $w = 0.5$ values of Equation (34).

Case II:

$$a_0 = -\frac{3r}{4q}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = \frac{9r}{4q\lambda}, \quad b_2 = 0, \quad \mu = 0, \quad w = -\frac{q^2}{12r\lambda}, \quad p = -\frac{2q^3}{27r^2}, \quad (35)$$

considering Equation (6), replacing Equation (35) into Equation (30), the following solution is attained

$$V = -\frac{3r}{4q} + \frac{9r \left(c_2 \sqrt{-\lambda} \cosh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] + c_1 \sqrt{-\lambda} \sinh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] \right)^2}{4q\lambda \left(c_1 \cosh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] + c_2 \sinh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] \right)^2}, \quad (36)$$

In addition, if Equation (36) is written instead of $u = \ln v$ transformation, the hyperbolic traveling wave solution of Equation (1) is as follows,

$$u_2(x, t) = \ln \left[-\frac{3r}{4q} + \frac{9r \left(c_2 \sqrt{-\lambda} \cosh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] + c_1 \sqrt{-\lambda} \sinh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] \right)^2}{4q\lambda \left(c_1 \cosh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] + c_2 \sinh \left[\left(x + \frac{q^2 t}{12r\lambda} \right) \sqrt{-\lambda} \right] \right)^2} \right]. \quad (37)$$

The hyperbolic traveling wave solution of Equation (37) produced from the $(G'/G, 1/G)$ -expansion method is as in Figure 4.

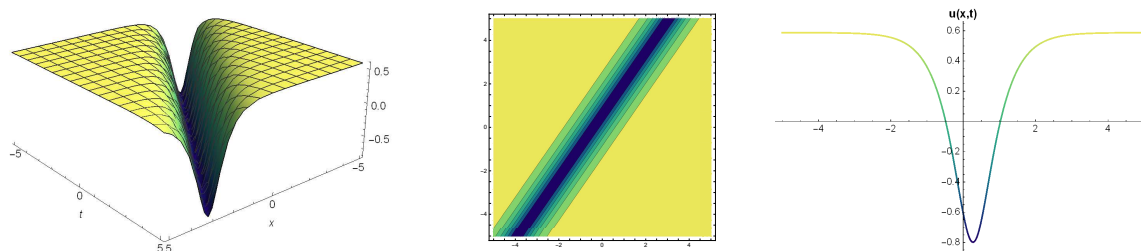


Figure 4. 3D, contour and 2D graphs respectively for $\lambda = -1$, $c_2 = 2$, $c_1 = 5$, $q = -5$, $r = 3$ values of Equation (37).

If $\lambda > 0$,

Case III:

$$a_0 = \frac{2p\lambda a_2 - \sqrt{-3pq + p^2\lambda^2 a_2^2}}{3p}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad \mu = 0, \quad w = \frac{pa_2}{2}, \quad (38)$$

$$r = \frac{2}{27} \left(2p\lambda^3 a_2^3 + \frac{3q\sqrt{p(-3q + p\lambda^2 a_2^2)}}{p} + 2\lambda^2 a_2^2 \sqrt{p(-3q + p\lambda^2 a_2^2)} \right),$$

considering Equation (6), replacing Equation (38) into Equation (30), the following solution is attained

$$V = \frac{\left(c_2 \sqrt{\lambda} \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] - c_1 \sqrt{\lambda} \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2 a_2}{\left(c_1 \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] + c_2 \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2} + \frac{2p\lambda a_2 - \sqrt{-3pq + p^2\lambda^2 a_2^2}}{3p}. \quad (39)$$

In addition, if Equation (39) is written instead of $u = \ln v$ transformation, the trigonometric traveling wave solution of Equation (1) is as follows,

$$u_3(x, t) = \ln \left[\frac{\left(c_2 \sqrt{\lambda} \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] - c_1 \sqrt{\lambda} \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2 a_2}{\left(c_1 \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] + c_2 \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2} + \frac{2p\lambda a_2 - \sqrt{-3pq + p^2\lambda^2 a_2^2}}{3p} \right]. \quad (40)$$

The trigonometric traveling wave solution of Equation (40) produced from the $(G'/G, 1/G)$ -expansion method is as in Figure 5.

