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Article

On a Class of Second-Order PDE&PDI Constrained Robust Modified Optimization Problems

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Abstract: In this paper, by using scalar multiple integral cost functionals and the notion of convexity associated with a multiple integral functional driven by an uncertain multi-time controlled second-order Lagrangian, we develop a new mathematical framework on multi-dimensional scalar variational control problems with mixed constraints implying second-order partial differential equations (PDEs) and inequations (PDIs). Concretely, we introduce and investigate an auxiliary (modified) variational control problem, which is much easier to study, and provide some equivalence results by using the notion of a normal weak robust optimal solution.

Keywords: uncertain data; controlled second-order Lagrangian; weak robust optimal solution; robust optimality conditions; modified objective function method

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1. Introduction

As we all know, the partial differential equations (PDEs) and inequations (PDIs) are very important in the study of many processes and phenomena in nature, science and engineering. In this regard, over time, several researchers have taken a particular interest in studying some optimization problems with ODE, PDE, or isoperimetric constraints. We mention here, for instance, the works of Mititelu [1], Treanţă [2], Olteanu and Treanţă [3], Mititelu and Treanţă [4], and Jayswal et al. [5]. Additionally, since the difficulty of the considered problems was increasing, several auxiliary (modified) optimization problems have been introduced to study the initial problem more easily (see, quite recently, Treanţă [6–9]). Moreover, the complexity of real-life processes and phenomena is very high and often involves uncertainty in initial data. In consequence, many researchers turned their attention to real problems involving higher-order PDEs, isoperimetric restrictions, uncertain data, or a combination thereof. In this respect, the reader is directed to the following research works: Liu and Yuan [10], Jeyakumar et al. [11], Wei et al. [12], Preeti et al. [13], Sun et al. [14], Treanţă [15], Lu et al. [16]. For other different but connected ideas on this subject, the reader is directed to [17–20].

In this paper, motivated and inspired by the above mentioned works, and based on a given class of constrained robust optimization problems, named (P), we introduce and investigate a new class of auxiliary (modified) control problems (which are much easier to study). More precisely, by considering multiple integral cost functionals and mixed (equality and inequality) constraints involving second-order partial derivatives and data uncertainty, we formulate and prove some equivalence results between the two considered classes of control problems. Compared to other research papers in this field, the elements of total novelty included in the paper are represented by the presence of second-order partial derivatives and also by the presence of uncertain data both in the objective functional and in the restrictions. Moreover, the proofs associated with the main results of the paper are presented in an innovative way. In addition, since the mathematical framework developed here is suitable for various approaches and scientific views on complex spatial and temporal behaviors, this paper could be seen as a fundamental work for a large community of researchers in science and engineering.

The paper is divided as follows. Section 2 introduces the preliminary tools that will be used to describe the problem under study. Section 3 contains the main results of this paper. The first main result, under convexity assumptions of the constraint functionals, formulates the equivalence between the considered second-order PDE&PDI constrained robust optimization problem (P) and the associated modified problem. The second main result represents the converse of the first main result, assuming only the convexity hyothesis of the objective functional. Section 4 contains the conclusions of the paper and further developments.

2. Preliminary Tools

Throughout the paper, we consider the following notations and working hypotheses:

- $\mathbb{R}^p, \mathbb{R}^q, \mathbb{R}^r$ and $\mathbb{R}^n$ are Euclidean spaces of dimensions p, q, r and n, respectively;
- $\Theta \subset \mathbb{R}^p$ is a compact domain, and the point $t = (t^\alpha) \in \Theta$ is a *multi-parameter of evolution* or *multi-time*;
- $\mathcal{S}$ is the space of C^4-class state functions $a = (a^\tau) : \Theta \to \mathbb{R}^q$, and $a_\alpha := \dfrac{\partial a}{\partial t^\alpha}$, $a_{\alpha\beta} := \dfrac{\partial^2 a}{\partial t^\alpha \partial t^\beta}$ denote the *partial speed* and *partial acceleration*, respectively;
- $\mathcal{C}$ is the space of C^1-class control functions $c = (c^j) : \Theta \to \mathbb{R}^r$;
- $dt = dt^1 \ldots dt^p$ is the volume element on $\mathbb{R}^p \supset \Theta$;
- T denotes the transpose of a vector;
- For two vectors $x, y \in \mathbb{R}^n$, we use the following convention for inequalities and equalities:
  (i) $x < y \Leftrightarrow x_i < y_i,\ \forall i = \overline{1,n}$;
  (ii) $x = y \Leftrightarrow x_i = y_i,\ \forall i = \overline{1,n}$;
  (iii) $x \leqq y \Leftrightarrow x_i \leq y_i,\ \forall i = \overline{1,n}$;
  (iv) $x \leq y \Leftrightarrow x_i \leq y_i,\ \forall i = \overline{1,n}$ and $x_i < y_i$ for some i.

