mathematics

Article

Fixed Point Theorem Based Solvability of 2-Dimensional Dissipative Cubic Nonlinear Klein-Gordon Equation

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Received: 17 April 2020; Accepted: 25 May 2020; Published: 5 July 2020

Abstract: The purpose of this article is to establish the solvability of the 2-Dimensional dissipative cubic nonlinear Klein-Gordon equation (2DDCNLKGE) through periodic boundary value conditions (PBVCs). The analysis of this study is founded on the Galerkin's method (GLK) and the Leray-Schauder's fixed point theorem (LS). First, the GLK method is used to construct some uniform priori estimates of approximate solution to the corresponding equation of 2DDCNLKGE. Finally, the LS fixed point theorem is applied to obtain the efficient and straightforward existence and uniqueness criteria of time periodic solution to the 2DDCNLKGE.

Keywords: 2-Dimensional dissipative cubic nonlinear Klein-Gordon equation; periodic solution; GLK method; LS fixed point theorem

MSC: 34A08; 34B10; 34B15

1. Introduction

The nonlinear Klein-Gordon equation (NLKGE for short) has been obtained by a modification of nonlinear Schrödinger equation $i\partial_t\psi = -\frac{1}{2}\partial_x{}^2\psi + k\left|\psi\right|^2\psi$, where $\psi(x,t)$ is a complex field. This equation has extensively been used for modeling of various nonlinear physical and environmental phenomena; see for instance [1–6] and their cited references.

As vital nonlinear partial differential equations (NLPDEs), the NLKG types equations have received great consideration for developing solutions by applying various types of techniques; see for instance [3,4,7,8] and their cited references.

Certain nonlinear physical systems expressed with NLPDEs may be transformed into nonlinear ordinary differential equations by using traveling wave transformations, and the travelling wave solutions of these NLPDEs is analogous to the exact solutions of corresponding nonlinear ordinary differential equations. The 2-Dimension dissipative NLKGE is a practical instance of the above mentioned nonlinear physical system.

Throughout this paper, R^+ denotes a set of positive real numbers.

The general form of a 2-Dimension dissipative NLKGE is:

$$u_{tt} - \Delta u + \alpha u_t + \beta u + \gamma|u|^k u = f(x,t),\ x \in \Omega,\ t \in R^+ \tag{1}$$

where u is a real valued, $\Omega = [0,\ L] \times [0,\ L]$, $\alpha,\ \beta,\ \gamma$ are real physical constants, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and k is a positive integer, which is used to measure the nonlinearity of 2D dissipative NLKGE. Here, it is also mentioned that α and β may be considered as continuous functions.

For $k = 2$, Equation (1) reduces to the following 2D dissipative quadratic NLKGE:

$$u_{tt} - \Delta u + \alpha u_t + \beta u + \gamma |u|^2 u = f(x,t),\ x \in \Omega,\ t \in R^+ \tag{2}$$

For $k = 3$ Equation (1), reduces to the following 2DDCNLKGE:

$$u_{tt} - \Delta u + \alpha u_t + \beta u + \gamma |u|^3 u = f(x,t),\ x \in \Omega,\ t \in R^+ \tag{3}$$

Equation (3) is used to explain relativistic quantum mechanics; see for instance [9].

In 2004, Gao and Guo [10] established solvability of the time-periodic solution of a 2D dissipative quadratic NLKG equation given by (2) with time periodic boundary value conditions using the GLK method [11,12] and the LS fixed point theorem [13]. There exists a wide range of solvability for Equations (2) and (3), in case of $\alpha = 0$ and $f = 0$; see for instance [1,14–16] and their cited references. After Gao and Guo [10], in 2006, Fu and Guo [17] established the time-periodic solution of the following one-dimensional viscous Camassa-Holm equation:

$$u_t - u_{xxt} - \mu(u_{xx} - u_{xxxx}) + 3uu_x = 2u_x u_{xx} + uu_{xxx} + f(x,\ t),\ x \in \Omega,\ t \in R^+, \mu > 0 \tag{4}$$

applying the GLK method and the LS fixed point theorem. Sequentially, in 2014, Gao et al. [16] proved the uniqueness of the time-periodic solution to 1D quadratic viscous modified Camassa-Holm equation:

$$u_t - u_{xxt} - \mu(u_{xx} - u_{xxxx}) + 3u^2 u_x - 2u_x u_{xx} - uu_{xxx} = f(x,\ t),\ x \in \Omega,\ t \in R^+, \mu > 0 \tag{5}$$

by means of the GLK method and the LS fixed point theorem.

In the last few decades, many researchers have devoted themselves to establishing the time-periodic solution for various nonlinear evolution equations; see for instance [10,11,17–20] and their cited references. Recently, Obinwanne and Collins [21] applied the LS fixed point theorem to obtain a solution of Duffing's equation. Moreover, there is a certain focus on the uniqueness of the time-periodic solution of 2DDCNLKGE given by Equation (5), applying the GLK method and the LS fixed point theorem. Inspired by the above-mentioned works in this paper, we establish a solvability for the following 2DDCNLKGE with PBVCs applying the GLK method and the LS fixed point theorem:

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t + \beta u + \gamma |u|^3 u = f(x,t),\ x \in \Omega, \\ u(x+L,t) = u(x,t),\ x \in \Omega, \\ f(x,t+\omega) = f(x,t), \end{cases} \tag{6}$$

where $t \in R^+,\ u = u(x,t)$ is a real value, $\Omega = [0,L] \times [0,L],\ \alpha,\ \beta,\ \gamma$ are real physical constants and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

The outline of this article is as follows: The present section provides an introduction to this article. In Section 2, we provide some notations, the GLK method, and the LS fixed point theorem. Section 3 is used to formulate uniform priori estimates of the approximate solution of 2DDCNLKGE given in Equation (6), which will be applied in the next section. Section 4 is devoted to establishing a unique time-periodic solution criterion for 2DDCNLKGE given in Equation (6). Finally, we provide a conclusion.

