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Article

Gottlieb Polynomials and Their q-Extensions

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Abstract: Since Gottlieb introduced and investigated the so-called Gottlieb polynomials in 1938, which are discrete orthogonal polynomials, many researchers have investigated these polynomials from diverse angles. In this paper, we aimed to investigate the q-extensions of these polynomials to provide certain q-generating functions for three sequences associated with a finite power series whose coefficients are products of the known q-extended multivariable and multiparameter Gottlieb polynomials and another non-vanishing multivariable function. Furthermore, numerous possible particular cases of our main identities are considered. Finally, we return to Khan and Asif's q-Gottlieb polynomials to highlight certain connections with several other known q-polynomials, and provide its q-integral representation. Furthermore, we conclude this paper by disclosing our future investigation plan.

Keywords: Gottlieb polynomials in several variables; q-Gottlieb polynomials in several variables; generating functions; generalized and generalized basic (or -q) hypergeometric function; Lauricella's multiple hypergeometric series in several variables; q-binomial theorem; q-exponential functions; q-calculus; q-Jacobi polynomials; q-Meixner polynomials

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1. Introduction and Preliminaries

Morris J. Gottlieb [1] introduced and investigated certain discrete orthogonal polynomials, i.e., the so-called Gottlieb polynomials, obtained by

$$\begin{aligned}\varphi_m(x;\eta) &= e^{-m\eta}\sum_{j=0}^{m}\binom{m}{j}\binom{x}{j}(1-e^{\eta})^j \\ &= e^{-m\eta}\,{}_2F_1[-m,-x;1;1-e^{\eta}],\end{aligned} \tag{1}$$

where $m \in \mathbb{N}_0$ and ${}_2F_1$ denote Gauss's hypergeometric series. Here and elsewhere, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ stand for the sets of complex numbers, integers, and positive integers, respectively, and also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_{\leq 0} := \mathbb{Z} \setminus \mathbb{N}$. A natural generalization of the ${}_2F_1$ in (1) is the generalized hypergeometric series ${}_uF_v$ $(u,\, v \in \mathbb{N}_0)$ with the v denominator and u numerator parameters defined by

$$\begin{aligned}{}_uF_v\left[\begin{matrix}\kappa_1,\ldots,\kappa_u\,; \\ \varrho_1,\ldots,\varrho_v\,;\end{matrix}\; z\right] &:= \sum_{\ell=0}^{\infty}\frac{(\kappa_1)_\ell\cdots(\kappa_u)_\ell}{(\varrho_1)_\ell\cdots(\varrho_v)_\ell}\frac{z^\ell}{\ell!} \\ &= {}_uF_v(\kappa_1,\ldots,\kappa_u;\varrho_1,\ldots,\varrho_v;z).\end{aligned} \tag{2}$$

Here, $(\kappa)_\eta$ $(\kappa,\, \eta \in \mathbb{C})$ indicates the Pochhammer symbol obtained through the use of the Gamma function Γ (see, e.g., [2] p. 2, 5), by

$$
\begin{aligned}
(\kappa)_\eta &= \frac{\Gamma(\kappa+\eta)}{\Gamma(\kappa)} \quad \left(\kappa+\eta \in \mathbb{C}\setminus\mathbb{Z}_0^-,\, \eta\in\mathbb{C}\setminus\{0\};\, \kappa\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}\right) \\
&= \begin{cases} 1 & (\eta=0) \\ \kappa(\kappa+1)\cdots(\kappa+n-1) & (\eta=n\in\mathbb{N}) \end{cases}
\end{aligned} \tag{3}
$$

where it is understood that $(0)_0 = 1$.

Gottlieb [1] offered a number of intriguing properties and formulas for the polynomials $\varphi_n(u;\xi)$, which were indicated by $l_n(u)$ in [1] and [3] (pp. 185–186). The following generating functions are known (as can be seen, e.g., in ([3] pp. 185–186), [4,5]):

$$
\sum_{n=0}^{\infty} \varphi_n(u;\xi)t^n = (1-t)^u\left(1-te^{-\xi}\right)^{-u-1} \qquad (|t|<1); \tag{4}
$$

$$
\sum_{n=0}^{\infty} \varphi_n(u;\xi)\frac{t^n}{n!} = e^t\, {}_1F_1\left[\begin{matrix} u+1\,; \\ 1\,; \end{matrix}\; -\left(1-e^{-\xi}\right)t\right]; \tag{5}
$$

$$
\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}\varphi_n(u;\xi)t^n = \left(1-te^{-\xi}\right)^{-\mu} {}_2F_1\left[\begin{matrix} \mu,\; -u\,; \\ 1\,; \end{matrix}\; \frac{\left(1-e^{-\xi}\right)t}{1-te^{-\xi}}\right]. \tag{6}
$$

Khan and Akhlaq [6] presented and studied the Gottlieb polynomials of two and three variables to afford their generating functions. Choi [7] modified Khan and Akhlaq's technique to provide the following generalization of the η variable and η parameter Gottlieb polynomials:

$$
\begin{aligned}
\varphi_n^{\eta}(u_1,\ldots,u_\eta;\rho_1,\ldots,\rho_\eta) &= \exp(-n\sigma_\eta) \\
&\times \sum_{\tau_1=0}^{n}\sum_{\tau_2=0}^{n-\tau_1}\sum_{\tau_3=0}^{n-\tau_1-\tau_2}\cdots\sum_{\tau_\eta=0}^{n-\tau_1-\tau_2-\cdots-\tau_{\eta-1}} \\
&\times \frac{(-n)_{\delta_\eta}\prod_{j=1}^{\eta}\left(-u_j\right)_{\tau_j}\prod_{j=1}^{\eta}\left(1-e^{\rho_j}\right)^{\tau_j}}{\delta_\eta!\prod_{j=1}^{\eta}\left(\tau_j\right),!}
\end{aligned} \tag{7}
$$

where $\eta\in\mathbb{N},\;\; n\in\mathbb{N}_0\;$ and:

$$
\sigma_\eta := \sum_{j=1}^{\eta}\rho_j \qquad \text{and} \qquad \delta_\eta := \sum_{j=1}^{\eta}\tau_j. \tag{8}
$$