The second-order PDE&PDI constrained controlled scalar optimization problem with uncertainty in the objective and constraint functionals is formulated as follows:

$$(P) \qquad \min_{(a(\cdot),c(\cdot))} \int_\Theta f(t, a(t), a_\sigma(t), a_{\alpha\beta}(t), c(t), w) dt$$

subject to

$$g(t, a(t), a_\sigma(t), a_{\alpha\beta}(t), c(t), u) \leqq 0, \quad t \in \Theta$$

$$h(t, a(t), a_\sigma(t), a_{\alpha\beta}(t), c(t), v) = 0, \quad t \in \Theta$$

$$a(t_0) = a_0,\ a(t_1) = a_1,\ a_\sigma(t_0) = a_{\sigma 0},\ a_\sigma(t_1) = a_{\sigma 1},$$

or

$$a(t)|_{\partial\Theta} = \text{given}, \quad a_\sigma(t)|_{\partial\Theta} = \text{given},$$

where $f : J^2\left(\Theta, \mathbb{R}^q\right) \times \mathcal{C} \times W \to \mathbb{R}$, $g = (g_1, \ldots, g_m) = (g_l) : J^2\left(\Theta, \mathbb{R}^q\right) \times \mathcal{C} \times U_l \to \mathbb{R}^m$, $l = \overline{1,m}$, $h = (h_1, \ldots, h_n) = (h_\zeta) : J^2\left(\Theta, \mathbb{R}^q\right) \times \mathcal{C} \times V_\zeta \to \mathbb{R}^n, \zeta = \overline{1,n}$, are C^3-class functionals, $w, u = (u_l)$ and $v = (v_\zeta)$ are the uncertain parameters for some convex compact subsets $W \subset \mathbb{R}, U = (U_l) \subset \mathbb{R}^m$ and $V = (V_\zeta) \subset \mathbb{R}^n$, respectively, and $J^2\left(\Theta, \mathbb{R}^q\right)$ denotes the second-order jet bundle associated to Θ and $\mathbb{R}^q$.

The associated robust counterpart of the aforementioned scalar optimization problem (P) is defined as:

$$(RP) \qquad \min_{(a(\cdot),c(\cdot))} \int_\Theta \max_{w \in W} f(t, a(t), a_\sigma(t), a_{\alpha\beta}(t), c(t), w) dt$$

subject to

$$g(t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t),u) \leqq 0, \quad t \in \Theta,\ \forall u \in U$$
$$h(t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t),v) = 0, \quad t \in \Theta,\ \forall v \in V$$
$$a(t_0) = a_0,\ a(t_1) = a_1,\ a_\sigma(t_0) = a_{\sigma 0},\ a_\sigma(t_1) = a_{\sigma 1},$$

or

$$a(t)|_{\partial\Theta} = \text{given}, \quad a_\sigma(t)|_{\partial\Theta} = \text{given}.$$

Furthermore, denoted by

$$D = \{(a,c) \in \mathcal{S} \times \mathcal{C} :\ g(t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t),u) \leqq 0,$$
$$h(t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t),v) = 0,\ a(t_0) = a_0,\ a(t_1) = a_1,$$
$$a_\sigma(t_0) = a_{\sigma 0},\ a_\sigma(t_1) = a_{\sigma 1},\ t \in \Theta, u \in U,\ v \in V\}$$

the set of all feasible solutions in (RP) and we say that it is the *robust feasible solution set* to the problem (P).

