mathematics

Article

A Study of Some Families of Multivalent q-Starlike Functions Involving Higher-Order q-Derivatives

Bilal Khan [1,*], Zhi-Guo Liu [1], Hari M. Srivastava [2,3,4], Nazar Khan [5], Maslina Darus [6] and Muhammad Tahir [5]

1 School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China; zgliu@math.ecnu.edu.cn
2 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
3 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
4 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
5 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; nazarmaths@gmail.com (N.K.); tahirmuhammad778@gmail.com (M.T.)
6 Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia; maslina@ukm.edu.my
* Correspondence: bilalmaths789@gmail.com

Received: 20 July 2020; Accepted: 21 August 2020; Published: 1 September 2020

Abstract: In the present investigation, by using certain higher-order q-derivatives, the authors introduce and investigate several new subclasses of the family of multivalent q-starlike functions in the open unit disk. For each of these newly-defined function classes, several interesting properties and characteristics are systematically derived. These properties and characteristics include (for example) distortion theorems and radius problems. A number of coefficient inequalities and a sufficient condition for functions belonging to the subclasses studied here are also discussed. Relevant connections of the various results presented in this investigation with those in earlier works on this subject are also pointed out.

Keywords: multivalent functions; q-difference (or q-derivative) operator; distortion theorems; radius problem

MSC: Primary 05A30; 30C45; Secondary 11B65; 47B38

1. Introduction, Definitions and Motivation

The class of functions, denoted by $\mathcal{H}(\mathbb{U})$, is a collection of the functions f which are holomorphic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In a domain $\mathbb{U} \subseteq \mathbb{C}$ an analytic function f is known as p-valent (or multivalent) in $\mathbb{U}$ $(p \in \mathbb{N} = \{1, 2, 3, \cdots\})$ if, for all $j \in \mathbb{C}$, the relation $f(z) = j$ has its roots not exceeding p in $\mathbb{U}$; Equivalently, one can state that there exists a number $j_0 \in \mathbb{C}$ such that the condition $f(z) = j_0$ has exactly p roots in $\mathbb{U}$. By $\mathcal{A}(p)$, we represent the class of functions with the following series representation:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}$. We notice that

$$\mathcal{A}(1) := \mathcal{A},$$

where $\mathcal{A}$ denotes the usual class of normalized analytic and univalent function in $\mathbb{U}$.

The class of functions, comprising all normalized univalent functions in the open unit disk $\mathbb{U}$, is represented by $\mathcal{S}$ which is a subclass of $\mathcal{A}$. In Geometric Function Theory of complex analysis, several researchers devoted their studies to the class of analytic functions and its subclasses. In the study of analytic functions, the roles of such geometric properties as (for example) convexity, starlikeness and close-to-convexity are specially notable.

A function $f \in \mathcal{A}(p)$ is said to be p-valently starlike in $\mathbb{U}$ whenever it fulfills the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

The family of all normalized p-valently starlike functions in $\mathbb{U}$ is represented by $\mathcal{S}^*(p)$. More generally, let $\mathcal{S}^*(p,\mu)$ be the class consisting of p-valently starlike functions of order μ $(0 \leq \mu < 1)$ in $\mathbb{U}$. In particular, we have

$$\mathcal{S}^*(p,0) = \mathcal{S}^*(p) \qquad (p \in \mathbb{N}).$$

Various articles have been dedicated to the study of subfamilies of analytic functions, specifically several subfamilies of p-valent functions. Coefficient bounds for p-valent functions were considered recently in [1] (see also [2]), whereas the neighborhoods of certain p-valently analytic functions with negative coefficients were studied in [3]. For some convolution (or Hadamard product) properties for the convexity and starlikeness of meromorphically p-valent functions, we refer the reader to [4].

In order to have a better understanding of the present article, some primary notion details and definitions of the q-difference calculus are evoked. Unless otherwise indicated, we assume throughout this article that

$$0 < q < 1 \quad \text{and} \quad p \in \mathbb{N} = \{1,2,3,\cdots\}.$$

For a function f defined on a q-geometric set, Jackson's q-derivative (or q-difference) D_q of a function defined on a subset of the complex space $\mathbb{C}$ is given by (see [5,6])

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0). \end{cases} \tag{2}$$

if $f'(0)$ exists.