2. Preliminary Notes

Here, we provide some introductory truths that are needed to describe the main results of this article.

Let B be a Banach space. For $1 \leq p \leq \infty$, the space $L^p(B;\ \omega)$ is defined as the set of ω-periodic B-measurable functions on $\Re$ (set of real numbers), such that:

$$\|u\|_{L^p(B;\ \omega)} = \begin{cases} \left(\int_0^\omega \|u\|_B^p ds\right)^{1/p} < \infty, & 1 \leq p < \infty \\ \sup\limits_{0 \leq t \leq \omega} \|u\|_B < \infty, & p = \infty. \end{cases}$$

The space $W^{k,\ p}(B;\ \omega)$ denote the set of functions that belong to $L^p(B;\ \omega)$ together with their derivatives up to order k; if B is a Hilbert space, we write $W^{k,\ 2}(B;\ \omega) = H^k(B;\ \omega)$.

During this study, we use these notations:

$$L^2_{per} = \left\{u \in L^2(\Omega) : u(x+L,t) = u(x,t)\right\},$$
$$H^k_{per} = \left\{u \in H^k : u(x+L,t) = u(x,t)\right\},\ k = 1,\ 2,$$

where $L^2(\Omega)$ is obtained from $L^p(B;\ \omega)$ by putting $p = 2$ and $H^k = H^k(B;\ \omega)$.

And

$$C^k(\omega,\ X) = \left\{f : [0,\ \infty) \rightarrow X : f^{(i)} \text{ is continuous},\ i = 0,\ 1,\ \cdots,\ k,\ f \text{ is an } \omega - \text{periodic function}\right\}.$$

where X may be a real or complex space.

For $k = 0$, we replace $C^0(\omega,\ X)$ with $C(\omega,\ X)$. The inner product and norm of $L^2(\Omega)$ are denoted by $(.,\ .)$ and $\|\ .\ \|$, respectively. We also denote that:

$$T = -\Delta,\ D(T) = H^2_{per},\ N(u) = -\gamma|u|^3 u, \text{ and } N$$

is a cubic nonlinear operator on $L^2(\Omega)$.

Now, we state the LS fixed point theorem, which will be used as the main tools of this study.

Theorem 1 [13]. *Let B be a Banach space and $T : B \rightarrow B$ be a continuous and compact mapping with property "there exists $R > 0$ such that the statement ($u = rTu$ with $r \in [0,\ 1)$) implies $\|u\|_B < R$". Then T has a fixed point u^* such that $\|u^*\| \leq R$.*

We now provide a brief discussion on the GLK method [11,12].

The GLK method is a strong and general method. Here, we introduce the GLK method with a nonconcrete problem modelled as a frail design on a Hilbert space H, V {\displaystyle V} specifically searching for $x \in H$:

$$y \in H, \text{ and } b(x,\ y) = h(y)$$

where $b(.,\ .)$ is bilinear and $h(y)$ is a bounded linear functional on H.

Select a n dimension subspace H_n of the Hilbert space H to solve the following problem: search $x_n \in H_n$ from:

$$b(x_n,\ y_n) = h(y_n),\ \text{for all } y_n \in H_n \tag{7}$$

Equation (7) is known as the GLK formula. The main theme of the GLK method is that the mistake is orthogonal to the preferred sub-spaces, since $H_n \subset H$ V n ⊂ V {\displaystyle V_{n}\subset V}, y_n is used v n {\displaystyle v_{n}} as a trial vector in the main problem. If the mistake between the solution of the main problem x u {\displaystyle u} and the solution of the GLK formula x_n is $m_n = x - x_n$, ϵ n = u − u n {\displaystyle \epsilon _{n} = u − u_{n}}thenthethen we have:

$$b(m_n,\ y_n) = b(x,\ y_n) - b(x_n,\ y_n) = h(y_n) - h(y_n) = 0 \tag{8}$$

In the GLK method, we can represent the problem in matrix form and calculate the solution algorithmically. Regarding this matrix representation, if we consider $e_1,\ e_2,\ e_3,\ \cdots,\ e_n$ as a basis of H_n, then from Equation (7), we can obtain $x_n \in H_n$ from $b(x_n,\ e_i) = h(e_i),\ i = 1,\ 2,\ 3,\ \cdots,\ n$.

Now, if we enlarge x_n according to this basis, we get $x_n = \sum\limits_{j=1}^{n} x_j e_j$ and hence obtain

$$b\left(\sum_{j=1}^{n} x_j e_j,\ e_i\right) = \sum_{j=1}^{n} x_j b\left(e_j,\ e_i\right) = h(e_i),\ i = 1,\ 2,\ 3,\ \cdots,\ n. \tag{9}$$

Equation (9) represents a system of equations given by $C_{ij} x_j = h_i$, where the coefficient matrix C_{ij} is given by $\sum\limits_{j=1}^{n} b\left(e_j,\ e_i\right)$ and $h_i = h(e_i)$.