From now on, to simplify our presentation, we introduce the following notation: $\pi = (t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t))$.

The first-order partial derivatives associated with f are defined as

$$f_a = \Big(\frac{\partial f}{\partial a^1}, \cdots, \frac{\partial f}{\partial a^q}\Big), \quad f_c = \Big(\frac{\partial f}{\partial c^1}, \cdots, \frac{\partial f}{\partial c^r}\Big).$$

Similarly, we have g_a and g_c using matrices with m rows and h_a and h_c using matrices with n rows.

In the following, we introduce the notion of a weak robust optimal solution for the considered class of constrained optimization problems. This concept will be used to formulate the associated robust necessary optimality conditions and the main results included in the present paper.

Definition 1. *A robust feasible solution $(\bar{a},\bar{c}) \in D$ is said to be a weak robust optimal solution to the multi-dimensional scalar optimization problem (P) if there does not exist another point $(a,c) \in D$ such that*

$$\int_\Theta \max_{w \in W} f(\pi, w)dt < \int_\Theta \max_{w \in W} f(\bar{\pi}, w)dt.$$

To formulate the concept of convexity and the robust optimality conditions associated with the aforementioned controlled optimization problem, we shall use the Saunders's multi-index notation (see Saunders [21], Treanţă [22]).

Definition 2. *A multiple integral functional (driven by an uncertain multi-time controlled second-order Lagrangian)*

$$F(a,c,\bar{w}) = \int_\Theta f(t,a(t),a_\sigma(t),a_{\alpha\beta}(t),c(t),\bar{w})dt = \int_\Theta f(\pi,\bar{w})dt$$

is said to be convex at $(\bar{a},\bar{c}) \in \mathcal{S} \times \mathcal{C}$ if the following inequality

$$F(a,c,\bar{w}) - F(\bar{a},\bar{c},\bar{w}) \geq \int_\Theta f_a(\bar{\pi},\bar{w})[a(t) - \bar{a}(t)]dt$$
$$+ \int_\Theta f_{a_\sigma}(\bar{\pi},\bar{w})[a_\sigma(t) - \bar{a}_\sigma(t)]dt$$
$$+ \frac{1}{n(\alpha,\beta)} \int_\Theta f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})[a_{\alpha\beta}(t) - \bar{a}_{\alpha\beta}(t)]dt$$

$$+\int_{\Theta} f_c(\bar{\pi}, \bar{w})[c(t) - \bar{c}(t)]dt$$

holds for all $(a, c) \in \mathcal{S} \times \mathcal{C}$.

In the following, according to Treanţă [22], we present the robust necessary optimality conditions for the scalar optimization problem (P).

Theorem 1. *Let* $(\bar{a}, \bar{c}) \in D$ *be a weak robust optimal solution to the problem* (P) *and* $\max_{w \in W} f(\pi, w) = f(\pi, \bar{w})$. *Then, then there exist the scalar* $\bar{\mu} \in \mathbb{R}$, *the piecewise smooth functions* $\bar{\upsilon} = (\bar{\upsilon}_l(t)) \in \mathbb{R}^m_+$, $\bar{\gamma} = (\bar{\gamma}_\zeta(t)) \in \mathbb{R}^n$, *and the uncertainty parameters* $\bar{u} \in U$ *and* $\bar{v} \in V$, *such that the following conditions*

$$\bar{\mu} f_a(\bar{\pi}, \bar{w}) + \bar{\upsilon}^T g_a(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_a(\bar{\pi}, \bar{v}) \tag{1}$$

$$-D_\sigma \left[\bar{\mu} f_{a_\sigma}(\bar{\pi}, \bar{w}) + \bar{\upsilon}^T g_{a_\sigma}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{a_\sigma}(\bar{\pi}, \bar{v})\right]$$

$$+\frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta} \left[\bar{\mu} f_{a_{\alpha\beta}}(\bar{\pi}, \bar{w}) + \bar{\upsilon}^T g_{a_{\alpha\beta}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{a_{\alpha\beta}}(\bar{\pi}, \bar{v})\right] = 0,$$

$$\bar{\mu} f_c(\bar{\pi}, \bar{w}) + \bar{\upsilon}^T g_c(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_c(\bar{\pi}, \bar{v}) = 0, \tag{2}$$

$$\bar{\upsilon}^T g(\bar{\pi}, \bar{u}) = 0, \ \bar{\upsilon} \geqq 0, \tag{3}$$

$$\bar{\mu} \geq 0 \tag{4}$$

hold for all $t \in \Theta$, *except at discontinuities.*

Remark 1. *The conditions (1)–(4) are known as robust necessary optimality conditions for the multi-dimensional scalar optimization problem* (P).