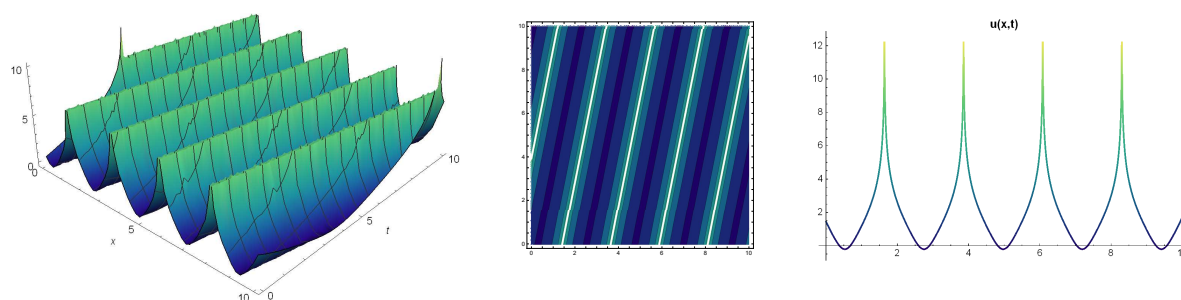


Figure 5. 3D, contour and 2D graphs respectively for $c_2 = -0.5$, $c_1 = -1$, $q = 1$, $a_0 = 0.5$, $p = -0.2$, $\lambda = 2$, $a_2 = -2$ values of Equation (40).

Case IV:

$$\begin{aligned} a_1 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad \mu = 0, \quad w = \frac{pa_2}{2}, \quad r = 2pa_0(-a_0 + \lambda a_2)^2, \\ q = -3pa_0^2 + 4p\lambda a_0 a_2 - p\lambda^2 a_2^2, \end{aligned} \quad (41)$$

considering Equation (6), replacing Equation (41) into Equation (30), the following solution is attained

$$V = a_0 - \lambda a_2 + \frac{(c_1^2 + c_2^2) \lambda a_2}{\left(c_1 \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] + c_2 \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2}. \quad (42)$$

In addition, if Equation (42) is written instead of $u = \ln v$ transformation, the analytical solution of Equation (1) is as follows,

$$u_4(x, t) = \ln \left[a_0 - \lambda a_2 + \frac{(c_1^2 + c_2^2) \lambda a_2}{\left(c_1 \cos \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] + c_2 \sin \left[\sqrt{\lambda} \left(x - \frac{1}{2} p t a_2 \right) \right] \right)^2} \right]. \quad (43)$$

The trigonometric traveling wave solution of Equation (43) produced from the $(G'/G, 1/G)$ -expansion method is as in Figure 6.

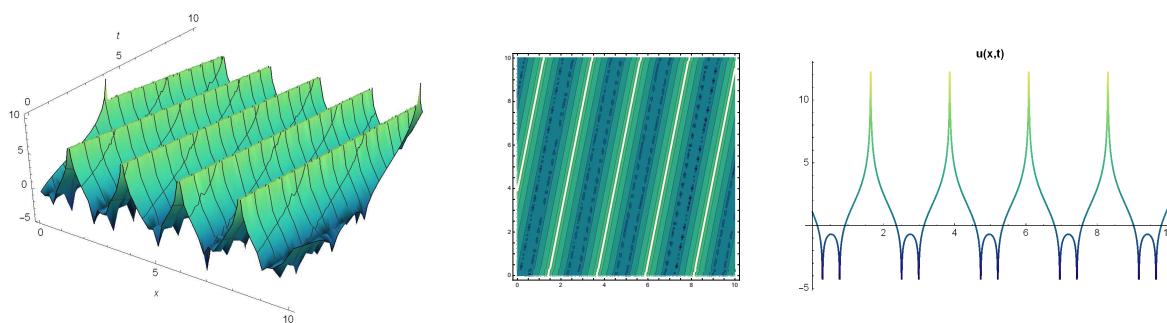


Figure 6. 3D, contour and 2D graphs respectively $c_2 = -0.5$, $c_1 = -1$, $a_0 = 0.5$, $p = -0.2$, $\lambda = 2$, $a_2 = -2$ values of Equation (43).

If $\lambda = 0$,

Case V:

$$a_0 = \frac{i\sqrt{q}}{\sqrt{3}\sqrt{p}}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = \frac{2w}{p}, \quad b_2 = 0, \quad \mu = 0, \quad r = -\frac{2iq^{3/2}}{3\sqrt{3}\sqrt{p}}, \quad (44)$$

considering Equation (6), replacing Equation (44) into Equation (30), the following solution is attained

$$V = \frac{i\sqrt{q}}{\sqrt{3}\sqrt{p}} + \frac{2c_2^2 w}{p(c_1 + c_2(-tw + x))^2}. \quad (45)$$

In addition, if Equation (45) is written instead of $u = \ln v$ transformation, the complex analytical solution of Equation (1) is as follows,

$$u_5(x, t) = \ln \left[\frac{i\sqrt{q}}{\sqrt{3}\sqrt{p}} + \frac{2c_2^2 w}{p(c_1 + c_2(-tw + x))^2} \right]. \quad (46)$$

The complex analytical solution of Equation (46) produced from the $(G'/G, 1/G)$ -expansion method is as in Figures 7 and 8.