3. Existence of Uniform Priori Estimates for the Solution of 2DDCNLKGE

In this section, applying the GLK method and Theorem 1, we formulate uniform priori estimates for an approximate solution to the 2DDCNLKGE.

In space $C\left(\omega,\ L^2_{per}\right)$, we write the problem given in Equation (6) as the following abstract problem:

$$u_{tt} + \alpha u_t + \beta u + Tu = N(u) + f,\ u(.,\ t) = u(.,\ t+\omega),\ \forall\ f \in C^1\left(\omega,\ H^1_{per}\right). \tag{10}$$

Now, we obtain an approximate solution of 2DDCNLKGE given in Equation (6) using the GLK method. Let $\left\{\omega_j\right\}_{j=0}^{\infty}$ be a normal orthogonal basis of the space L^2_{per} and satisfy $T\omega_j = \lambda_j \omega_j,\ (\ j = 1,\ 2,\ \cdots)$, where λ_j are eigenvalues for the map T and the eigenvectors $\omega_j,\ (j = 1,\ 2,\ \cdots)$. We denote $H_n = \text{span}\{\omega_1,\ \omega_2,\ \cdots,\ \omega_n\},\ \forall\ n \in \aleph$ (set of natural numbers).

Now, by the GLK method, for any $n \in \aleph$ and any sequence of functions $\left\{a_{jn}(t)\right\}_{j=1}^{n}$, where $a_{jn}(t) \in C^2(\ \omega,\ \Re),\ (j = 1, 2,\ 3,\ \cdots,\ n)$ and $\Re$ denotes the set of real numbers, we can say that the function $u_n = \sum\limits_{j=1}^{n} a_{jn}(t)\omega_j \in C^2(\ \omega,\ H_n)$ is an approximate solution of Equation (10), if the following system holds:

$$\left(u_{ntt} + \alpha u_{nt} + \beta u_n + Tu_n,\ \omega_j\right) = \left(N(u_n) + f,\ \omega_j\right),\ j = 1,\ 2,\ 3,\ \cdots,\ n, \tag{11}$$

where,

$$N(u_n) = -\gamma |u_n|^3 u_n \text{ and } H_n = span\{\omega_1,\ \omega_2,\ \omega_3,\ \cdots,\ \omega_n\}.$$

In order to demonstrate that Equation (10) has an approximate solution, we use Theorem 1. A solution $u(x,t)$ of Equation (10) is said to be unique if it has a fixed value $u_1(x,t)$, which satisfies Equation (10) uniquely, that is the solution $u(x,t)$ has no any value that is not equal to $u_1(x,t)$ and this solution will be ω-periodic if $u(x,t) = u(x, t+\omega)$.

Now, from the classical viewpoint of ordinary differential equations, it is clear that for any fixed $v_n(t) = \sum\limits_{j=1}^{n} b_{jn}(t)\omega_j \in C^2(\ \omega,\ H_n)$, the following linear ordinary equation system

$$\left(u_{ntt} + \alpha u_{nt} + \beta u_n + Tu_n,\ \omega_j\right) = \left(\mu N(v_n) + f,\ \omega_j\right),\ 0 \le \mu \le 1,\ j = 1,\ 2,\ 3,\ \cdots,\ n, \tag{12}$$

offers a unique ω-periodic solution $a_{jn}(t)$ and the map $F_\mu : v_n(t) \to u_n(t)$ is continuous and compact on $C^2(\omega,\ H_n)$. Furthermore, the map F_μ is completely continuous and hence uniform for $0 \le \mu \le 1$. Clearly for $\mu = 0$, the linear ordinary equation system given by Equation (12) has a unique solution. Therefore, to prove the existence of the time periodic solution of Equation (12) by applying Theorem 2, it is enough to show that the inequality

$$\sup_{0 \le t \le \omega} \|u_{ntt}(t)\| \le c \tag{13}$$

holds for all possible solutions of Equation (12), and the nonlinear term $N(u_n)$ is replaced by $\mu N(u_n)$, $(0 \le \mu \le 1)$, and c is a constant function of $\alpha,\ \beta,\ \gamma,\ \omega, f$ and L.

Now, we establish some lemmas that convey the required uniform priori estimators for the time periodic solution of Equation (11).

Lemma 1. *If* $f \in C^1\left(\omega,\ H^1_{per}\right)$, then

$$\|u_{nt}(t)\|^2 + \|\nabla u_n(t)\|^2 + \|u_n(t)\|^2 + \|u_n(t)\|_5^5 \le d_1$$

where d_1 is a positive constant function of $\alpha,\ \beta,\ \gamma,\ \omega, f$ and L .