Definition 3. *A robust feasible solution* $(\bar{a}, \bar{c})$ *is said to be a normal weak robust optimal solution to the problem* (P) *if* $\bar{\mu} > 0$ *in Theorem 1. Without loss of generality, we can consider* $\bar{\mu} = 1$.

3. The Associated Modified Optimization Problem

In this section, we use the modified objective function method to reduce the complexity associated with the considered multi-dimensional scalar optimization problem (P). In this regard, let $(\bar{a}, \bar{c})$ be an arbitrary given robust feasible solution to the multi-dimensional scalar optimization problem (P). The modified multi-dimensional scalar optimization problem associated with the original variational control problem (P) is defined as:

$$(P)_{(\bar{a}, \bar{c})} \quad \min_{(a(\cdot), c(\cdot))} \int_{\Theta} \Big\{ f_a(\bar{\pi}, w)(a(t) - \bar{a}(t)) + f_{a_\sigma}(\bar{\pi}, w)(a_\sigma(t) - \bar{a}_\sigma(t))$$

$$+\frac{1}{n(\alpha, \beta)} f_{a_{\alpha\beta}}(\bar{\pi}, w)(a_{\alpha\beta}(t) - \bar{a}_{\alpha\beta}(t)) + f_c(\bar{\pi}, w)(c(t) - \bar{c}(t)) \Big\} dt$$

subject to

$$g(\pi, u) \leqq 0, \quad t \in \Theta$$

$$h(\pi, v) = 0, \quad t \in \Theta$$

$$a(t_0) = a_0, \ a(t_1) = a_1, \ a_\sigma(t_0) = a_{\sigma 0}, \ a_\sigma(t_1) = a_{\sigma 1},$$

or

$$a(t)|_{\partial\Theta} = \text{given}, \quad a_\sigma(t)|_{\partial\Theta} = \text{given},$$

where the functionals f, g and h are given as in the original variational control problem (P).

The associated robust counterpart of the multi-dimensional scalar optimization problem $(P)_{(\bar{a},\bar{c})}$ is defined as:

$$(RP)_{(\bar{a},\bar{c})} \quad \min_{(a(\cdot),c(\cdot))} \int_{\Theta} \max_{w \in W} \Big\{ f_a(\bar{\pi},w)(a(t)-\bar{a}(t)) + f_{a_\sigma}(\bar{\pi},w)(a_\sigma(t)-\bar{a}_\sigma(t))$$

$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},w)(a_{\alpha\beta}(t)-\bar{a}_{\alpha\beta}(t)) + f_c(\bar{\pi},w)(c(t)-\bar{c}(t)) \Big\} dt$$

subject to

$$g(\pi,u) \leqq 0, \quad t \in \Theta,\ \forall u \in U$$

$$h(\pi,v) = 0, \quad t \in \Theta,\ \forall v \in V$$

$$a(t_0) = a_0,\ a(t_1) = a_1,\ a_\sigma(t_0) = a_{\sigma 0},\ a_\sigma(t_1) = a_{\sigma 1},$$

or

$$a(t)|_{\partial\Theta} = \text{given}, \quad a_\sigma(t)|_{\partial\Theta} = \text{given}.$$

Remark 2. *Let us remark that the robust feasible solution set to the problem $(P)_{(\bar{a},\bar{c})}$ is the same as in the problem (P), and, in consequence, it is also denoted by D.*

Definition 4. *A point $(\hat{a},\hat{c}) \in D$ is said to be a weak robust optimal solution to the modified multi-dimensional scalar optimization problem $(P)_{(\bar{a},\bar{c})}$ if there does not exist another point $(a,c) \in D$ such that*

$$\int_{\Theta} \max_{w \in W} \Big[f_a(\bar{\pi},w)(a-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(a_\sigma-\bar{a}_\sigma)$$