Then, Choi [7] presented the following two generating functions:

$$
\sum_{n=0}^{\infty}\varphi_n^{\eta}(u_1,\ldots,u_\eta;\rho_1,\ldots,\rho_\eta)t^n = \left(1-te^{-\sigma_\eta}\right)^{-1-\sum_{j=1}^{\eta}u_j}\prod_{j=1}^{\eta}\left(1-te^{\rho_j-\sigma_\eta}\right)^{u_j} \tag{9}
$$

and:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\varphi_n^{\eta}(u_1,\ldots,u_\eta;\rho_1,\ldots,\rho_\eta)t^n = \left(1-te^{-\sigma_\eta}\right)^{-\mu} \\
&\times F_D^{(\eta)}\left[\mu,-u_1,\ldots,-u_\eta;1;\frac{\left(e^{\rho_1}-1\right)te^{-\sigma_\eta}}{1-te^{-\sigma_\eta}},\ldots,\frac{\left(e^{\rho_\eta}-1\right)te^{-\sigma_\eta}}{1-te^{-\sigma_\eta}}\right],
\end{aligned} \tag{10}
$$

where $F_D^{(\eta)}[\cdot]$ denotes Lauricella's multiple hypergeometric series in η variables defined by (see, e.g., [8] p. 33, Equation (4)):

$$\begin{aligned} &F_D^{(\eta)}[a, b_1, \ldots, b_\eta; c; u_1, \ldots, u_\eta] \\ &\quad = \sum_{\tau_1, \tau_2, \ldots, \tau_\eta = 0}^{\infty} \frac{(a)_{\delta_\eta} (b_1)_{\tau_1} \cdots (b_\eta)_{\tau_\eta}}{(c)_{\delta_\eta}} \frac{u_1^{\tau_1}}{\tau_1!} \cdots \frac{u_\eta^{\tau_\eta}}{\tau_\eta!} \\ &\qquad \left(\max\{|u_1|, \ldots, |u_\eta|\} < 1\right), \end{aligned} \tag{11}$$

and δ_η is the same as in (8). Some other properties of univariate and multivariate Gottlieb polynomials have recently been provided (see [9,10]).

A great surge in activities of q-extensions (or analogues) of polynomials, series, identities, functions, and their related theories in association with q-calculus (quantum calculus) has recently been observed in a variety of research areas of, for example, pure and applied mathematics, physics, and engineering. Some q-notations and q-identities are recalled (see, e.g., ([2] Section 6), [11,12]). The q-shifted factorial $(\lambda; q)_m$ is given by

$$(\lambda; q)_m := \begin{cases} 1 & (m = 0) \\ \prod_{j=0}^{m-1} (1 - \lambda q^j) & (m \in \mathbb{N}), \end{cases} \tag{12}$$

where λ, $q \in \mathbb{C}$ are such that $\lambda \neq q^{-\ell}$ $(\ell \in \mathbb{N}_0)$. The q-shifted factorial for a non-positive integer subscript is given by

$$(\lambda; q)_{-m} := \frac{1}{\prod_{j=1}^{m} (1 - \lambda q^{-j})} \quad (m \in \mathbb{N}_0), \tag{13}$$

which gives:

$$(\lambda; q)_{-m} = \frac{1}{(\lambda q^{-m}; q)_m} = \frac{(-q/\lambda)^m \, q^{\binom{m}{2}}}{(q/\lambda; q)_m} \quad (m \in \mathbb{N}_0). \tag{14}$$

One also defines:

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda q^j\right) \quad (\lambda,\, q \in \mathbb{C},\, |q| < 1). \tag{15}$$

One finds from (12), (13) and (15) that:

$$(\lambda; q)_m = \frac{(\lambda; q)_\infty}{(\lambda\, q^m; q)_\infty} \quad (m \in \mathbb{Z}), \tag{16}$$

which is able to be generalized to $m = \mu \in \mathbb{C}$:

$$(\lambda; q)_\mu = \frac{(\lambda; q)_\infty}{(\lambda\, q^\mu; q)_\infty} \quad (\mu \in \mathbb{C},\, |q| < 1), \tag{17}$$

where the multiple-valued q^μ is assumed to take its principal value. One can easily see that:

$$\lim_{q \to 1} \frac{(q^\lambda; q)_m}{(q^\mu; q)_m} = \frac{(\lambda)_m}{(\mu)_m} \quad (m \in \mathbb{N}_0,\ \lambda \in \mathbb{C},\ \ \mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \tag{18}$$

A q-analogue of the generalized hypergeometric series ${}_uF_v$ in (2) is given by

$$\begin{aligned} {}_u\phi_v \left[\begin{matrix} \kappa_1, \ldots, \kappa_u\, ; \\ \varrho_1, \ldots, \varrho_v\, ; \end{matrix} \; q,\, z \right] &= {}_u\phi_v(\kappa_1, \ldots, \kappa_u; \varrho_1, \ldots, \varrho_v; z) \\ &:= \sum_{\ell=0}^{\infty} (-1)^{(1-u+v)\ell} q^{(1-u+v)\binom{\ell}{2}} \frac{(\kappa_1; q)_\ell \cdots (\kappa_u; q)_\ell}{(\varrho_1; q)_\ell \cdots (\varrho_v; q)_\ell} \frac{z^\ell}{(q; q)_\ell}, \end{aligned} \tag{19}$$

provided that the series converges. It is noted that:

$$(q^{-m};q)_\ell = \begin{cases} \frac{(q;q)_m}{(q;q)_{m-\ell}}(-1)^\ell q^{\binom{\ell}{2}-m\ell} & (m,\ \ell \in \mathbb{N}_0,\ 0 \leq \ell \leq m) \\ 0 & (m,\ \ell \in \mathbb{N}_0,\ m \geq \ell+1). \end{cases} \tag{20}$$

The use of (20) in (19) would terminate and yield a polynomial of degree m in z whenever any of the numerator parameters $\kappa_1, \ldots, \kappa_u$ is of the form q^{-m} $(m \in \mathbb{N}_0)$.