It is readily observed from Equation (2) that

$$\lim_{q \to 1-} \left(D_q f\right)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a differentiable function f in a given subset of the set $\mathbb{C}$. Further, on account of (1) and (2), we obtain

$$\left(D_q^{(1)} f\right)(z) = [p]_q\, z^{p-1} + \sum_{n=1}^{\infty} [n+p]_q\, a_{n+p} z^{n+p-1} \tag{3}$$

$$\left(D_q^{(2)} f\right)(z) = [p]_q\, [p-1]_q\, z^{p-2} + \sum_{n=1}^{\infty} [n+p]_q\, [n+p-1]_q\, a_{n+p} z^{n+p-2} \tag{4}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$\left(D_q^{(p)} f\right)(z) = [p]_q! + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n]_q!} a_{n+p} z^n, \tag{5}$$

where $\left(D_q^{(p)} f\right)(z)$ is the q-derivative of $f(z)$ of order p.

For any non-negative integer n, the q-number $[n]_q$ is given as follows:

$$[n]_q = \sum_{j=0}^{n-1} q^j = 1 + q + q^2 + \cdots + q^{n-1} \qquad \text{and} \qquad [0]_q = 0.$$

In general, for $\lambda \in \mathbb{C}$, we write

$$[\lambda]_q = \frac{1-q^\lambda}{1-q}.$$

The q-factorial $[n]_q!$ is stated as

$$[0]_q! = 0 \quad \text{and} \quad [n]_q! = \prod_{k=1}^{n} [k]_q .$$

It is straightforward to observe that

$$\lim_{q \to 1-} [\lambda]_q = \lambda \quad \text{and} \quad \lim_{q \to 1-} [n]_q! = n!.$$

Now, for each $f \in \mathcal{A}(p)$, it is easily seen by means of the operator D_q when applied s times on both sides of (1) with respect to z that

$$\left(D_q^{(s)} f\right)(z) = \frac{[p]_q!}{[p-s]_q!} z^{p-s} + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n+p-s]_q!} a_{n+p} z^{n+p-s}.$$

Since the q-calculus is being vastly used in different areas of mathematics and physics, it is of great interest to researchers. In the study of Geometric Function Theory, the versatile applications of the q-derivative operator D_q make it remarkably significant. Historically speaking, it was Ismail et al. [7] who first presented the idea of a q-extension of the class of starlike functions in 1990. However, in his work published in 1989, Srivastava applied the concepts of the q-calculus by systematically using the basic (or q-) hypergeometric functions:

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}} \quad (\mathfrak{r}, \mathfrak{s} \in \mathbb{N}_0 = \{0, 1, 2, \cdots\})$$

in Geometric Function Theory (GFT) (see, for details, [8]). More recently, In a survey-cum-expository review article by Srivastava [9], the state-of-the-art survey and applications of the q-calculus, the q-derivative operator, the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory of Complex Analysis were investigated and, at the same time, the obvious triviality of the so-called $(\mathfrak{p}, q)$-calculus involving a redundant parameter $\mathfrak{p}$ was exposed.

Inspired by the above-mentioned works, in recent years, important researches have played a significant part in the development of geometric function theory of complex analysis. Several convolutional and fractional calculus q-operators were defined by many researchers, which were surveyed in the above-cited work by Srivastava [9]. We first briefly describe some of the recent developments. Mahmood et al. [10] (see also [11]) found the estimate of the third Hankel determinant. In [12] several interesting results for q-starlike functions related to conic region were obtained. For related results, one may refer to [13–16] and the references cited therein. Additionally, the recently-published review article by Srivastava [9] is potentially useful for researchers and scholars working on these topics. For other recent investigations involving the q-calculus, one may refer to [17–28]. In this paper, we propose mainly to generalize the work presented in [29].

Definition 1 (see [7])**.** *A function $f \in \mathcal{A}$ is said to be in the function class $\mathcal{S}_q^*$ if*

$$f(0) = f'(0) - 1 = 0 \tag{6}$$

and

$$\left| \frac{z}{f(z)} \left(D_q f\right)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \tag{7}$$

In the light of the relation given in (7), it is clear that, in the limit case when $q \to 1-$, we have

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

The closed disk defined by the above formula converges in some sense, as $q \to 1-$, to the right-half plane and $\mathcal{S}_q^*$ given by Definition 1 becomes the well-known class $\mathcal{S}^*$.