Proof. After multiplication by $a'_{jn}(t)$ and taking sum over j from 1 to n on both sides of Equation (12), we get:

$$\begin{array}{c} \frac{1}{2}\frac{d}{dt}\left(\|u_{nt}\|^2 + \|\nabla u_n\|^2 + \beta\|u_n\|^2\right) + \alpha\|u_{nt}\|^2 + \frac{\mu\gamma}{5}\frac{d}{dt}\int_\Omega \left|u_n(t)\right|^5 dx \\ = \int_\Omega f\ u_{nt} dx \le \frac{\alpha}{2}\|u_{nt}\|^2 + \frac{2}{\alpha}\|f\|, \end{array} \tag{14}$$

That is

$$\frac{d}{dt}\left(\|u_{nt}\|^2 + \|\nabla u_n\|^2 + \beta\|u_n\|^2 + \frac{2\mu\gamma}{5}\|u_n\|_5^5\right) + \alpha\|u_{nt}\|^2 \le \frac{4}{\alpha}\|f\|. \tag{15}$$

Now, after multiplication by $a_{jn}(t)$ and taking sum over j from 1 to n on both sides of Equation (12), we yield:

$$(u_{ntt} + \alpha u_{nt} + \beta u_n + T u_n,\ u_n) = (\mu N(u_n) + f,\ u_n),$$

This implies that

$$\frac{d}{dt}\alpha\|u_n\|^2 + 2\|\nabla u_n\|^2 + \beta\|u_n\|^2 + 2\int_\Omega u_{ntt} u_n dx + \frac{5\mu\gamma}{2}\|u_n\|_5^5 \le \frac{4}{\beta}\|f\|. \tag{16}$$

Multiplying both sides of Equation (16) by δ and adding Equation (15), we have:

$$\begin{array}{c} \frac{d}{dt}\left(\|u_{nt}\|^2 + \|\nabla u_n\|^2 + (\beta + \delta\alpha)\|u_n\|^2 + \frac{2\mu\gamma}{5}\|u_n\|_5^5\right) + \alpha\|u_{nt}\|^2 + 2\delta\|\nabla u_n\|^2 \\ + \delta\beta\|u_n\|^2 + \delta\mu\gamma\|u_n\|_5^5 + 2\delta\int_\Omega u_{ntt}(t) u_n dx \le 4\left(\frac{1}{\alpha} + \frac{\delta}{\beta}\right)\|f\| = c_1 \end{array} \tag{17}$$

Integrating inequality (17) over the closed interval $[0,\ \omega]$, we get:

$$\int_0^\omega \left[(\alpha - 2\delta)\|u_{nt}\|^2 + 2\delta\|\nabla u_n\|^2 + \delta\beta\|u_n\|^2 + \frac{5\delta\mu\gamma}{2}\|u_n\|_5^5\right] dt \le \omega c_1.$$

Now, if we take $\delta < \alpha/2$, then for $t^* \in (0,\ \omega)$, $b > 0$, we have:

$$\|u_{nt}(t^*)\|^2 + \|\nabla u_n(t^*)\|^2 + \|u_n(t^*)\|^2 + \|u_n(t^*)\|_5^5 \le \frac{c_1}{b}.$$

Again, integrating inequality (17) from t^* to $t \in (t^*,\ t^* + \omega)$ and for any $L > 0$, we have:

$$\|u_{nt}(t)\|^2 + \|\nabla u_n(t)\|^2 + \|u_n(t)\|^2 + \|u_n(t)\|_5^5 \le 2c_1\omega + (c_1 L/b) = d_1$$

Therefore, we deduce that:

$$\sup_{0\le t\le\omega}\left(\|u_{nt}(t)\|^2+\|\nabla u_n(t)\|^2+\|u_n(t)\|^2+\|u_n(t)\|_5^5\right)\le d_1. \tag{18}$$

Which finishes the proof. □

Remark 1. *From inequality (18), we can obtain the estimate* $\sup_{0\le t\le\omega}\|u_{ntt}(t)\|\le d$. *Hence, LS fixed point Theorem 2 and Lemma 1 offers the following result:*

"If $f\in C\left(\omega,\ H^1_{per}\right)$*, then for any positive integer n Equation (10) has an approximate solution* $(u_n(t),\ u'_n(t))\in C^2(\omega,\ H_n)\times C^1(\omega,\ H_n)$ *".*

From the above established results, it is clear that $\{u_n\}_{n=1}^{\infty}$ is the sequence of an approximate solution of Equation (10). Now we have to prove that the sequence $\{u_n\}_{n=1}^{\infty}$ is convergent and that the converging point is a solution of Equation (10) and to fulfill the requirement, we have to establish a priori estimates for u_n.

Lemma 2. *If* $f\in C\left(\omega,\ H^1_{per}\right)$*, then there exists a positive constant* $d_2=d_2(\alpha,\ \beta,\ \gamma,\ \omega,\ L,\ f)$ such that:

$$\sup_{0\le t\le\omega}\left(\|\nabla u_{nt}(t)\|^2+\|\Delta u_n(t)\|^2+\|\nabla u_n(t)\|^2\right)\le d_2$$

Proof. The following inequalities, which are obtained from Ladyzhenskaya's inequality [22–25], will be needed to prove this lemma:

$$\begin{cases}\|u_n\|_4\le c\|\nabla u_n\|^{1/2}\|u_n\|^{2/2}, & u_n\in H^1(\Omega),\\ \|u_n\|_8\le c\|\Delta u_n\|^{1/6}\|u_n\|^{5/6}, & u_n\in H^2(\Omega).\end{cases} \tag{19}$$