$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},w)(a_{\alpha\beta}-\bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(c-\bar{c}) \Big] dt$$

$$< \int_{\Theta} \max_{w \in W} \Big[f_a(\bar{\pi},w)(\hat{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(\hat{a}_\sigma-\bar{a}_\sigma)$$

$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},w)(\hat{a}_{\alpha\beta}-\bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(\hat{c}-\bar{c}) \Big] dt.$$

In the following, we establish some equivalence results between the problems (P) and $(P)_{(\bar{a},\bar{c})}$, under some appropriate convexity assumptions.

Theorem 2. *Let $(\bar{a},\bar{c})$ be a normal weak robust optimal solution to the scalar optimization problem (P). Furthermore, assume that $\int_{\Theta} \bar{v}^T g(\pi,\bar{u}) dt$, $\int_{\Theta} \bar{\gamma}^T h(\pi,\bar{v}) dt$ are convex at $(\bar{a},\bar{c})$. Then, $(\bar{a},\bar{c})$ is also a weak robust optimal solution to the modified multi-dimensional scalar optimization problem $(P)_{(\bar{a},\bar{c})}$.*

Proof. By hypothesis, the relations (1)–(4), with $\bar{\mu} = 1$, are fulfilled for all $t \in \Theta$, except at discontinuities. Hence, the conditions (1) and (2) yield

$$\int_{\Theta} (\tilde{a}-\bar{a}) \{ f_a(\bar{\pi},\bar{w}) + \bar{v}^T g_a(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_a(\bar{\pi},\bar{v}) \tag{5}$$

$$-D_\sigma \Big[f_{a_\sigma}(\bar{\pi},\bar{w}) + \bar{v}^T g_{a_\sigma}(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_{a_\sigma}(\bar{\pi},\bar{v}) \Big]$$

$$+\frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \Big[f_{a_{\alpha\beta}}(\bar{\pi},\bar{w}) + \bar{v}^T g_{a_{\alpha\beta}}(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_{a_{\alpha\beta}}(\bar{\pi},\bar{v}) \Big] \} dt$$

$$+ \int_{\Theta} (\tilde{c}-\bar{c}) \{ f_c(\bar{\pi},\bar{w}) + \bar{v}^T g_c(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_c(\bar{\pi},\bar{v}) \} dt$$

$$= \int_{\Theta} \Big[(\tilde{a}-\bar{a}) \{ f_a(\bar{\pi},\bar{w}) + \bar{v}^T g_a(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_a(\bar{\pi},\bar{v}) \}$$

$$+(\tilde{a}_\sigma - \bar{a}_\sigma)\{f_{a_\sigma}(\bar{\pi},\bar{w}) + \bar{v}^T g_{a_\sigma}(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_{a_\sigma}(\bar{\pi},\bar{v})\}$$
$$+\frac{1}{n(\alpha,\beta)}(\tilde{a}_{\alpha\beta} - \bar{a}_{\alpha\beta})\{f_{a_{\alpha\beta}}(\bar{\pi},\bar{w}) + \bar{v}^T g_{a_{\alpha\beta}}(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_{a_{\alpha\beta}}(\bar{\pi},\bar{v})\}\Big]dt$$
$$+\int_\Theta (\tilde{c} - \bar{c})\{f_c(\bar{\pi},\bar{w}) + \bar{v}^T g_c(\bar{\pi},\bar{u}) + \bar{\gamma}^T h_c(\bar{\pi},\bar{v})\}dt = 0,$$

where we used the formula of integration by parts, the divergence formula and the boundary conditions formulated in the considered problem.