We now define the following subclasses of the family of multivalent q-starlike functions.

Definition 2. *A function $f \in \mathcal{A}(p)$ is said to belong to the class $\mathcal{S}_q^*(1, p, m, s, \mu)$ if and only if*

$$\Re \left(\frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} \right) \geq \mu.$$

We call $\mathcal{S}_q^*(1, p, m, s, \mu)$ the class of higher-order q-starlike function of Type 1.

Definition 3. *A function $f \in \mathcal{A}(p)$ is said to belong to the class $\mathcal{S}_q^*(2, p, m, s, \mu)$ if and only if*

$$\left| \frac{\frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - \mu}{1-\mu} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

We call $\mathcal{S}_q^*(2, p, m, s, \mu)$ the class of higher-order q-starlike functions of Type 2.

Definition 4. *A function $f \in \mathcal{A}(p)$ is said to belong to the class $\mathcal{S}_q^*(3, p, m, s, \mu)$ if and only if*

$$\left| \frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - 1 \right| < 1 - \mu.$$

We call $\mathcal{S}_q^*(3, p, m, s, \mu)$ the class of higher-order q-starlike function of Type 3.

Remark 1. *One can easily seen that*

$$\mathcal{S}_q^*(1,1,1,0,\mu) = \mathcal{S}_{(q,1)}^*(\mu),$$

$$\mathcal{S}_q^*(2,1,1,0,\mu) = \mathcal{S}_{(q,2)}^*(\mu)$$

and

$$\mathcal{S}_q^*(3,1,1,0,\mu) = \mathcal{S}_{(q,3)}^*(\mu),$$

where

$$\mathcal{S}_{(q,1)}^*(\mu), \quad \mathcal{S}_{(q,2)}^*(\mu) \qquad \text{and} \qquad \mathcal{S}_{(q,3)}^*(\mu)$$

are the classes of functions introduced and studied by Wongsaijai and Sukantamala (see [29]). Furthermore, we have

$$\mathcal{S}_q^*(2,1,1,0,0) = \mathcal{S}_{(q,2)}^*(0) = \mathcal{S}_q^*(0) = \mathcal{S}_q^*,$$

where $\mathcal{S}_q^*$ is the class of functions introduced and studied by Ismail et al. [7].

2. A Set of Main Results

We first derive the inclusion results for the following generalized multivalent q-starlike function classes:

$$\mathcal{S}_q^*(1,p,m,s,\mu), \quad \mathcal{S}_q^*(2,p,m,s,\mu) \qquad \text{and} \qquad \mathcal{S}_q^*(3,p,m,s,\mu),$$

each of which involves higher-order q-derivatives.

Theorem 1. *For $0 < \mu < 1$, it is asserted that*

$$\mathcal{S}_q^*(3,p,m,s,\mu) \subset \mathcal{S}_q^*(2,p,m,s,\mu) \subset \mathcal{S}_q^*(1,p,m,s,\mu).$$

Proof. First of all, we suppose that $f \in \mathcal{S}_q^*(3,p,m,s,\mu)$. Then, by Definition 4, we have

$$\left| \frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - 1 \right| < 1 - \mu.$$

Moreover, by using the triangle inequality, we find that

$$\begin{aligned}
&\left| \frac{\frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - \mu}{1-\mu} - \frac{1}{1-q} \right| \\
&= \frac{1}{1-\mu} \left| \frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - \mu - \frac{1-\mu}{1-q} \right| \\
&\leq \frac{1}{1-\mu} \left| \frac{z^m \left(D_q^{(m+s)} f\right)(z)}{\left(D_q^{(s)} f\right)(z)} - 1 \right| + \frac{q}{1-q} \\
&\leq 1 + \frac{q}{1-q} \leq \frac{1}{1-q}.
\end{aligned} \tag{8}$$