Multiplying both sides of Equation (12) by $-\lambda_j a_{jn}(t)$ and taking the sum over j from 1 to n, we have $(u_{ntt}+Tu_n+\alpha u_{nt}+\beta u_m,\ -\Delta\ u_n)=(\mu N(u_n)+f,\ -\Delta u_n)$, and hence applying inequalities of (19), Hölder's inequality, and Young's inequality, we yield the following:

$$\begin{gathered}\tfrac{1}{2}\tfrac{d}{dt}\alpha\|\nabla u_n\|^2+\|\Delta u_n\|^2+\beta\|\nabla u_n\|^2+\int_\Omega\nabla u_n\nabla u_{ntt}(t)dx\\ \le\mu\gamma d_2\|\Delta u_n\|\|u_n\|_4\|u_n\|_8^3+\|f\|\|\Delta u_n\|\\ \le\mu\gamma d_2c^2\|\Delta u_n\|\|\nabla u_n\|^{1/2}\|u_n\|^{1/2}\|\Delta u_n\|^{1/2}\|u_n\|^{5/2}+\|f\|\|\Delta u_n\|\\ \le\mu\gamma e d_3\|\Delta u_n\|^{3/2}+\|f\|\|\Delta u_n\|\\ \le 2\varepsilon\|\Delta u_n\|^2+e(\varepsilon,\ \gamma,\ c,\ d_3)+\tfrac{\|f\|^2}{\varepsilon}.\end{gathered}$$

Taking ε to be small enough, we get:

$$\frac{1}{2}\frac{d}{dt}\alpha\|\nabla u_n\|^2+\frac{3}{4}\|\Delta u_n\|^2+\beta\|\nabla u_n\|^2+\int_\Omega\nabla u_n\nabla u_{ntt}(t)dx\le d_4. \tag{20}$$

Multiplying both sides of Equation (12) by $-\lambda_j a'_{jn}(t)$ and taking the sum over j from 1 to n, we obtain $(u_{ntt}+Tu_n+\alpha u_{nt}+\beta u_m,\ -\Delta\ u_{nt})=(\mu N(u_n)+f,\ -\Delta u_{nt})$, and hence applying inequalities of (19), Hölder's inequality, and Young's inequality, we get the following:

$$\begin{gathered}\tfrac{1}{2}\tfrac{d}{dt}\Big[\beta\|\nabla u_n\|^2+\|\Delta u_n\|^2+\|\nabla u_{nt}\|^2\Big]+\alpha\|\nabla u_{nt}\|^2\\
=-\mu\gamma\mathrm{Re}\textstyle\int_\Omega |u_n|^3\nabla u_n.\nabla u_{nt}dx+\int_\Omega \nabla f.\nabla u_{nt}dx,\quad (0\le\mu\le 1)\\
\le e\gamma\|\nabla u_{nt}\|\|u_n\|_8^3\|\nabla u_n\|_4+\|\nabla f\|\|\nabla u_{nt}\|\\
\le ed_5c^2\|\nabla u_{nt}\|\|\Delta u_n\|^{1/2}\|u_n\|^{5/2}\|\nabla u_n\|^{1/2}\|\Delta u_n\|^{1/2}+\|\nabla f\|\|\nabla u_{nt}\|\\
\le \varepsilon\|\nabla u_{nt}\|^2+e(\varepsilon,\ \gamma,\ c,\ d_5)\|\Delta u_n\|+\varepsilon\|\nabla u_{nt}\|^2+e\|\nabla f\|^2.\end{gathered}$$

Taking ε to be small enough, we get:

$$\frac{1}{2}\frac{d}{dt}\Big[\beta\|\nabla u_n\|^2+\|\Delta u_n\|^2+\|\nabla u_{ntt}\|^2\Big]+\frac{3\alpha}{4}\|\nabla u_{nt}\|^2\le e\|\Delta u_n\|+e\|\nabla f\|^2. \tag{21}$$

Multiplying both sides of (20) by δ and by adding with (21), we have:

$$\begin{gathered}\tfrac{1}{2}\tfrac{d}{dt}\Big[\|\nabla u_{nt}\|^2+\|\Delta u_n\|^2+(\beta+\alpha\delta)\|\nabla u_n\|^2\Big]+\tfrac{3\alpha}{4}\|\nabla u_{nt}\|^2+\tfrac{3\delta}{4}\|\Delta u_n\|^2\\
+\beta\delta\|\nabla u_n\|^2+\delta\textstyle\int_\Omega \nabla u_n\nabla u_{ntt}(t)dx\\
\le \delta d_4+e\|\Delta u_n\|+e\|\nabla f\|^2\le d_5+\tfrac{1}{4}\delta\|\Delta u_n\|^2+d_6,\end{gathered} \tag{22}$$

where, d_5 and d_6 are constants.

Integrating both sides of (22) with respect to t from 0 to ω, we get:

$$\int_0^\omega \Big[(3\alpha-4\delta)\|\nabla u_{nt}\|^2+2\delta\|\Delta u_n\|^2+4\delta\beta\|\nabla u_n\|^2\Big]dt\le\int_0^\omega 4(d_5+d_6)dt=d_7\omega.$$

Now, for $0<\delta<\frac{\alpha}{4}$ and $t^{**}\in(0,\ \omega)$, we obtain:

$$\|\nabla u_{nt}(t^{**})\|^2+\|\Delta u_n(t^{**})\|^2+\|\nabla u_n(t^{**})\|^2\le d_8.$$

Again, integrating both sides of (22) from t^{**} to $t\in(t^{**},\ t^{**}+\omega)$, we have:

$$\|\nabla u_{nt}(t)\|^2+\|\Delta u_n(t)\|^2+\|\nabla u_n(t)\|^2\le d_9.$$

Thus, for a constant $d_2=d_2(\alpha,\ \beta,\ \gamma,\ \omega,\ f,\ L)$, we have:

$$\sup_{0\le t\le\omega}\Big(\|\nabla u_{nt}(t)\|^2+\|\Delta u_n(t)\|^2+\|\nabla u_n(t)\|^2\Big)\le d_2$$

which finishes the proof. □

The next lemma will establish the priori estimates of a higher order for u_n.