The notation $[z]_q$ is defined by

$$[z]_q := \frac{1-q^z}{1-q} \quad (q \in \mathbb{C} \setminus \{1\},\ z \in \mathbb{C},\ q^z \neq 1). \tag{21}$$

The case $z = m \in \mathbb{N}$ of (21) gives the q-extension (or q-analogue) of $m \in \mathbb{N}$:

$$[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1} \quad (m \in \mathbb{N}), \tag{22}$$

since:

$$\lim_{q \to 1} [m]_q = m.$$

The q-analogue of $m!$ is defined by

$$[m]_q! := \begin{cases} 1 & (m = 0), \\ \prod_{k=1}^{m} [k]_q & (m \in \mathbb{N}). \end{cases} \tag{23}$$

From (12) and (23), we have:

$$(q;q)_m = (1-q)^m\, [m]_q! \quad (m \in \mathbb{N}_0). \tag{24}$$

The Gaussian polynomial analogous to $\binom{m}{\ell}$ (or the q-binomial coefficient) is given by

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{[m]_q!}{[\ell]_q![m-\ell]_q!} = \frac{(q;q)_m}{(q;q)_\ell\,(q;q)_{m-\ell}} \quad (m,\ \ell \in \mathbb{N}_0,\ 0 \leq \ell \leq m), \tag{25}$$

which can be generalized as follows:

$$\begin{bmatrix} \mu \\ \ell \end{bmatrix}_q := \frac{[\mu]_q[\mu-1]_q \cdots [\mu-\ell+1]_q}{[\ell]_q!} \quad (\mu \in \mathbb{C},\ \ell \in \mathbb{N}_0). \tag{26}$$

The generalized q-binomial coefficient (26) is the q-extension of the generalized binomial coefficient:

$$\binom{\mu}{\ell} = \frac{\mu(\mu-1)\cdots(\mu-\ell+1)}{\ell!} = \frac{(-1)^\ell\,(-\mu)_\ell}{\ell!} \quad (\mu \in \mathbb{C},\ \ell \in \mathbb{N}_0).$$

It follows from (24) and (26) that:

$$\begin{bmatrix} \mu \\ \ell \end{bmatrix}_q = \frac{\left(q^{\mu-\ell};q\right)_\ell}{(q;q)_\ell} = \frac{(q^{-\mu};q)_\ell}{(q;q)_\ell}(-q^\mu)^\ell q^{-\binom{\ell}{2}} \quad (\mu \in \mathbb{C},\ \ell \in \mathbb{N}_0). \tag{27}$$

The q-binomial theorem is given by

$${}_1\phi_0(\lambda;-;q,z) = \sum_{k=0}^{\infty} \frac{(\lambda;q)_k}{(q;q)_k} z^k = \frac{(\lambda z;q)_\infty}{(z;q)_\infty} \quad (|q| < 1,\ |z| < 1). \tag{28}$$

A q-extension of the classical exponential function e^z is given by

$$e_q(z) := \sum_{\ell=0}^{\infty} \frac{z^\ell}{[\ell]_q!} = \sum_{\ell=0}^{\infty} \frac{[(1-q)z]^\ell}{(q;q)_\ell} = \frac{1}{((1-q)z;q)_\infty} \quad \left(|z| < \frac{1}{1-q}\right), \tag{29}$$

and another q-extension of the classical exponential function e^z is given by

$$E_q(z) := \sum_{\ell=0}^{\infty} q^{\binom{\ell}{2}} \frac{z^\ell}{[\ell]_q!} = \sum_{\ell=0}^{\infty} q^{\binom{\ell}{2}} \frac{[(1-q)z]^\ell}{(q;q)_\ell} = (-(1-q)z;q)_\infty \quad (|z| < \infty). \tag{30}$$

The q-exponential functions are related as follows:

$$e_q(-z)E_q(z) = e_q(z)E_q(-z) = 1, \tag{31}$$

and:

$$e_{1/q}(z) = E_q(z). \tag{32}$$

F. H. Jackson [13] may be accepted as the first systematic developer of q-calculus. The q-derivative of a function $f(u)$ is given by

$$D_q\{f(u)\} := \frac{d_q f(u)}{d_q u} = \frac{f(qu) - f(u)}{(q-1)u}. \tag{33}$$

Obviously:

$$\lim_{q \to 1} D_q\{f(u)\} = \frac{d}{du}\{f(u)\},$$

if $f(u)$ is differentiable. Suppose that $0 < a < b$. The (Jackson's) definite q-integral (see also J. Thomae [14]) is defined as follows (see, e.g., [2] Chapter 6, [13,15] Section 19):

$$\int_0^b f(u)\, d_q u = (1-q) \sum_{j=0}^{\infty} q^j b\, f\left(q^j b\right) \tag{34}$$

and:

$$\int_a^b f(u)\, d_q u = \int_0^b f(u)\, d_q u - \int_0^a f(u)\, d_q u. \tag{35}$$

Recall the q-Erkus–Srivastava polynomials $u_{n,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$ which are generated by (see [16])

$$\begin{gathered} \sum_{n=0}^{\infty} u_{n,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)t^n = \prod_{j=1}^{r} \frac{1}{\left(x_j t^{m_j};q\right)_{\alpha_j}} \\ \left(|t| < \min\left\{|x_1|^{-1/m_1},\ldots,|x_r|^{-1/m_r}\right\}\right). \end{gathered} \tag{36}$$

Here, $r \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_r \in \mathbb{C}$, and $m_1,\ldots,m_r \in \mathbb{N}$.

Recently, Khan and Asif [17] derived certain interesting generating functions for a q-extension of the Gottlieb polynomials. Subsequently, Choi [18,19] investigated several generating functions for the q-extensions of the two and three-variable Gottlieb polynomials. In the sequel, Choi and Srivastava [20] introduced and investigated a q-analogue of a multiparameter and multivariable extension of the Gottlieb polynomials, which is recalled in Section 2.

In this paper, by using a similar method as those in [9,10,21], we investigate certain q-generating functions for three sequences associated with finite power series whose coefficients are products of the q-extended multivariable and multiparameter Gottlieb polynomials $\varphi_{n,q}^{m}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ in (37) and another non-vanishing multivariable function, which are asserted in Theorems 1–3. Furthermore, several numerous possible particular cases of the main identities in Theorems 1–3 are considered in Section 4. Finally, we return to Khan and Asif's q-Gottlieb polynomials $\varphi_{n,q}(x;\lambda)$ in (63) to give certain connections with several other q-polynomials, and provide its q-integral representation. We close this paper by disclosing our future investigation plan.