The inequalities in (8) shows that $f \in \mathcal{S}_q^*(2, p, m, s, \mu)$, Thus, clearly,

$$\mathcal{S}_q^*(3, p, m, s, \mu) \subset \mathcal{S}_q^*(2, p, m, s, \mu).$$

Next, we let $f \in \mathcal{S}_q^*(2, p, m, s, \mu)$. Then. by Definition 3, we have

$$\left| \frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} - \frac{1 - \mu q}{1 - q} \right| < \frac{1 - \mu}{1 - q}. \tag{9}$$

Furthermore, from (9), we see that

$$\frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)}$$

lies in the circle of radius $\frac{1-\mu}{1-q}$ with its center at $\frac{1-\mu q}{1-q}$ and we observe that

$$\frac{1 - \mu q}{1 - q} - \frac{1 - \mu}{1 - q} = \mu,$$

which implies that

$$\Re \left(\frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} \right) > \mu.$$

Consequently, $f \in \mathcal{S}_q^*(1, p, m, s, \mu)$, that is, $\mathcal{S}_q^*(2, p, m, s, \mu) \subset \mathcal{S}_q^*(1, p, m, s, \mu)$. This completes the proof of Theorem 1. □

If we put $m = 1 = s + 1$ in Theorem 1, we arrive at the following known result.

Corollary 1 (see [29]). *For* $0 \leq \mu < 1$,

$$\mathcal{S}_{q,3}^*(\mu) \subset \mathcal{S}_{q,2}^*(\mu) \subset \mathcal{S}_{q,1}^*(\mu).$$

Finally, in the next result in this section, we settle a sufficient condition for the function class $\mathcal{S}_q^*(3, p, m, s, \mu)$ consisting of generalized q-starlike functions of Type 3. Luckily, for the classes $\mathcal{S}_q^*(1, p, m, s, \mu)$ and $\mathcal{S}_q^*(2, p, m, s, \mu)$ of Type 1 and Type 2, respectively, this result also provides the corresponding sufficient condition.

Theorem 2. *A function* $f \in \mathcal{A}(p)$ *and of the form* (1) *is in the class* $\mathcal{S}_q^*(3, p, m, s, \mu)$ *if it satisfy the following coefficient inequality*:

$$\sum_{n=1}^{\infty} \left(\Upsilon_{(2,n)} + (1 - \mu) \frac{[n + p]_q!}{[n + p - s]_q!} \right) |a_{n+p}| < \frac{(1 - \mu)[p]_q!}{[p - s]_q!} - \Upsilon_1, \tag{10}$$

where

$$\Upsilon_1 = \left(\frac{[p]_q!}{[p - s - m]_q!} - \frac{[p]_q!}{[p - s]_q!} \right) \tag{11}$$

and

$$\Upsilon_{(2,n)} = \left(\frac{[n + p]_q!}{[n + p - s - m]_q!} - \frac{[n + p]_q!}{[n + p - s]_q!} \right). \tag{12}$$

Proof. Assuming that (10) holds true, it suffices to show that

$$\left| \frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} - 1 \right| < 1 - \mu.$$

We observe that

$$\begin{aligned}
\left| \frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} - 1 \right| &= \left| \frac{z^m \left(D_q^{(m+s)} f \right)(z) - \left(D_q^{(s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} \right| \\
&= \left| \frac{\Upsilon_1 z^{p-s} + \sum_{n=1}^{\infty} \Upsilon_{(2,n)} a_{n+p} z^{n+p-s}}{\frac{[p]_q!}{[p-s]_q!} z^{p-s} + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n+p-s]_q!} a_{n+p} z^{n+p-s}} \right| \\
&\leq \frac{\Upsilon_1 + \sum_{n=1}^{\infty} \Upsilon_{(2,n)} \left| a_{n+p} \right| |z^n|}{\frac{[p]_q!}{[p-s]_q!} - \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n+p-s]_q!} \left| a_{n+p} \right| |z^n|} \\
&\leq \frac{\Upsilon_1 + \sum_{n=1}^{\infty} \Upsilon_{(2,n)} \left| a_{n+p} \right|}{\frac{[p]_q!}{[p-s]_q!} - \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n+p-s]_q!} \left| a_{n+p} \right|},
\end{aligned} \tag{13}$$

where Υ_1 and $\Upsilon_{(2,n)}$ are given by (11) and (12), respectively. We see that $1 - \mu$ is the upper bound of the last expression in (13) if the condition in (10) is satisfied. This completes the proof of Theorem 2. □