Now, let us proceed by contradiction and consider $(\bar{a},\bar{c})$ is not a weak robust optimal solution to the modified multi-dimensional scalar optimization problem $(P)_{(\bar{a},\bar{c})}$. Then, there exists $(\tilde{a},\tilde{c}) \in D$ such that

$$\int_\Theta \max_{w\in W}\Big[f_a(\bar{\pi},w)(\tilde{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(\tilde{a}_\sigma - \bar{a}_\sigma)$$
$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},w)(\tilde{a}_{\alpha\beta} - \bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(\tilde{c}-\bar{c})\Big]dt$$
$$< \int_\Theta \max_{w\in W}\Big[f_a(\bar{\pi},w)(\bar{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(\bar{a}_\sigma - \bar{a}_\sigma)$$
$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},w)(\bar{a}_{\alpha\beta} - \bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(\bar{c}-\bar{c})\Big]dt.$$

Since $\max_{w\in W} f(\pi,w) = f(\pi,\bar{w})$, we obtain

$$\int_\Theta \Big[f_a(\bar{\pi},\bar{w})(\tilde{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},\bar{w})(\tilde{a}_\sigma - \bar{a}_\sigma)$$
$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\tilde{a}_{\alpha\beta} - \bar{a}_{\alpha\beta}) + f_c(\bar{\pi},\bar{w})(\tilde{c}-\bar{c})\Big]dt$$
$$< \int_\Theta \Big[f_a(\bar{\pi},\bar{w})(\bar{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},\bar{w})(\bar{a}_\sigma - \bar{a}_\sigma)$$
$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\bar{a}_{\alpha\beta} - \bar{a}_{\alpha\beta}) + f_c(\bar{\pi},\bar{w})(\bar{c}-\bar{c})\Big]dt,$$

equivalent with

$$\int_\Theta \Big[f_a(\bar{\pi},\bar{w})(\tilde{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},\bar{w})(\tilde{a}_\sigma - \bar{a}_\sigma) \tag{6}$$
$$+\frac{1}{n(\alpha,\beta)} f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\tilde{a}_{\alpha\beta} - \bar{a}_{\alpha\beta}) + f_c(\bar{\pi},\bar{w})(\tilde{c}-\bar{c})\Big]dt < 0.$$

The robust feasibility of $(\tilde{a},\tilde{c})$ in the problem (P) and the robust necessary efficiency condition (3) yield

$$\int_\Theta \bar{v}^T g(\tilde{\pi},\bar{u})dt - \int_\Theta \bar{v}^T g(\bar{\pi},\bar{u})dt \le 0. \tag{7}$$

By hypothesis, since $\int_\Theta \bar{v}^T g(\pi,\bar{u})dt$ is convex at $(\bar{a},\bar{c})$, we obtain

$$\int_\Theta \Big\{\bar{v}^T g(\tilde{\pi},\bar{u}) - \bar{v}^T g(\bar{\pi},\bar{u})\Big\}dt \ge \int_\Theta (\tilde{a}-\bar{a})\bar{v}^T g_a(\bar{\pi},\bar{u})dt$$
$$+\int_\Theta (\tilde{a}_\sigma - \bar{a}_\sigma)\bar{v}^T g_{a_\sigma}(\bar{\pi},\bar{u})dt$$
$$+\frac{1}{n(\alpha,\beta)}\int_\Theta (\tilde{a}_{\alpha\beta} - \bar{a}_{\alpha\beta})\bar{v}^T g_{a_{\alpha\beta}}(\bar{\pi},\bar{u})dt + \int_\Theta (\tilde{c}-\bar{c})\bar{v}^T g_c(\bar{\pi},\bar{u})dt.$$

The above inequality together with relation (7) yield

$$\int_{\Theta}(\tilde{a}-\bar{a})\bar{\nu}^T g_a(\bar{\pi},\bar{u})dt+\int_{\Theta}(\tilde{a}_{\sigma}-\bar{a}_{\sigma})\bar{\nu}^T g_{a_{\sigma}}(\bar{\pi},\bar{u})dt \tag{8}$$

$$+\frac{1}{n(\alpha,\beta)}\int_{\Theta}(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})\bar{\nu}^T g_{a_{\alpha\beta}}(\bar{\pi},\bar{u})dt+\int_{\Theta}(\tilde{c}-\bar{c})\bar{\nu}^T g_c(\bar{\pi},\bar{u})dt\leq 0.$$