2. q-Extension of the Gottlieb Polynomials in Several Variables

In this section, for an easier reference of our main results in the next section, we recall the q-extension of the multivariable and multiparameter Gottlieb polynomials $\varphi_n^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ in (7), which is given in [20].

Definition 1. *(See [20]) A q-extension of the generalized (multivariable and multiparameter) Gottlieb polynomials $\varphi_n^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ is defined as follows:*

$$\begin{aligned}\varphi_{n,q}^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m) &:= \left(\prod_{l=1}^m E_q(-\lambda_l)\right)^n \\ &\times \sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}\cdots\sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}} \begin{bmatrix} n \\ \delta_m \end{bmatrix}_q \\ &\times \prod_{j=1}^m \begin{bmatrix} x_j \\ r_j \end{bmatrix}_q q^{\sum_{j=1}^m \binom{r_j}{2} - \sum_{j=1}^m x_j r_j} \prod_{j=1}^m [1-e_q(\lambda_j)]^{r_j},\end{aligned} \tag{37}$$

where $m\in\mathbb{N},\ n\in\mathbb{N}_0$. By using (20) and (25), this definition can be expressed in the following form:

$$\begin{aligned}\varphi_{n,q}^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m) &= \left(\prod_{l=1}^m E_q(-\lambda_l)\right)^n \\ &\times \sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}\cdots\sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}} q^{-\binom{\delta_m}{2}} \frac{(q^{-n};q)_{\delta_m}}{(q;q)_{\delta_m}} \\ &\times \prod_{j=1}^m \frac{(q^{-x_j};q)_{r_j}}{(q;q)_{r_j}} \prod_{j=1}^m \{[1-e_q(\lambda_j)]q^n\}^{r_j},\end{aligned} \tag{38}$$

where δ_m is the same as in (8).

By using some of the q-identities, Choi and Srivastava [20] derived a set of three generating functions for $\varphi_{n,q}^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ in terms of Srivastava's general basic (or q-) hypergeometric series in several variables defined by (see [8] p. 350, Equation (284)):

$$\begin{aligned}&\Phi_{G:H';\cdots;H^{(m)}}^{E:F';\cdots;F^{(m)}}\left(\begin{matrix}\left[(e):\vartheta',\ldots,\vartheta^{(m)}\right]:\ [(f'):\phi'];\ \cdots;\ \left[(f^{(m)}):\phi^{(m)}\right]; \\ \left[(g):\psi^{(1)},\ldots,\psi^{(m)}\right]:\ [(h'):\delta'];\ \cdots;\ \left[(h^{(m)}):\delta^{(m)}\right];\end{matrix}\quad q;\, w_1,\ldots,w_m\right) \\ &= \sum_{r_1,\ldots,r_m=0}^{\infty}\Omega(r_1,\ldots,r_m)\frac{w_1^{r_1}}{(q;q)_{r_1}}\cdots\frac{w_m^{r_m}}{(q;q)_{r_m}},\end{aligned} \tag{39}$$

where for convenience:

$$\Omega(r_1,\ldots,r_m) := \frac{\prod_{\ell=1}^{E}(e_\ell;q)_{r_1\vartheta'_\ell+\cdots+r_m\vartheta_\ell^{(m)}}\prod_{\ell=1}^{F'}(f'_\ell;q)_{r_1\phi'_\ell}\cdots\prod_{\ell=1}^{F^{(m)}}\left(f_\ell^{(m)};q\right)_{r_m\phi_\ell^{(m)}}}{\prod_{\ell=1}^{G}(g_\ell;q)_{r_1\psi'_\ell+\cdots+r_m\psi_\ell^{(m)}}\prod_{\ell=1}^{H'}(h'_\ell;q)_{r_1\delta'_\ell}\cdots\prod_{\ell=1}^{H^{(m)}}\left(h_\ell^{(m)};q\right)_{r_m\delta_\ell^{(m)}}},$$

the (real or complex) arguments $w_1,\ldots,w_m$, the complex parameters:

$$\begin{cases} e_\ell\quad (\ell=1,\ldots,E);\quad f_\ell^{(j)}\quad \left(\ell=1,\ldots,F^{(j)}\right); \\ g_\ell\quad (\ell=1,\ldots,G);\quad h_\ell^{(j)}\quad \left(\ell=1,\ldots,H^{(j)}\right);\quad (j=1,\ldots,m)\end{cases}$$

and the combined coefficients:

$$\begin{cases} \vartheta_\ell^{(j)} \quad (\ell = 1, \ldots, E); \quad \phi_\ell^{(j)} \quad \left(\ell = 1, \ldots, F^{(j)}\right); \\ \psi_\ell^{(j)} \quad (\ell = 1, \ldots, G); \quad \delta_\ell^{(j)} \quad \left(\ell = 1, \ldots, H^{(j)}\right); \quad (j = 1, \ldots, m) \end{cases}$$

are so constrained that the multiple q-series in (39) converges.

Choi and Srivastava [20] presented each of the following q-generating functions for the multiparameter and multivariable polynomials $\varphi_{n,q}^{m}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m)$ in (37):

$$\begin{aligned} \sum_{n=0}^{\infty} \varphi_{n,q}^{m}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m) t^n &= \frac{\left(qt \prod_{l=1}^{m} E_q(-\lambda_l); q\right)_\infty}{\left(t \prod_{l=1}^{m} E_q(-\lambda_l); q\right)_\infty} \\ &\times \Phi_{1:0;\cdots;0}^{0:1;\cdots;1}\left(\begin{array}{c} \overline{\qquad\qquad} : \quad [q^{-x_1} : 1]; \\ \left[qt \prod_{j=1}^{m} E_q(-\lambda_j) : 1, \ldots, 1\right] : \quad \overline{\qquad} ; \end{array} \right. \\ &\qquad \left. \begin{array}{cc} \cdots; & [q^{-x_m} : 1]; \\ \cdots; & \overline{\qquad}; \end{array} q; \Xi_1, \ldots, \Xi_m \right), \end{aligned} \tag{40}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \varphi_{n,q}^{m}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m) \frac{t^n}{(q;q)_n} &= \frac{1}{\left(t \prod_{l=1}^{m} E_q(-\lambda_l); q\right)_\infty} \\ \times \Phi_{1:0;\ldots;0}^{0:1;\ldots;1}\left(\begin{array}{c} \overline{\qquad} : \quad [q^{-x_1} : 1]; \cdots; [q^{-x_m} : 1]; \\ [q : 1, \ldots, 1] : \quad \overline{\qquad}; \cdots; \overline{\qquad}; \end{array} q; \Xi_1, \ldots, \Xi_m \right) \end{aligned} \tag{41}$$

and:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q^\mu;q)_n}{(q;q)_n} \varphi_{n,q}^{m}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m) t^n &= \frac{\left(q^\mu t \prod_{l=1}^{m} E_q(-\lambda_l); q\right)_\infty}{\left(t \prod_{l=1}^{m} E_q(-\lambda_l); q\right)_\infty} \\ \times \Phi_{2:0;\ldots;0}^{1:1;\ldots;1}\left(\begin{array}{c} [q^\mu : 1, \ldots, 1] : \\ [q : 1, \ldots, 1], \left[q^\mu t \prod_{j=1}^{m} E_q(-\lambda_j) : 1, \ldots, 1\right] : \end{array} \right. \\ \left. \begin{array}{ccc} [q^{-x_1} : 1]; & \cdots; & [q^{-x_m} : 1]; \\ \overline{\qquad}; & \cdots; & \overline{\qquad}; \end{array} q; \Xi_1, \ldots, \Xi_m \right), \end{aligned} \tag{42}$$

where:

$$\begin{cases} \Xi_1 := \left(\prod_{l=2}^{m} E_q(-\lambda_l)\right) [1 - E_q(-\lambda_1)] t \\ \Xi_2 := \left(\prod_{\substack{1 \leq l \leq m \\ l \neq 2}} E_q(-\lambda_l)\right) [1 - E_q(-\lambda_2)] t \\ \quad \vdots \\ \Xi_m := \left(\prod_{l=1}^{m-1} E_q(-\lambda_l)\right) [1 - E_q(-\lambda_m)] t, \end{cases} \tag{43}$$

provided that both sides of each of the assertions (40)–(42) exist. It is understood in (43) that an empty product is to be interpreted as 1.

3. Main Results

In this section, by employing a method similar to those in [9,10,21], we present certain q-generating functions for three sequences associated with finite power series whose coefficients are products of the q-extended multivariable and multiparameter Gottlieb polynomials $\varphi^{m}_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ in (37) and another non-vanishing multivariable function, which are asserted in the following theorems.

Theorem 1. *Let η, ν, τ, $\psi \in \mathbb{C}$. Furthermore, let $x_1,\ldots,x_m$, $\lambda_1,\ldots,\lambda_m \in \mathbb{C}$ $(m \in \mathbb{N})$ and $\xi_1,\ldots,\xi_r \in \mathbb{C}$ $(r \in \mathbb{N})$. Furthermore, let $\varphi^{m}_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ be given in (37). Conforming to an identically nonzero function $\Omega_\nu(\xi_1,\ldots,\xi_r)$ of order ν and of r variables $\xi_1,\ldots,\xi_r$, set:*

$$\Lambda_{\nu,\psi}(\xi_1,\ldots,\xi_r;\tau) := \sum_{k=0}^{\infty} a_k \Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\tau^k \tag{44}$$

$$(a_k \in \mathbb{C}\setminus\{0\} \text{ for } k \in \mathbb{N}_0)$$

and:

$$\begin{aligned} &R^{\nu,\psi}_{n,p;q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\eta) \\ &:= \sum_{k=0}^{[n/p]} a_k \varphi^{m}_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\,\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\,\eta^k \end{aligned} \tag{45}$$

$$(n \in \mathbb{N}_0,\ p \in \mathbb{N}).$$

Then, the sequence (45) is generated as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} R^{\nu,\psi}_{n,p;q}\left(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\frac{\eta}{t^p}\right)t^n &= \frac{\left(qt\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty}{\left(t\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty} \\ \times\,\Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{matrix} \rule{3cm}{0.4pt} \\ \left[qt\prod_{j=1}^{m}E_q(-\lambda_j):1,\ldots,1\right] \end{matrix} \begin{matrix} : [q^{-x_1}:1];\ \cdots; \\ : \rule{1cm}{0.4pt}\ ;\ \cdots; \end{matrix}\right. & \\ \left.\begin{matrix} [q^{-x_m}:1]; \\ \rule{1cm}{0.4pt}\ ; \end{matrix}\ q;\Xi_1,\ldots,\Xi_m\right)\Lambda_{\nu,\psi}(\xi_1,\ldots,\xi_r;\eta),& \end{aligned} \tag{46}$$

provided both sides of (46) exist.

Proof. Put $\mathcal{L}$ to indicate the left-hand side of the statement (46). Then, we have:

$$\mathcal{L} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \varphi^{m}_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k t^{n-pk}. \tag{47}$$

Recall a series rearrangement technique (see, e.g., [22]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \mathscr{A}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathscr{A}(k,n+pk), \tag{48}$$

where $\mathscr{A} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ is a bounded function such that the involved double series is absolutely convergent. Using (48) in (47) would lead to:

$$\begin{aligned} \mathcal{L} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \varphi^{m}_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} \varphi^{m}_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)t^n \sum_{k=0}^{\infty} a_k \Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k. \end{aligned}$$

Finally, by applying (40) and (44) to the last two summations, we obtain the right member of (46). $\square$

Theorem 2. *Enable all of the constraints in* Theorem 1 *including the polynomials and the functions be assumed. Furthermore, let:*

$$\begin{aligned} &S^{\nu,\psi}_{n,p;q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\eta) \\ &\quad := \sum_{k=0}^{[n/p]} \frac{a_k}{(q;q)_{n-pk}} \varphi^m_{n-pk,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r)\,\eta^k \end{aligned} \tag{49}$$

$$(n \in \mathbb{N}_0,\ p \in \mathbb{N}).$$

Then, sequence (49) is generated as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} S^{\nu,\psi}_{n,p;q}\left(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\frac{\eta}{t^p}\right)t^n = \frac{1}{\left(t\prod_{l=1}^{m} E_q(-\lambda_l);q\right)_\infty} \\ &\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{matrix} \rule{2em}{0.4pt} : [q^{-x_1}:1];\ \cdots; \\ [q:1,\dots,1]: \rule{2em}{0.4pt};\ \cdots; \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} [q^{-x_m}:1]; \\ \rule{2em}{0.4pt}; \end{matrix}\ q;\Xi_1,\dots,\Xi_m\right)\Lambda_{\nu,\psi}(\xi_1,\dots,\xi_r;\eta), \end{aligned} \tag{50}$$

provided both sides of (50) exist.