3. Analytic Functions with Negative Coefficients

This section is devoted to a new family of subclasses $\mathcal{TS}_q^* (k, p, m, s, \mu)$ $(k = 1, 2, 3)$ of multivalent q-starlike functions with negative coefficients. Let a subset of $\mathcal{A}(p)$, which consists of functions with negative coefficients, be $\mathcal{T}(p)$ and have the following series representation:

$$f(z) = z^p - \sum_{n=1}^{\infty} \left| a_{n+p} \right| z^{n+p} \qquad (z \in \mathbb{U};\ p \in \mathbb{N}). \tag{14}$$

We also let

$$\mathcal{TS}_q^* (k, p, m, s, \mu) := \mathcal{S}_q^* (k, p, m, s, \mu) \cap \mathcal{T}(p) \qquad (k = 1, 2, 3). \tag{15}$$

Theorem 3. *Let* $0 < \mu < 1$. *Then*

$$\mathcal{TS}_q^* (1, p, m, s, \mu) \equiv \mathcal{TS}_q^* (2, p, m, s, \mu) \equiv \mathcal{TS}_q^* (3, p, m, s, \mu).$$

Proof. In view of Theorem 1. it is sufficient here to show that

$$\mathcal{TS}_q^* (1, p, m, s, \mu) \subset \mathcal{TS}_q^* (3, p, m, s, \mu).$$

Indeed, if we assume that $f \in \mathcal{TS}_q^* (1, p, m, s, \mu)$, then we have

$$\Re \left(\frac{z^m \left(D_q^{(m+s)} f \right)(z)}{\left(D_q^{(s)} f \right)(z)} \right) \geq \mu.$$

We now consider

$$\Re\left(\frac{z^m\left(D_q^{(m+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}\right)=\Re\left(\frac{\frac{[p]_q!}{[p-m-s]_q!}z^{p-s}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-m-s]_q!}a_{n+p}z^{n+p-s}}{\frac{[p]_q!}{[p-s]_q!}z^{p-s}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-s]_q!}a_{n+p}z^{n+p-s}}\right)$$

$$=\Re\left(\frac{\frac{[p]_q!}{[p-m-s]_q!}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-m-s]_q!}a_{n+p}z^{n}}{\frac{[p]_q!}{[p-s]_q!}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-s]_q!}a_{n+p}z^{n}}\right)\geq\mu. \tag{16}$$

If we let z lie on the real axis, then the value of

$$\frac{z^m\left(D_q^{(m+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}$$

is real. In this case, upon letting $z\to 1^-$ along the real line, we get

$$\frac{[p]_q!}{[p-m-s]_q!}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-m-s]_q!}\left|a_{n+p}\right|\geq\mu\left(\frac{[p]_q!}{[p-s]_q!}-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+p-s]_q!}\left|a_{n+p}\right|\right). \tag{17}$$

We see that (17) satisfies the inequality in (10). And so, by applying Theorem 2, the proof of Theorem 3 is completed. □

If we put $m=1=s+1$ in Theorem 3, we are led to the following results.

Corollary 2 (see [29], Theorem 8). *If* $0\leq\mu<1$, *then*

$$\mathcal{TS}^*_{(q,1)}(\mu)\equiv\mathcal{TS}^*_{(q,2)}(\mu)\equiv\mathcal{TS}^*_{(q,3)}(\mu).$$

Corollary 3. *Let the function f of the form (14) be in the class $\mathcal{TS}^*_q(k,p,m,s,\mu)$ $(k=1,2,3)$. Then*

$$a_{n+p}\leq\frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!}-\Upsilon_1}{\left(\Upsilon_{(2,n)}+(1-\mu)\frac{[n+p]_q!}{[n+p-s]_q!}\right)}. \tag{18}$$

The result is sharp for the function $f_t(z)$ given by

$$f_t(z)=z^p-\frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!}-\Upsilon_1}{\left(\Upsilon_{(2,1)}+(1-\mu)\frac{[n+p]_q!}{[n+p-s]_q!}\right)}z^{p+1},$$

where Υ_1 and $\Upsilon_{(2,1)}$ are given by (11) and (12), respectively.