Furthermore, from the assumption, $\int_{\Theta}\bar{\gamma}^T h(\pi,\bar{v})dt$ is convex at $(\bar{a},\bar{c})$ and the robust feasibility of $(\tilde{a},\tilde{c})$ and $(\bar{a},\bar{c})$ in the multi-dimensional scalar optimization problem (P), we obtain

$$\int_{\Theta}(\tilde{a}-\bar{a})\bar{\gamma}^T h_a(\bar{\pi},\bar{v})dt+\int_{\Theta}(\tilde{a}_{\sigma}-\bar{a}_{\sigma})\bar{\gamma}^T h_{a_{\sigma}}(\bar{\pi},\bar{v})dt \tag{9}$$

$$+\frac{1}{n(\alpha,\beta)}\int_{\Theta}(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})\bar{\gamma}^T h_{a_{\alpha\beta}}(\bar{\pi},\bar{v})dt+\int_{\Theta}(\tilde{c}-\bar{c})\bar{\gamma}^T h_c(\bar{\pi},\bar{v})dt\leq 0.$$

On combining the inequalities (6), (8) and (9), we obtain

$$\int_{\Theta}\Big[(\tilde{a}-\bar{a})\{f_a(\bar{\pi},\bar{w})+\bar{\nu}^T g_a(\bar{\pi},\bar{u})+\bar{\gamma}^T h_a(\bar{\pi},\bar{v})\}$$

$$+(\tilde{a}_{\sigma}-\bar{a}_{\sigma})\{f_{a_{\sigma}}(\bar{\pi},\bar{w})+\bar{\nu}^T g_{a_{\sigma}}(\bar{\pi},\bar{u})+\bar{\gamma}^T h_{a_{\sigma}}(\bar{\pi},\bar{v})\}$$

$$+\frac{1}{n(\alpha,\beta)}(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})\{f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})+\bar{\nu}^T g_{a_{\alpha\beta}}(\bar{\pi},\bar{u})+\bar{\gamma}^T h_{a_{\alpha\beta}}(\bar{\pi},\bar{v})\}\Big]dt$$

$$+\int_{\Theta}(\tilde{c}-\bar{c})\{f_c(\bar{\pi},\bar{w})+\bar{\nu}^T g_c(\bar{\pi},\bar{u})+\bar{\gamma}^T h_c(\bar{\pi},\bar{v})\}dt<0,$$

which contradicts the relation (5). Consequently, $(\bar{a},\bar{c})$ is a weak robust optimal solution to the problem $(P)_{(\bar{a},\bar{c})}$, and this completes the proof. □

The next result represents the reciprocal of the above theorem.

Theorem 3. *Let $(\bar{a},\bar{c})$ be a weak robust optimal solution to the modified scalar optimization problem $(P)_{(\bar{a},\bar{c})}$. Furthermore, assume that $\max_{w\in W} f(\pi,w)=f(\pi,\bar{w})$ and the functional $\int_{\Theta} f(\pi,\bar{w})dt$ is convex at $(\bar{a},\bar{c})$. Then, $(\bar{a},\bar{c})$ is a weak robust optimal solution to the multi-dimensional scalar optimization problem (P).*

Proof. Contrary to the result, we assume that $(\bar{a},\bar{c})$ is not a weak robust optimal solution to multi-dimensional scalar optimization problem (P). Then, there exists another point $(\tilde{a},\tilde{c})\in D$ such that

$$\int_{\Theta}\max_{w\in W} f(\tilde{\pi},w)dt<\int_{\Theta}\max_{w\in W} f(\bar{\pi},w)dt.$$

Since $\max_{w\in W} f(\pi,w)=f(\pi,\bar{w})$, we obtain

$$\int_{\Theta} f(\tilde{\pi},\bar{w})dt<\int_{\Theta} f(\bar{\pi},\bar{w})dt. \tag{10}$$