Lemma 3. If $f\in C^1\Big(\ \omega,\ H^1_{per}\Big)$ and $d_{10}=d_{10}(\alpha,\ \beta,\ \gamma,\ \omega,\ L,\ f)$ is positive constant, then

$$\sup_{0\le t\le\omega}\|u_{ntt}(t)\|\le d_{10}$$

Proof. Differentiating Equation (11), we get:

$$\Big(u_{nttt}+\alpha u_{ntt}+\beta u_{nt}-\Delta u_{nt},\ \omega_j\Big)=\Big(-4\gamma|u_n|^3u_{nt}+f',\ \omega_j\Big),\ j=1,\ 2,\ 3,\ \cdots,\ n. \tag{23}$$

After multiplying both sides of (23) with $2a''_{jn}(t)$ and taking the sum over j from 1 to n, we get:

$$\begin{gathered}\frac{d}{dt}\left(\|u_{ntt}\|^2+\beta\|u_{nt}\|^2+\|\nabla u_{nt}\|^2+\|\nabla u_{nt}\|^2\right)+2\alpha\|u_{ntt}\|^2 \\ \leq 8\gamma\|u_{ntt}\|\|u_{nt}\|_4\|u_n\|_8^3+2\|u_{ntt}\|\|f'\|\leq\alpha\|u_{ntt}\|^2+\frac{1}{\alpha}\left[8\|f'\|^2+8\gamma\left(\|u_{nt}\|_4\|u_n\|_8^3\right)^2\right].\end{gathered}$$

Using inequality (19) and Lemma 2, we have:

$$\frac{d}{dt}\left(\|u_{ntt}\|^2+\beta\|u_{nt}\|^2+\|\nabla u_{nt}\|^2+\|\nabla u_{nt}\|^2\right)+2\alpha\|u_{ntt}\|^2\leq d_{11}. \tag{24}$$

Again multiplying both sides of (23) with $2a'_{jn}(t)$ and taking the sum over j from 1 to n and by lemma 2, we get:

$$\begin{gathered}\frac{d}{dt}\alpha\|u_{nt}\|^2+2\|\nabla u_{nt}\|^2+2\beta\|u_{nt}\|^2+2\int_\Omega u_{nttt}u_{nt}dx \\ \leq 8\gamma\int_\Omega|u_n|^3u_{nt}^2dx+2\int_\Omega f'\ u_{nt}dx\leq 8\gamma\|u_n\|_4^3\|u_{nt}\|_4^2+2\|u_{nt}\|\|f'\|=d_{12}\end{gathered} \tag{25}$$

Multiplying both sides of (25) by δ and on adding with (24), we have:

$$\begin{gathered}\frac{d}{dt}\left(\|u_{nttt}\|^2+\|\nabla u_{nt}\|^2+(\beta+\alpha\delta)\|u_{nt}\|^2\right)+2\alpha\|u_{ntt}\|^2+2\delta\|\nabla u_{nt}\|^2 \\ +2\beta\delta\|u_{nt}\|^2+2\delta\int_\Omega u_{nttt}u_{nt}dx\leq d_{11}+\delta d_{12}\doteq d_{13}.\end{gathered} \tag{26}$$

Integrating both sides of (26) from 0 to ω, we get:

$$\int_0^\omega\left[2(\alpha-\delta)\|u_{ntt}\|^2+2\delta\|\nabla u_{nt}\|^2+2\delta\beta\|u_{nt}\|^2\right]dt\leq d_{14}\omega.$$

Now, for $0<\delta<\frac{\alpha}{2}$ and $t^{***}\in(0,\ \omega)$, we have:

$$\|u_{ntt}(t^{***})\|^2+\|\nabla u_{nt}(t^{***})\|^2+\|\nabla u_{nt}(t^{***})\|^2\leq\frac{d_{15}}{n}.$$

Again, integrating Inequality (26) from t^{***} to $t\in[t^{***},\ t^{***}+\omega]$, we have:

$$\|u_{ntt}(t)\|^2+\|\nabla u_{nt}(t)\|^2+\|u_{nt}(t)\|^2\leq d_{16}. \tag{27}$$

Thus, for a constant $d_{10}=d_{10}(\alpha,\ \beta,\ \gamma,\ \omega,\ f,\ L)$, we obtain:

$$\sup_{0\leq t\leq\omega}\|u_{ntt}(t)\|^2\leq d_{10}.$$

This completes the proof. □

4. Solvability of Periodic Solution to the 2DDCNLKGE

In this section, we establish the existence and uniqueness of time periodic solutions to 2DDCNLKGE given in Equation (6).

The next theorem leads the existence criteria of a time periodic solution for 2DDCNLKGE given in Equation (6).