On the account of Theorem 3, it should be noted that Type 1, Type 2 and Type 3 of the multivalent q-starlike functions are essentally the same. Consequently, for simplicity, we state and prove the following distortion theorem for the function class $\mathcal{TS}^*_q(k,p,m,s,\mu)$ in which it is assumed that $k=1,2,3$.

Theorem 4. *If* $f \in \mathcal{TS}_q^* (k, p, m, s, \mu) \quad (k = 1, 2, 3)$, *then*

$$|f(z)| \geq r^p - \left(\frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!}} \right) r^{p+1} \quad (n \in \mathbb{N}) \quad |z^p| = r^p \quad (0 < r < 1) \tag{19}$$

and

$$|f(z)| \leq r^p + \left(\frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!}} \right) r^{p+1} \quad (p \in \mathbb{N}) \quad |z^p| = r^p \quad (0 < r < 1). \tag{20}$$

The equalities in (19) *and* (20) *are attained for the function* $f(z)$ *given by*

$$f(z) = z^p - \frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\left(\Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!} \right)} z^{p+1} \tag{21}$$

$$\text{at} \quad z = r \text{ and } z = r \exp(i(2\ell+1)\pi) \quad \ell \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\},$$

where Υ_1 *and* $\Upsilon_{(2,1)}$ *are given by* (11) *and* (12), *respectively.*

Proof. We can see that the following inequality follows from Theorem 2:

$$\begin{aligned} \Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!} \sum_{n=1}^{\infty} |a_{n+p}| &\leq \sum_{n=1}^{\infty} \left(\Upsilon_{(2,n)} + (1-\mu) \frac{[n+p]_q!}{[n+p-s]_q!} \right) |a_{n+p}| \\ &< \frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1, \end{aligned}$$

which yields

$$|f(z)| \leq r^p + \sum_{n=1}^{\infty} |a_{n+p}| r^{n+p} \leq r^p + r^{p+1} \sum_{n=1}^{\infty} |a_{n+p}| \leq r^p + \frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!}} r^{p+1}.$$

Similarly, we have

$$|f(z)| \geq r^p - \sum_{n=1}^{\infty} |a_{n+p}| r^{n+p} \geq r^p - r^{p+1} \sum_{n=1}^{\infty} |a_{n+p}| \geq r^p - \frac{\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu) \frac{[1+p]_q!}{[1+p-s]_q!}} r^{p+1}.$$

We have thus completed the proof of Theorem 4. □

In its special case when $m = 1 = s + 1 = p$ and if we let $q \longrightarrow 1-$, Theorem 4 coincides with a similar result (see [30]) given as follows.

Corollary 4 (see [30]). *If* $f \in \mathcal{TS}^* (\mu)$, *then*

$$r - \left(\frac{1-\mu}{2-\mu} \right) r^2 \leq |f(z)| \leq r + \left(\frac{1-\mu}{2-\mu} \right) r^2 \qquad (|z| = r \quad (0 < r < 1)).$$

The proof of the following result (Theorem 5 below) is similar to the proof of Theorem 4, so the analogous details of our proof of Theorem 5 have been omitted.

Theorem 5. *If $f \in \mathcal{TS}_q^*(k,p,m,s,\mu)$ $(k = 1,2,3)$, then*

$$\left|f'(z)\right| \geq pr^{p-1} - \left(\frac{\frac{(p+1)(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu)\frac{[1+p]_q!}{[1+p-s]_q!}}\right) r^p \quad (p \in \mathbb{N};\ |z| = r) \quad |z^p| = r^p \quad (0 < r < 1)$$

and

$$\left|f'(z)\right| \leq pr^{p-1} + \left(\frac{\frac{(p+1)(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1}{\Upsilon_{(2,1)} + (1-\mu)\frac{[1+p]_q!}{[1+p-s]_q!}}\right) r^p \quad (p \in \mathbb{N};\ |z| = r) \quad |z^p| = r^p \quad (0 < r < 1).$$

The result is sharp for the function $f(z)$ given by (21).

In its special case when we put $m = 1 = s + 1 = p$ and let $q \longrightarrow 1-$, Theorem 4 reduces to the following known result.