Proof. Let $\mathcal{M}$ denote the left member of the assertion (50). Then, using (48), we obtain:

$$\begin{aligned} \mathcal{M} &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \frac{a_k}{(q;q)_{n-pk}} \varphi^m_{n-pk,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r)\eta^k t^{n-pk} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a_k}{(q;q)_n} \varphi^m_{n,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} \varphi^m_{n,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty} a_k\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r)\eta^k. \end{aligned}$$

Now, employing (41) and (44) to the last two summations leads to the right member of (50). This completes the proof. □

Theorem 3. *Let all of the constraints in* Theorem 1 *including the polynomials and the functions be assumed. Furthermore, let:*

$$\begin{aligned} &T^{\nu,\psi}_{n,p;q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\eta) \\ &\quad := \sum_{k=0}^{[n/p]} a_k \frac{(q^\mu;q)_{n-pk}}{(q;q)_{n-pk}} \varphi^m_{n-pk,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r)\,\eta^k. \end{aligned} \tag{51}$$

$$(n \in \mathbb{N}_0,\ p \in \mathbb{N}).$$

Then, the sequence (51) is generated as follows:

$$\sum_{n=0}^{\infty} T^{\nu,\psi}_{n,p;q}\left(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\frac{\eta}{t^p}\right)t^n = \frac{\left(q^{\mu}t\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_{\infty}}{\left(t\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_{\infty}}$$
$$\times\,\Phi^{1:1;\ldots;1}_{2:0;\ldots;0}\left(\begin{array}{c} [q^{\mu}:1,\ldots,1]: \;\; [q^{-x_1}:1];\;\ldots;\;[q^{-x_m}:1]; \\ [q:1,\ldots,1],\left[q^{\mu}t\prod_{j=1}^{m}E_q(-\lambda_j):1,\ldots,1\right]: \;\; \text{———};\;\ldots;\;\text{———}; \end{array} q;\Xi_1,\ldots,\Xi_m\right)\Lambda_{\nu,\psi}(\xi_1,\ldots,\xi_r;\eta), \tag{52}$$

provided both sides of (52) exist.

Proof. Let $\mathcal{N}$ stand for the left-hand side of the assertion (52). Then, employing (48), we obtain:

$$\begin{aligned}\mathcal{N} &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k\frac{(q^{\mu};q)_{n-pk}}{(q;q)_{n-pk}}\varphi^m_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k t^{n-pk} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\frac{(q^{\mu};q)_{n}}{(q;q)_{n}}\varphi^m_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k t^{n} \\ &= \sum_{n=0}^{\infty}\frac{(q^{\mu};q)_{n}}{(q;q)_{n}}\varphi^m_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)t^n\sum_{k=0}^{\infty}a_k\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r)\eta^k.\end{aligned}$$

Finally, applying (42) and (44) to the last two summations, we obtain the right member of (52). The proof is complete. □

4. Particular Cases

In this section, among numerous possible particular cases of Theorems 1 and 2, we choose to only demonstrate several ones.

Setting:

$$\Omega_{\nu+\psi k}(\xi_1,\ldots,\xi_r) = \varphi^r_{\nu+\psi k,q}(\xi_1,\ldots,\xi_r;\lambda_1,\ldots,\lambda_r) \quad (k\in\mathbb{N}_0,\ r\in\mathbb{N})$$

in Theorem 1, we obtain a generating relation in the following corollary.

Corollary 1. *Let all of the constraints in* Theorem 1 *including the polynomials and the functions be assumed. Furthermore, let:*

$$\Lambda^{(1)}_{\nu,\psi}(\xi_1,\ldots,\xi_r;\tau) := \sum_{k=0}^{\infty}a_k\varphi^r_{\nu+\psi k,q}(\xi_1,\ldots,\xi_r;\lambda_1,\ldots,\lambda_r)\tau^k \tag{53}$$

and:

$$\begin{aligned}&{}_{(1)}R^{\nu,\psi}_{n,p;q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\eta) \\ &:= \sum_{k=0}^{[n/p]}a_k\varphi^m_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\,\varphi^r_{\nu+\psi k,q}(\xi_1,\ldots,\xi_r;\lambda_1,\ldots,\lambda_r)\,\eta^k\end{aligned} \tag{54}$$

Then, the sequence (54) is generated as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{(1)}R^{\nu,\psi}_{n,p;q}\left(x_1,...,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\frac{\eta}{t^p}\right)t^n = \frac{\left(qt\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty}{\left(t\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty} \\
&\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{array}{l} \overline{} : [q^{-x_1}:1];\ \cdots; \\ \left[qt\prod_{j=1}^{m}E_q(-\lambda_j):1,\dots,1\right] : \overline{};\ \cdots; \end{array}\right. \\
&\qquad\left.\begin{array}{l}[q^{-x_m}:1]; \\ \overline{};\end{array}\ q;\Xi_1,\dots,\Xi_m\right)\Lambda^{(1)}_{\nu,\psi}(\xi_1,\dots,\xi_r;\eta),
\end{aligned} \tag{55}$$

provided that each member of (55) exists.

Furthermore, putting $a_k = 1$, $\nu = 0$, $\psi = 1$ in Corollary 1, in terms of (40), we obtain the subsequent result.