Theorem 2. For any $f\in C^1\left(\omega,\ H^1_{per}\right)$, the time periodic solution $(u(x,\ t),\ u'(x,\ t))$ of 2DDCNLKGE given in Equation (6), is expressed in the following way:

$$u(x,t)\in C^2\left(\omega,\ H^2_{per}\right),\ u'(x,t)\in L^2_{per}$$

Proof. For all positive integers, we have proven that Equation (6) has an approximate solution $u_n(t)$, i.e., the system given by Equation (11) holds and we have estimates of the norm of $u_n(t)$. It is possible to consider a subsequence $\left\{\left(u_{n_k}(t),\ u'_{n_k}(t)\ \right)\right\}$ converging weakly to $(u(t),\ u'(t)) \in H^1_{per} \times L^2_{per}$, for fixed t and uniform boundedness norms $\|u_n(t)\|_{H^1_{per}}$ and $\|u'_n(t)\|^2_{per}$. We have to prove that $(u(t),\ u'(t))$ is a solution of the 2DDCNLKGE given in Equation (6). In fact, by weak convergence of $\left\{u_{n_k}(t)\right\}$ and $\left\{u'_{n_k}(t)\right\}$ to $u(t)$ and $u'(t)$ in spaces H^1_{per} and L^2_{per}, respectively, we mean that the following are true:

$$\mathrm{u_{n_k}(t) \rightharpoonup u(t)} \text{ as } k \to \infty, \text{ weakly in } H^1_{per}, \tag{28}$$

$$\mathrm{u'_{n_k}(t) \rightharpoonup u'(t)} \text{ as } k \to \infty, \text{ weakly in } L^2_{per}. \tag{29}$$

Since $H^1_{per} \hookrightarrow L^2_{per}$ is compact, hence for a subsequence of $\left\{u_{n_k}(t)\right\}$ which is again denoted by $\left\{u_{n_k}(t)\right\}$ for convenience and for any $t \in [0,\ \omega)$, we have:

$$u_{n_k}(t) \text{ converges strongly to } u(t) \text{ as } k \to \infty \text{ in } L^2_{per}(\Omega), \tag{30}$$

and

$$u_{n_k}(t) \text{ converges to } u(t) \text{ as } k \to \infty, \text{ a.e. (almost every where) } \Omega. \tag{31}$$

By inequality (13), lemmas 2 and 3 and for any $t \in [0,\ \omega)$, we have $\left\{u_{n_k}(t)\right\}$ is uniformly bounded in H^1_{per}. Consequently, for a subsequence of $\left\{u_{n_k}(t)\right\}$, which is again denoted by $\left\{u_{n_k}(t)\right\}$ for convenience and for any $t \in [0,\ \omega)$, we get:

$$u_{n_k}(t) \text{ converges to } u(t) \text{ as } k \to \infty, \text{ a.e. (almost every where) } \Omega.$$

$$\left\{u_{n_k}(t)\right\} \text{converges weakly to } u(t) \text{ as } k \to \infty \text{ in } H^2_{per},$$

and

$$\Delta \mathrm{u_{n_k}(t)} \rightharpoonup \Delta \mathrm{u(t)} \text{ as } k \to \infty, \text{ converges weakly in } L^2_{per}. \tag{32}$$

Similar to (30), for a subsequence of $\left\{u_{n_k}(t)\right\}$, which is still denoted by $\left\{u_{n_k}(t)\right\}$, for convenience and for any $t \in [0,\ \omega)$, we obtain:

$$\nabla u_{n_k}(t) \text{converges strongly to} w \in H^1_{per} \text{ as } k \to \infty \text{ in } L^2_{per}(\Omega), \tag{33}$$

and

$$\nabla u_{n_k}(\ t) \to w \text{ as } k \to \infty, \text{ a.e. } \Omega. \tag{34}$$

Combining Equations (28), (29) and (34), we obtain:

$$\nabla u_{n_k}(\ t) \to \nabla u(t) \text{ as } k \to \infty, \text{ a.e. } \Omega. \tag{35}$$

Since $\left\{u'_{n_k}(t)\right\}$ is uniform bound in H^1_{per}, similar to the above procedure, we obtain:

$$\nabla u'_{n_k}(t) \text{ converges strongly to } u'(t) \text{ as } k \to \infty \text{ in } L^2_{per}(\Omega), \tag{36}$$

and

$$u'_{n_k}(\ t) \text{ converges to } u'(t) \text{ as } k \to \infty, \text{ a.e. } \Omega. \tag{37}$$

According as inequality (13) and lemma 2, we have:

$$\|N\left(u_{n_k}(\ t)\right)\| \le d_{17}, \tag{38}$$

where $d_{17} = d_{17}(\alpha,\ \beta,\ \gamma,\ \omega,\ f,\ L)$ is a constant.

Now, combining (31) and (35), we get:

$$N\Big(u_{n_k}(\ t)\Big) \text{ converges to } N(u(t)) \text{ as } k \to \infty,\ \text{a.e. } \Omega. \tag{39}$$

Applying lemma 1.3 of Lions [16], we get:

$$N\Big(u_{n_k}(\ t)\Big) \text{ converges weakly to } N(u(\ t)) \text{ as } k \to \infty \text{ in } L^4_{per}. \tag{40}$$

Since $(u_n(t),\ u'_n(t)) \in C^2\Big(\ \omega,\ H^2_{per}\Big) \times C^1\Big(\omega,\ L^2_{per}\Big)$, then from lemma 3, Equations (28) and(29), we get $(u(t),\ u'(t)) \in C^2\Big(\ \omega,\ H^2_{per}\Big) \times C^1\Big(\omega,\ L^2_{per}\Big)$, and hence for some $t \in [0,\ \omega)$, we obtain:

$$u_{ntt}(t) \text{ converges weakly to } u_{tt}(t) \text{ as } k \to \infty \text{ in } L^2_{per}(\Omega). \tag{41}$$