By assumption, since $\int_{\Theta} f(\pi,\bar{w})dt$ is a convex functional at $(\bar{a},\bar{c})$, we have

$$\int_{\Theta}\Big\{f(\tilde{\pi},\bar{w})-f(\bar{\pi},\bar{w})\Big\}dt\geqq\int_{\Theta} f_a(\bar{\pi},\bar{w})(\tilde{a}-\bar{a})dt+\int_{\Theta} f_{a_{\sigma}}(\bar{\pi},\bar{w})(\tilde{a}_{\sigma}-\bar{a}_{\sigma})dt$$

$$+\frac{1}{n(\alpha,\beta)}\int_{\Theta} f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})dt+\int_{\Theta} f_c(\bar{\pi},\bar{w})(\tilde{c}-\bar{c})dt,$$

which along with the inequality (10), yield

$$\int_\Theta f_a(\bar{\pi},\bar{w})(\tilde{a}-\bar{a})dt + \int_\Theta f_{a_\sigma}(\bar{\pi},\bar{w})(\tilde{a}_\sigma-\bar{a}_\sigma)dt$$

$$+\frac{1}{n(\alpha,\beta)}\int_\Theta f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})dt + \int_\Theta f_c(\bar{\pi},\bar{w})(\tilde{c}-\bar{c})dt < 0,$$

equivalently with

$$\int_\Theta f_a(\bar{\pi},\bar{w})(\tilde{a}-\bar{a})dt + \int_\Theta f_{a_\sigma}(\bar{\pi},\bar{w})(\tilde{a}_\sigma-\bar{a}_\sigma)dt$$

$$+\frac{1}{n(\alpha,\beta)}\int_\Theta f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})dt + \int_\Theta f_c(\bar{\pi},\bar{w})(\tilde{c}-\bar{c})dt$$

$$< \int_\Theta f_a(\bar{\pi},\bar{w})(\bar{a}-\bar{a})dt + \int_\Theta f_{a_\sigma}(\bar{\pi},\bar{w})(\bar{a}_\sigma-\bar{a}_\sigma)dt$$

$$+\frac{1}{n(\alpha,\beta)}\int_\Theta f_{a_{\alpha\beta}}(\bar{\pi},\bar{w})(\bar{a}_{\alpha\beta}-\bar{a}_{\alpha\beta})dt + \int_\Theta f_c(\bar{\pi},\bar{w})(\bar{c}-\bar{c})dt.$$

The above inequality can be rewritten as

$$\int_\Theta \max_{w\in W}\Big[f_a(\bar{\pi},w)(\tilde{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(\tilde{a}_\sigma-\bar{a}_\sigma)$$

$$+\frac{1}{n(\alpha,\beta)}f_{a_{\alpha\beta}}(\bar{\pi},w)(\tilde{a}_{\alpha\beta}-\bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(\tilde{c}-\bar{c})\Big]dt$$

$$< \int_\Theta \max_{w\in W}\Big[f_a(\bar{\pi},w)(\bar{a}-\bar{a}) + f_{a_\sigma}(\bar{\pi},w)(\bar{a}_\sigma-\bar{a}_\sigma)$$

$$+\frac{1}{n(\alpha,\beta)}f_{a_{\alpha\beta}}(\bar{\pi},w)(\bar{a}_{\alpha\beta}-\bar{a}_{\alpha\beta}) + f_c(\bar{\pi},w)(\bar{c}-\bar{c})\Big]dt,$$

which contradicts our assumption that $(\bar{a},\bar{c})$ is a weak robust optimal solution to the modified multi-dimensional scalar optimization problem $(P)_{(\bar{a},\bar{c})}$. Hence, the proof is completed. □

Remark 3. *As can be easily seen, to derive the equivalence between the original problem (P) and its associated modified problem $(P)_{(\bar{a},\bar{c})}$, we imposed the convexity assumption only on the constraint functionals; whereas, to establish the converse of Theorem 2, it is sufficient to consider only the convexity hypothesis of the objective functional.*

4. Conclusions

In this paper, for a given class of constrained robust optimization problems, named (P), we have introduced and investigated an auxiliary (modified) robust optimization problem. More specifically, by considering multiple integral cost functionals and mixed constraints involving second-order partial derivatives and data uncertainty, we have formulated and proved some equivalence results for the considered control problems. For this aim, we used the concept of convexity associated with multiple integral functionals and the notion of a normal weak robust optimal solution.

Some developments of the results presented in this paper, which will be investigated in future papers, are provided by the study of the saddle-point optimality criteria and the associated (Wolfe, Mond-Weir, mixed) duality theory for the considered class of control problems.

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