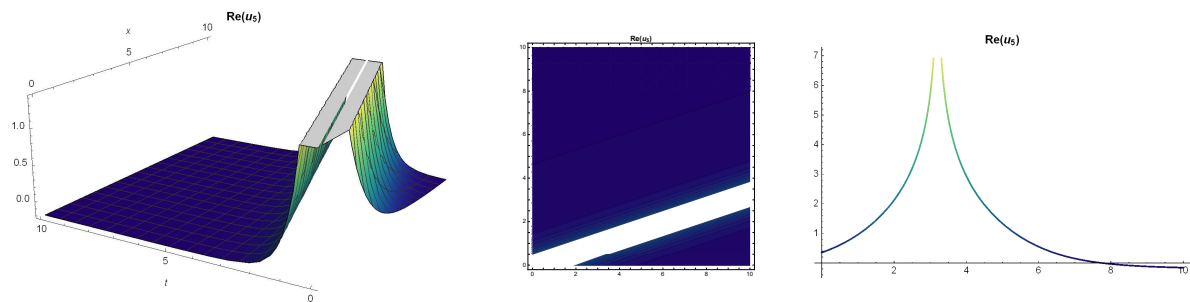


Figure 7. The real part of the 3D, contour and 2D graphics respectively for $c_2 = 5$, $c_1 = -1$, $q = -1$, $p = -0.5$, $w = 3$ values of Equation (46).

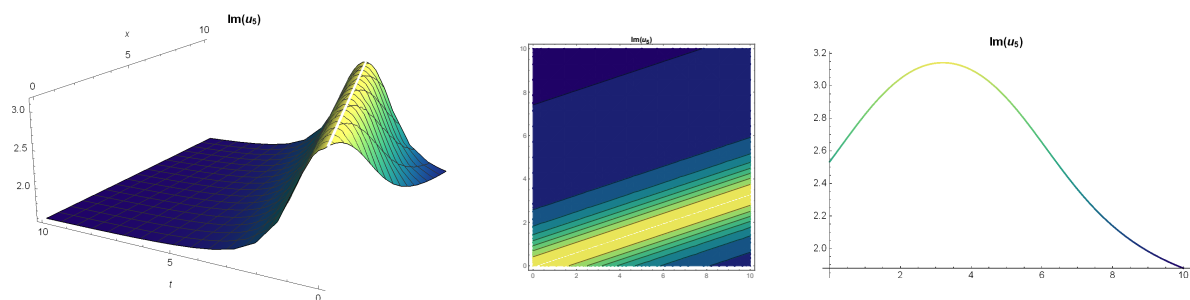


Figure 8. The imaginary part of 3D, contour and 2D graphs respectively for $c_2 = 5$, $c_1 = -1$, $q = -1$, $p = -0.5$, $w = 3$ values of Equation (46).

Case VI:

$$a_1 = 0, b_1 = 0, b_2 = 0, \mu = 0, r = 2pa_0^3, w = \frac{pa_2}{2}, q = -3pa_0^2, \quad (47)$$

considering Equation (6), replacing Equation (47) into Equation (30), the following solution is attained

$$V = a_0 + \frac{c_2^2 a_2}{\left(c_1 + c_2 \left(x - \frac{1}{2} p t a_2\right)\right)^2}. \quad (48)$$

In addition, if Equation (48) is written instead of $u = \ln v$ transformation, the analytical solution of Equation (1) is as follows,

$$u_5(x, t) = \ln \left[a_0 + \frac{c_2^2 a_2}{\left(c_1 + c_2 \left(x - \frac{1}{2} p t a_2\right)\right)^2} \right]. \quad (49)$$

The analytical solution of Equation (49) produced from the $(G'/G, 1/G)$ -expansion method is as in Figure 9.

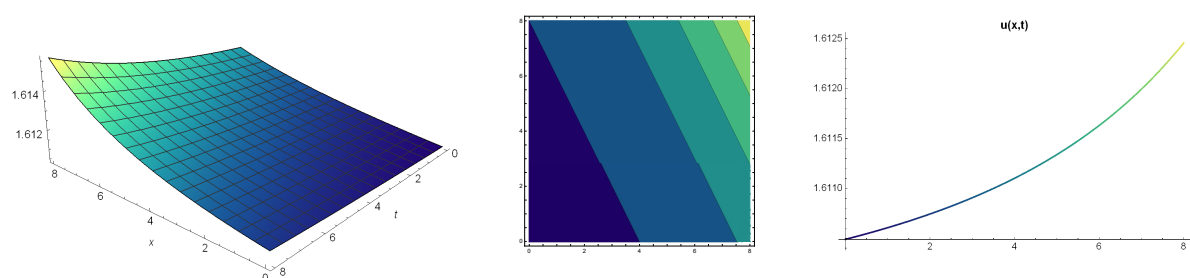


Figure 9. 3D, contour and 2D graphs respectively for $c_2 = -0.1$, $c_1 = 2$, $a_2 = 2$, $p = -0.5$, $a_0 = 5$ values of Equation (49).