Corollary 2. *Let all of the constraints in* Theorem 1, *including the polynomials and the functions, be assumed. Furthermore, let:*

$$\begin{aligned}
&{}_{(2)}R_{n,p;q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\eta) \\
&:= \sum_{k=0}^{[n/p]} \varphi^m_{n-pk,q}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\,\varphi^r_{k,q}(\xi_1,\dots,\xi_r;\lambda_1,\dots,\lambda_r)\,\eta^k
\end{aligned} \tag{56}$$

$$(n \in \mathbb{N}_0,\ p \in \mathbb{N}).$$

Then, the sequence (56) is generated as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{(2)}R_{n,p;q}\left(x_1,...,x_m;\lambda_1,\dots,\lambda_m;\xi_1,\dots,\xi_r;\frac{\eta}{t^p}\right)t^n \\
&= \frac{\left(qt\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty}{\left(t\prod_{l=1}^{m}E_q(-\lambda_l);q\right)_\infty}\frac{\left(q\eta\prod_{l=1}^{r}E_q(-\lambda_l);q\right)_\infty}{\left(\eta\prod_{l=1}^{r}E_q(-\lambda_l);q\right)_\infty} \\
&\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{array}{l} \overline{} : [q^{-x_1}:1];\ \cdots; \\ \left[qt\prod_{j=1}^{m}E_q(-\lambda_j):1,\dots,1\right] : \overline{};\ \cdots; \end{array}\right. \\
&\qquad\left.\begin{array}{l}[q^{-x_m}:1]; \\ \overline{};\end{array}\ q;\Xi_1,\dots,\Xi_m\right) \\
&\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{array}{l} \overline{} : [q^{-\xi_1}:1]; \\ \left[q\eta\prod_{j=1}^{r}E_q(-\lambda_j):1,\dots,1\right] : \overline{}; \end{array}\right. \\
&\qquad\left.\begin{array}{l}\cdots;\ [q^{-\xi_r}:1]; \\ \cdots;\ \overline{};\end{array}\ q;\Xi_1,\dots,\Xi_r\right),
\end{aligned} \tag{57}$$

provided that each member of (57) exists.

Setting:

$$\Omega_{\nu+\psi k}(\xi_1,\dots,\xi_r) = u^{(\alpha_1,\dots,\alpha_r)}_{\nu+\psi k,q}(\xi_1,\dots,\xi_r)$$

in Theorem 2, we obtain the following.

Corollary 3. *Let all of the constraints in* Theorem 2 *including the polynomials and the functions be assumed. Furthermore, let:*

$$\Lambda^{(2)}_{\nu,\psi}(\xi_1,\ldots,\xi_r;\tau) := \sum_{k=0}^{\infty} a_k u^{(\alpha_1,\ldots,\alpha_r)}_{\nu+\psi k,q}(\xi_1,\ldots,\xi_r)\tau^k \tag{58}$$

and:

$$\begin{aligned} &{}_{(1)}S^{\nu,\psi}_{n,p;q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\eta) \\ &:= \sum_{k=0}^{[n/p]} \frac{a_k}{(q;q)_{n-pk}} \varphi^m_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\, u^{(\alpha_1,\ldots,\alpha_r)}_{\nu+\psi k,q}(\xi_1,\ldots,\xi_r)\eta^k. \end{aligned} \tag{59}$$

Then, the sequence (59) is generated as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{(1)}S^{\nu,\psi}_{n,p;q}\left(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\frac{\eta}{t^p}\right)t^n = \frac{1}{\left(t\prod_{l=1}^{m} E_q(-\lambda_l);q\right)_\infty} \\ &\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{array}{c} \text{———} : [q^{-x_1}:1];\ \cdots; \\ {[q:1,\ldots,1]} : \text{———};\ \cdots; \end{array}\right. \\ &\qquad\qquad \left.\begin{array}{c} [q^{-x_m}:1]; \\ \text{———}; \end{array} q;\Xi_1,\ldots,\Xi_m\right)\Lambda^{(2)}_{\nu,\psi}(\xi_1,\ldots,\xi_r;\eta), \end{aligned} \tag{60}$$

provided that each member of (60) exists.

Putting $a_k = 1$, $\nu = 0$, and $\psi = 1$ in Corollary 3 gives the following generating relation in Corollary 4.

Corollary 4. *Let all of the constraints in* Theorem 2 *including the polynomials and the functions be assumed. Furthermore, let:*

$$\begin{aligned} &{}_{(1)}S^{\nu,\psi}_{n,p;q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\eta) \\ &:= \sum_{k=0}^{[n/p]} \frac{1}{(q;q)_{n-pk}} \varphi^m_{n-pk,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m) \\ &\qquad \times u^{(\alpha_1,\ldots,\alpha_r)}_{k,q}(\xi_1,\ldots,\xi_r)\eta^k. \end{aligned} \tag{61}$$

Then, the sequence (61) is generated as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{(1)}S^{\nu,\psi}_{n,p;q}\left(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m;\xi_1,\ldots,\xi_r;\frac{\eta}{t^p}\right)t^n \\ &= \frac{1}{\left(t\prod_{l=1}^{m} E_q(-\lambda_l);q\right)_\infty} \prod_{j=1}^{r} \frac{1}{(\xi_j\eta^{m_j};q)_{\alpha_j}} \\ &\times \Phi^{0:1;\cdots;1}_{1:0;\cdots;0}\left(\begin{array}{c} \text{———} : [q^{-x_1}:1];\ \cdots; \\ {[q:1,\ldots,1]} : \text{———};\ \cdots; \end{array}\right. \\ &\qquad\qquad \left.\begin{array}{c} [q^{-x_m}:1]; \\ \text{———}; \end{array} q;\Xi_1,\ldots,\Xi_m\right), \end{aligned} \tag{62}$$

provided that each member of (62) exists.