Corollary 5 (see [30]). *If $f \in \mathcal{TS}^*(\mu)$, then*

$$1 - \left(\frac{2(1-\mu)}{2-\mu}\right) r \leq \left|f'(z)\right| \leq 1 + \left(\frac{2(1-\mu)}{2-\mu}\right) r \qquad (|z| = r \ \ (0 < r < 1)).$$

Finally, we find the radii of close-to-convexity, starlikeness and convexity for functions belonging to the family $\mathcal{TS}_q^*(k,p,m,s,\mu)$ $(k = 1,2,3)$.

Theorem 6. *Let the function f, given by (14), be in the class $\mathcal{TS}_q^*(k,p,m,s,\mu)$ $(k = 1,2,3)$. Then $f(z)$ is a p-valent close-to-convex function of order χ $(0 \leq \chi < p)$ for $|z| \leq r_0(p,n,\eta,\chi)$, where*

$$r_0 = \inf_{n \geq 1}\left[\frac{\left(\Upsilon_{(2,n)} + (1-\mu)\frac{[n+p]_q!}{[n+p-s]_q!}\right)(p-\chi)}{\left(\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1\right)(n+p)}\right]^{\frac{1}{n}}. \tag{22}$$

The result is sharp for the function $f_t(z)$ given by (18).

Proof. By applying Theorem 2 and the form (14), we see for $|z| < r_0$ that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p - \chi \qquad (|z| \leq r_0).$$

This completes the proof of Theorem 6. □

Theorem 7. *Let the function f, given by (14), be in the class $\mathcal{TS}_q^*(k,p,m,s,\mu)$ $(k = 1,2,3)$. Then $f(z)$ is a p-valent starlike function of order χ $(0 \leq \chi < p)$ for $|z| \leq r_1(p,n,\eta,\chi)$, where*

$$r_1 = \inf_{n \geq 1}\left[\frac{\left(\Upsilon_{(2,n)} + (1-\mu)\frac{[n+p]_q!}{[n+p-s]_q!}\right)(p-\chi)}{\left(\frac{(1-\mu)[p]_q!}{[p-s]_q!} - \Upsilon_1\right)(n+p-\chi)}\right]^{\frac{1}{n}}. \tag{23}$$

The result is sharp for the function $f_t(z)$ given by (18).

Proof. Using the same steps as in the proof of Theorem 6, it is seen that

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p-\chi \qquad (|z|\leq r_1),$$

which evidently proves Theorem 7. □

Corollary 6. *Let the function f, given by (14), be in the class $\mathcal{TS}_q^*(k,p,m,s,\mu)$ $(k=1,2,3)$. Then $f(z)$ is a p-valent convex function of order χ $(0\leq\chi<p)$ for $|z|\leq r_2(p,n,\eta,\chi)$, where*

$$r_2=\inf_{n\geq 1}\left[\frac{\left(\Upsilon_{(2,n)}+(1-\mu)\frac{[n+p]_q!}{[n+p-s]_q!}\right)p(p-\chi)}{\left(\frac{(1-\mu)[p]_q!}{[p-s]_q!}-\Upsilon_1\right)(n+p)(n+p-\chi)}\right]^{\frac{1}{n}}. \tag{24}$$

The result is sharp for the function $f_t(z)$ given by (18).

4. Conclusions

Our present investigation is motivated by the well-established potential for the usages of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory as described in a recently-published survey-cum-expository review article by Srivastava [9]. Here, we have introduced and studied systematically some interesting subclasses of multivalent (or p-valent) q-starlike functions in the open unit disk $\mathbb{U}$. We have also provided relevant connections of the various results, which we have demonstrated in this paper, with those derived in many earlier works cited here.

Author Contributions: All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This work here was supported by UKM Grant: GUP-2019-032.

Acknowledgments: The authors would like to express their gratitude to the anonymous referees for many valuable suggestions regarding a previous version of this paper.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Yang, D.-G.; Liu, J.-L. Some sufficient conditions for p-valent strongly starlike functions. *Comput. Math. Appl.* **2010**, *59*, 2018–2025. [CrossRef]
2. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Kiran, S.; Khan, B. Some applications of higher-order derivatives involving certain subclass of analytic and multivalent functions. *J. Nonlinear Var. Anal.* **2018**, *2*, 343–353.
3. Altintaş, O. Neighborhoods of certain p-valently analytic functions with negative coefficients. *Appl. Math. Comput.* **2007**, *187*, 47–53. [CrossRef]
4. Liu, J.-L.; Srivastava, H.M. Some convolution conditions for starlikeness and convexity of meromorphically multivalent functions. *Appl. Math. Lett.* **2003**, *16*, 13–16. [CrossRef]
5. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
6. Jackson, F.H. q-Difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
7. Ismail, M.E.-H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
8. Srivastava, H.M. Univalent functions, fractional calculus and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA, 1989; pp. 329–354.
9. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]

10. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q-starlike functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
11. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. *Mathematics* **2019**, *7*, 181. [CrossRef]
12. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z.; Tahir, M. A generalized conic domain and its applications to certain subclasses of analytic functions. *Rocky Mountain J. Math.* **2019**, *49*, 2325–2346. [CrossRef]
13. Mahmood, S.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Khan, B.; Tahir, M. A certain subclass of meromorphically q-starlike functions associated with the Janowski functions. *J. Inequal. Appl.* **2019**, *2019*, 88. [CrossRef]
14. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general classes of q-starlike functions associated with the Janowski functions. *Symmetry* **2019**, *11*, 292. [CrossRef]
15. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [CrossRef]
16. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general families of q-starlike functions associated with the Janowski functions. *Filomat* **2019**, *33*, 2613–2626. [CrossRef]
17. Khan, Q.; Arif, M.; Raza, M.; Srivastava, G.; Tang, H. Some applications of a new integral operator in q-analog for multivalent functions. *Mathematics* **2019**, *7*, 1178. [CrossRef]
18. Khan, B.; Srivastava, H.M.; Khan, N.; Darus, M.; Tahir, M.; Ahmad, Q.Z. Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain. *Mathematics* **2020**, *8*, 1334. [CrossRef]
19. Liu, Z.-G. Two q-difference equations and q-operator identities. *J. Differ. Equ. Appl.* **2010**, *16*, 1293–1307. [CrossRef]
20. Liu, Z.-G. Some operator identities and q-series transformation formulas. *Discrete Math.* **2003**, *265*, 119–139. [CrossRef]
21. Liu, Z.-G. An expansion formula for q-series and applications. *Ramanujan J.* **2002**, *6*, 429–447. [CrossRef]
22. Mahmood, S.; Raza, N.; AbuJarad, E.S.; Srivastava, G.; Srivastava, H.M.; Malik, S.N. Geometric properties of certain classes of analytic functions associated with a q-integral operator. *Symmetry* **2019**, *11*, 719. [CrossRef]
23. Srivastava, H.M.; Bansal, D. Close-to-convexity of a certain family of q-Mittag-Leffler functions. *J. Nonlinear Var. Anal.* **2017**, *1*, 61–69.
24. Rehman, M.S.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, B.; Khan, N. Partial sums of generalized q-Mittag-Leffler functions. *AIMS Math.* **2019**, *5*, 408–420. [CrossRef]
25. Srivastava, H.M.; Raza, N.; AbuJarad, E.S.A.; Srivastava, G.; AbuJarad, M.H. Fekete-Szegö inequality for classes of (p,q)-starlike and (p,q)-convex functions. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM)* **2019**, *113*, 3563–3584. [CrossRef]
26. Srivastava, H.M.; Aouf, M.K.; Mostafa, A.O. Some properties of analytic functions associated with fractional q-calculus operators. *Miskolc Math. Notes* **2019**, *20*, 1245–1260. [CrossRef]
27. Srivastava, H.M.; El-Deeb, S.M. A certain class of analytic functions of complex order with a q-analogue of integral operators. *Miskolc Math. Notes* **2020**, 417–433. [CrossRef]
28. Shi, L.; Khan, Q.; Srivastava, G.; J.-L. Liu, J.-L.; Arif, M. A study of multivalent q-starlike functions connected with circular domain. *Mathematics* **2019**, *7*, 1–12. [CrossRef]
29. Wongsaijai, B.; Sukantamala, N. Certain properties of some families of generalized starlike functions with respect to q-calculus. *Abstr. Appl. Anal.* **2016**, *2016*, 6180140. [CrossRef]
30. Silverman, H. Univalent functions with negative coefficients. *Proc. Am. Math. Soc.* **1975**, *51*, 109–116. [CrossRef]