6. Results and Discussion

There are various methods for obtaining the exact solution of NLEEs. Analytic solutions of the Z-S equations were successfully constructed by using both methods. When $u = \ln v$ transformation is performed in both methods, solution functions are logarithmic. This is a result of the exponential functions in the structure of the Z-S equation. In different cases of p, q, r coefficients of Z-S equation, this equation is recognized by a different name. The main purpose of this article is to present the solution of Z-S equation and also to solve the equations of the sinh–Gordon, (DBM), (TDB) and the Liouville equations. For example; The Z-S equation for $r = q = 0$ is called the Liouville equation. In this study, if the values $r = q = 0$ in Equation (24) are written, the traveling wave solution of the Liouville equation is obtained as

$$u(x, y) = \ln \left[\frac{a_0 + \frac{a_1^2}{6a_0 \left(A \cosh \left[\lambda \left(x - \frac{3pta_0}{\lambda^2} \right) \right] - A \sinh \left[\lambda \left(x - \frac{3pta_0}{\lambda^2} \right) \right] - \frac{a_1}{6a_0} \right)^2}}{+ \frac{a_1}{A \cosh \left[\lambda \left(x - \frac{3pta_0}{\lambda^2} \right) \right] - A \sinh \left[\lambda \left(x - \frac{3pta_0}{\lambda^2} \right) \right] - \frac{a_1}{6a_0}}} \right], \quad (50)$$

Similarly, the solutions of the sinh–Gordon, DBM, TDB and Liouville equations can be obtained by using both the methods. The V solution described above is presented in hyperbolic form in $(1/G')$ -expansion method, and in hyperbolic, trigonometric and rational forms in $(G'/G, 1/G)$ -expansion method. In this case, $(G'/G, 1/G)$ -expansion method is advantageous in terms of solution. However, in $(G'/G, 1/G)$ -expansion method, the process complexity is higher. This can be observed in the system Equation (31), $(1/G')$ -expansion method is more advantageous in terms of process. 2D, 3D and contour graphics which we consider will help in traveling wave solutions which have considerable importance in applied sciences, are presented. In order to draw these graphs, real values are given to arbitrary constants in the analytical solution.

This problem contains the properties of many equations. For the different states of the coefficients, to offer the solution of the equation which includes equations with different names, also to offer the solution of the subclass equations. It is known that each equation has different meanings. For example, with the interaction of solitons produced by sinh–Gordon equation, kink and antikink solutions came to the fore. We can make the same comments for the solutions obtained in these studies for. In this case, the equation we dealt with is the umbrella task. It makes the wave solutions valuable because it will carry the properties of the equations under the umbrella. The most important factor that stands out in this study is to take a different solutions from the solution in classical $(G'/G, 1/G)$ -expansion method. This results in obtaining different types of traveling wave solutions from the classical method. It also creates a basis for a new study. This is an improved method that can produce different solutions by adding the solution we offer with the Equation (17) to the classical $(G'/G, 1/G)$ solution. Because the Equation (17) presented and the equilibrium term 2 in the equation discussed are different from the solutions offered in the classical $(G'/G, 1/G)$ -expansion method. Different types of solutions were obtained with both methods and both methods can be used as important instruments to get traveling wave solution for many different NLEEs.

7. Conclusions

In this article, we have applied the $(1/G')$ -expansion and $(G'/G, 1/G)$ -expansion methods to derive analytic solutions for the Zhiber-Shabat equation. The solutions obtained are complex rational, hyperbolic, rational and trigonometric type traveling wave solutions. The 2D, 3D and contour graphics of these solutions were presented by giving value to arbitrary parameters. These graphs represent the stationary wave at any given moment. As it is very difficult to obtain the solutions of NLEEs, in this study traveling wave solutions of Zhiber-Shabat equation are presented applying two complex methods using many complex operations and transformations. These are very effective and powerful

methods for obtaining analytical solutions and can be used to obtain solutions of many mathematical models representing physical phenomena. The accuracy of the attained solutions has been assured by putting them back into the original equations with the help of the computer package program.

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