Multiplying each equation in (11) by any $a_{jn}(t) \in C^2(\ \omega,\ \Re)$, and summing up over j from 1 to n, we get:

$$(u_{ntt} + \alpha u_{nt} + \beta u_n + Tu_n,\ \tau) = (N(u_n) + f,\ \tau),\ \text{for all } \tau \in C^2(\omega,\ H_n).$$

For any fixed $k_0 \le k$, by $H_{n_{k_0}} \subset H_{n_{k_0}+1} \subset \cdots$, we have:

$$\Big(u_{n_k tt} + \alpha u_{n_k t} + \beta u_{n_k} + Tu_{n_k},\ \tau\Big) = \Big(N\Big(u_{n_k}\Big) + f,\ \tau\Big),\ \text{for all } \tau \in C^2\Big(\omega,\ H_{n_{k_0}}\Big). \tag{42}$$

Combining (32), (40), (41), and (42), we deduce:

$$(u_{tt} + \alpha u_t + \beta u + Tu,\ \tau) = (N(u) + f,\ \tau),\ \text{for all } \tau \in C^2\Big(\omega,\ H_{n_{k_0}}\Big). \tag{43}$$

Here k_0 is an arbitrarily chosen number such that (43) holds for all $\tau \in C^2\Big(\omega,\ \bigcup_{n=1}^{\infty} H_n\Big)$. Since $\bigcup_{n=1}^{\infty} H_n$ is dense in L^2_{per}, then $(u(t),\ u'(t))$ is a solution of (43), where $\tau \in C^2\Big(\omega,\ L^2_{per}\Big)$, i.e., $(u(x,\ t),\ u'(x,\ t))$ is a solution of 2DDCNLKGE given by (6).

This completes the proof. □

The next theorem will form a new uniqueness criteria of time periodic solution to 2DDCNLKGE given in Equation (6).

Theorem 3. *If the hypothesis of Theorem 2 holds, then the 2DDCNLKGE given in Equation (6) has a unique time periodic solution.*

Proof. Let $(u(x,\ t),\ u'(x,\ t)\ )$ and $(u^*(x,\ t),\ u^{*\prime}(x,\ t)\ )$ be distinct time periodic solutions of (6).

If we set $(v(x,\ t), v'(x,\ t)\ ) = (u(x,\ t),\ u'(x,\ t)) - \ (u^*(x,\ t),\ u^{*\prime}(x,\ t))$ then from (11), we get:

$$v_{tt} + \alpha v_t + \beta v + Tv = Nu - Nu^*. \tag{44}$$

From (44), we obtain:

$$\frac{1}{2}\frac{d}{dt}\Big(\|v_t\|^2 + \alpha\|v\|^2\Big) + \beta\|v\|^2 + \|\nabla v\|^2 = (Nu - Nu^*,\ v). \tag{45}$$

Using lemmas 1, 2, and 3 in (45), we get:

$$\frac{d}{dt}\Big(\|v_t\|^2 + \|v\|^2\Big) + \delta\Big(\|v\|^2 + \|\nabla v\|^2\Big) \le 0, \tag{46}$$

where, $\delta \ge 0$.

Now, using Gronwall's Inequality [26] in (46), we obtain:

$$\left(\|v_t(t)\|^2 + \|v(t)\|^2\right) \leq \left(\|v_t(0)\|^2 + \|v(0)\|^2\right)e^{-\delta t},\ \forall\ t \geq 0 \tag{47}$$

.

From ω-periodicity of v, we get:

$$\left(\|v_t(t)\|^2 + \|v(t)\|^2\right) = \left(\|v_t(t+\kappa\omega)\|^2 + \|v(t+\kappa\omega)\|^2\right). \tag{48}$$

where, κ is any positive integer.

Using (47) and (48), we get:

$$\left(\|v_t(t)\|^2 + \|v(t)\|^2\right) \leq \left(\|v_t(0)\|^2 + \|v(0)\|^2\right)e^{-\delta(t+\kappa\omega)}.$$

This gives us:

$$v_t(0) = v(0) = 0$$

.

Hence:

$$(u(x,\ t),\ u'(x,\ t)\ )\ =\ (u^*(x,\ t),\ u^{*\prime}(x,\ t)\ ),$$

i.e., the time periodic solution of 2DDCNLKGE given by (6) is unique. This completes the proof. □

5. Conclusions

This article has proven a new solvability criterion for a time periodic solution for 2DDCNLKGE given in Equation (6) with the help of the GLK method and the LS fixed point theorem. The LS fixed point theorem helps us to determine the existence of approximate solution points within uniform priori estimates, whereas uniform priori estimates of the approximate solution of 2DDCNLKGE is constructed by using the GLK method. Theorem 2 provided an easy procedure to check the presence of a time periodic solution of 2DDCNLKGE given in Equation (6) and Theorem 3 ensured the uniqueness of that time periodic solution. The results of this article provided an easy and straightforward technique to identify a unique time periodic solution of 2DDCNLKGE given by Equation (6). Furthermore, these results extend the corresponding results of Gao and Guo [10], Kosecki [14], Geoggiev [15], Ozawa et al. [16], and Gao et al. [18].

Author Contributions: All the authors worked together and equally contributed to the preparation and writing of this paper. They also read and agreed to the final copy of the manuscript. M.A.: Conceptualization, methodology, writing—original draft preparation and editing. A.K.: Visualization, investigation, formal analysis, writing—reviewing and editing. M.Z.A.: Investigation, supervision, validation. S.H.S.: Formal analysis, writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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