5. Return to the q-Gottlieb Polynomials

Khan and Asif [17] presented the following q-extension of the Gottlieb polynomials $\varphi_{m,q}(u;\lambda)$:

$$\begin{aligned}\varphi_{m,q}(u;\lambda) &= \left[E_q(-\lambda)\right]^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} u \\ k \end{bmatrix}_q q^{k(k-1)-uk}\left[1-e_q(\lambda)\right]^k \\ &= \left[E_q(-\lambda)\right]^m {}_2\phi_1\left(\begin{array}{c} q^{-m},\ q^{-u}; \\ q; \end{array} q,\ \left[1-e_q(\lambda)\right]q^m \right).\end{aligned} \tag{63}$$

In this section, we find certain relationships between the q-Gottlieb polynomials $\varphi_{m,q}(u;\lambda)$ in (63) and some other known q-polynomials. We also give an integral representation for the $\varphi_{m,q}(u;\lambda)$. Recall two q-extensions of Jacobi polynomials given as follows (see, e.g., [23]):

$$p_m(u;\mathfrak{a},\mathfrak{b}\backslash q) = {}_2\phi_1\left(\begin{array}{c} q^{-m},\ \mathfrak{a}\mathfrak{b}q^{m+1}; \\ \mathfrak{a}q; \end{array} q,\ qu \right) \tag{64}$$

and:

$$P_m(u;\mathfrak{a},\mathfrak{b},\mathfrak{c};q) = {}_3\phi_2\left(\begin{array}{c} q^{-m},\ \mathfrak{a}\mathfrak{b}q^{m+1},\ u; \\ \mathfrak{a}q,\ \mathfrak{c}q; \end{array} q,\ q \right), \tag{65}$$

which, for distinction, are called little and big q-Jacobi polynomials, respectively. The little and big q-Jacobi polynomials are found to be connected as follows:

$$\mathfrak{b}^m q^{m+\binom{m}{2}} p_m(u;\mathfrak{a},\mathfrak{b}\backslash q) = \frac{(\mathfrak{b}q;q)_m}{(\mathfrak{a}q;q)_m}(-1)^m P_m(\mathfrak{b}qu;\mathfrak{b},\mathfrak{a},0;q). \tag{66}$$

Comparison of (63)–(65) together with (66) provides the following relationships between them:

$$\varphi_{m,q}(u;\lambda) = \left[E_q(-\lambda)\right]^m p_m\left(\left[1-e_q(\lambda)\right]q^{m-1};1,q^{-u-m-1}\backslash q\right) \tag{67}$$

and:

$$\begin{aligned}\varphi_{m,q}(u;\lambda) =& \left[-q^{\frac{1}{2}(m+2u+1)}E_q(-\lambda)\right]^m \frac{(q^{-u-m};q)_m}{(q;q)_m} \\ & \times P_m\left(q^{-u-1}\left[1-e_q(\lambda)\right];q^{-u-m-1},1,0;q\right).\end{aligned} \tag{68}$$

Comparing the q-Meixner polynomials explicitly defined by (see, e.g., [23] Equation (14.13.1)):

$$M_m(q^{-u};\mathfrak{b},\mathfrak{c};q) = {}_2\phi_1\left(\begin{array}{c} q^{-m},\ q^{-u}; \\ \mathfrak{b}q; \end{array} q,\ -\frac{q^{m+1}}{\mathfrak{c}} \right) \tag{69}$$

with the q-Gottlieb polynomials in (63) gives the following relation:

$$\varphi_{m,q}(u;\lambda) = \left[E_q(-\lambda)\right]^m M_m\left(q^{-u};1,q\left[1-e_q(\lambda)\right]^{-1};q\right). \tag{70}$$

Recall the following Heine's q-integral representation for ${}_2\phi_1$ (see [11] p. 521):

$${}_2\phi_1\left(\begin{array}{c} q^{\alpha},\ q^{\beta}; \\ q^{\gamma}; \end{array} q,\ u \right) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(qt;q)_{\gamma-\beta-1}}{(ut;q)_{\alpha}}\, d_q t, \tag{71}$$

where $\Gamma_q(z)$ is the q-analogue of the classical Gamma function $\Gamma(z)$ defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \quad (|q|<1,\ z\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}), \tag{72}$$

where it is assumed that q^z and $(1-q)^{1-z}$ take their principal values (see, e.g., [11] Section 10.3; see also [2] Section 6.4). We recall the correct integral representation of $\Gamma_q(z)$ given by (e.g., [24] Equation (1.11))

$$\Gamma_q(z) = \int_0^{\frac{1}{1-q}} u^{z-1}\, E_q(-qu)\, d_q u \quad (0 < q < 1,\, \Re(z) > 0). \tag{73}$$

Employing (71) in the q-Gottlieb polynomials $\varphi_{m,q}(u;\lambda)$ in (63) gives the following integral representation:

$$\varphi_{m,q}(u;\lambda) = \frac{(1-q)\left[E_q(-\lambda)\right]^m}{(1-q^u)\Gamma_q(u)\Gamma_q(-u)} \int_0^1 \frac{t^{-u-1}(qt;q)_u}{\left(\left[1-e_q(\lambda)\right]q^m t;q\right)_{-m}}\, d_q t, \tag{74}$$

where $-1 < u < 0$ and $\left|q^m\left[1-e_q(\lambda)\right]\right| < 1$.

6. Concluding Remarks and a Future Research Plan

Gottlieb [1] introduced and investigated certain discrete orthogonal polynomials in (1), which were named Gottlieb polynomials. Since then, Gottlieb polynomials have often appeared in the literature (see, e.g., [3]). Khan and Akhlaq [6] extended the Gottlieb polynomials in two and three variables. Choi [7] generalized the Gottlieb polynomials in several variables as in (7). Khan and Asif [17] provided a q-extension of the Gottlieb polynomials. Choi [18,19] investigated several generating functions for the q-extensions of the two- and three-variable Gottlieb polynomials. Choi and Srivastava [20] introduced and investigated a q-analogue of a multiparameter and multivariable extension of the Gottlieb polynomials. In this paper, by using a method similar to those in [9,10,21], we investigated certain q-generating functions for three sequences associated with finite power series whose coefficients are products of the q-extended multivariable and multiparameter Gottlieb polynomials $\varphi^m_{n,q}(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)$ in (37) and another non-vanishing multivariable function, which were in Theorems 1–3. Furthermore, several numerous possible particular cases of the main identities in Theorems 1–3 were considered in Section 4. Finally, we came back to Khan and Asif's q-Gottlieb polynomials $\varphi_{n,q}(x;\lambda)$ in (63) to give certain connections with several other q-polynomials, and presented its q-integral representation.

We conclude this paper by revealing our future investigation plan: finding certain partial differential equations for the Gottlieb and q-Gottlieb polynomials, which may be used for some physical problems.

In fact, we tried to apply the polynomials presented herein to some physical problems. However, it is currently difficult for us to find such applications. This remains a future investigation.

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