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Perishable Inventory System with N-Policy, MAP Arrivals, and Impatient Customers

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Abstract: In this study, we consider a perishable inventory system that has an (s, Q) ordering policy, along with a finite waiting hall. The single server, which provides an item to the customer after completing the required service performance for that item, only begins serving after N customers have arrived. Impatient demand is assumed in that the customers waiting to be served lose patience and leave the system if the server's idle time overextends or if the arriving customers find the system to be full and will not enter the system. This article analyzes the impatient demands caused by the N-policy server to an inventory system. In the steadystate, we obtain the joint probability distribution of the level of inventory and the number of customers in the system. We analyze some measures of system performance and get the total expected cost rate in the steadystate. We present a beneficial cost function and confer the numerical illustration that describes the impact of impatient customers caused by N-policy on the inventory system's total expected cost rate.

Keywords: (s, Q)-policy; Markovian Arrival Process; N-policy; impatient customers

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1. Introduction

Perishable inventory system research draws inspiration from Nahmias' [1] seminal piece on ordering policies for perishable inventory. Nahmias studied the ordering policies for fixed and random shelf lifetime perishable inventory. Earlier inventory systems research usually assumed that the stock items are non-perishable. However, this is not realistic, thus creating the need to study perishable inventory systems. For more details on perishable inventory, we refer interested readers to Aijun Liu et al. [2], Darestani [3], Ioannidis [4], Kalpakam and Arivarignan [5], Liu and Lian [6], Sung-Seok Ko [7], Weiss [8], and Zhang et al. [9].

Generally, in the literature on inventory models, customers receive the stock demanded instantaneously only when the stock is available; otherwise, waiting is the norm. In the case of the inventory maintained at a service facility, customers usually wait for the item demanded because some service is performed on it, for instance, a fast food outlet or hospital dispensary. Further, due to the complexity and uniqueness of a customer's order, the service time may stretch and be variable, such as special medicinal preparation for liver-impaired patients or gluten-free dietary requests. This then builds a queue in the service system, often leading to impatient customers, with those customers sometimes reneging or balking from the service. Recognizing that queues can form during stock-out situations, Berman et al. [10] examined an inventory model with a service facility where both the demand and service rates are known and constant. They determined the optimal order quantity for the minimal expected total cost. Since then, there has been keen interest in the perishable queueing-inventory system and impatient customers (see, for example, Amirthakodi and Sivakumar [11], Arivarignan et al. [12], Hamadi et al. [13], Manuel et al. [14], and Lawrence et al. [15]).

For many inventory systems with service activities, the setup can require several minutes, and these setup activities incur costs to the inventory system. One way to reduce the setup cost is to employ an N-policy, i.e., if the system is empty, the server is on vacation. When there are at least N customers in the system, the server begins service. Yadin and Naor [16] suggested the N-policy concept. Heyman [17] first analyzed the N-policy system with an M/G/1 queue. The N-policy has been extended by others, such as Ke [18], Kella [19], and Wang and Ke [20], to a queueing network. Krishnamoorthy and Anbazhagan [21] have considered a finite waiting hall perishable inventory system under an N-policy. Similarly, Jeganathan et al. [22] considered a perishable inventory system with a finite waiting hall and customer service under an N-policy, but they allowed the server to take multiple vacations, assuming that the customers reach the service station in a Poisson manner and inventory replenishment is instantaneous.

All previous references about N-policy in the inventory system focused on the setup cost reduced in the system. Herein, we examine another fact that the cost of customers lost. It is a significant component of the total expected cost rate.

Despite the fact that the N-policy successfully lessens the inventory system's general arrangement cost, it can nevertheless bring about waiting time vulnerability for the primary *N-1* customers. For instance, the first customer arrives at a vacant waiting hall, and the service channel withholds the service until the other *N-1* customers arrive into the system. Assuming the customer appearance rate is moderate, there is a probability of developing customer impatience. Our work is motivated by this perception. Specifically, we investigated the effect of N-policy on the arriving customers to the inventory system and focused on showing the possible results of increasingly impatient customers' impact on the total expected cost rate of the system.

In real life, you can see some rides in theme and amusement parks, theaters in malls, as well as adventure activities like skydiving, scuba diving, rafting, and parasailing starting to sell tickets to customers after some customers come to their systems. In these systems, the first customers have to wait for other arrivals. They become easily impatient, so they go for other systems.

We examine a perishable inventory system with a finite waiting capacity, and the customers arrive as a Markovian Arrival Process. We assume that the server provides service only when there are N customers in the system; otherwise, the server remains idle. If the customers arrive and find the system to be full, they will not enter the system. At the same time, the customer who is waiting for service and finds the server to be idle becomes impatient and may exit the system.

The remainder of this paper is structured as follows. Section 2 presents the notation used in the paper and the corresponding model development. In Section 3, the steady-state analysis of the model is presented. In Section 4, we derive the measures of system performance under steady-state analysis. In Section 5, the total expected system cost rate is obtained. A cost analysis is provided in Section 6. Section 7 presents the numerical illustration. Section 8 concludes the paper.

2. Model

The following notation will be used in this paper:

$\mathbf{0}$: Zero matrix.
$\boldsymbol{I}$: Identity matrix.
$\boldsymbol{I}_{\mathbf{x}}$: Identity matrix of order x.

$[\mathbb{P}]_{ij}$: Entry at $(i,j)^{th}$ position of a matrix $\mathbb{P}$.
$F_{i_{(\mathbf{x}\times\mathbf{y})}}$: Size of matrix F_i is x row and y column.
$\boldsymbol{e}$: Unit column vector of appropriate dimension.
$I(t)$: Inventory level at time t.
$T(t)$: Server status at time t.
$C(t)$: Number of customers waiting and being served at time t.
$J(t)$: Phase of the arrival process at time t.

$$T(t) = \begin{cases} 0, & \text{if server is idle} \\ 1, & \text{if server is busy.} \end{cases}$$

Consider that a perishable inventory system contains a limited waiting hall size $H(<\infty)$ (including the service receiver) with at most S items as inventory and a single server. When the customer demand reaches a predetermined level N $(0 < N < H)$, the server begins service. The customers request for one item each. The customer only receives the requested item after certain service activities are performed on that item. Service time is a negative exponential distribution with parameter $\mu(>0)$. For replenishment, an order quantity $Q(= S - s > s + 1)$ is placed when the inventory level drops to the reorder level s and the items are received only after a random time, which has a negative exponential distribution with parameter $\beta(>0)$. The customers who are waiting for service may exit the system while the server is idle, these impatient(reneging) customers are assumed to leave the system after a random time, which is distributed as a negative exponential with parameter $\alpha(>0)$. If the waiting hall is full, then all new arriving customers are considered to be lost. The lifetime of each item has a negative exponential distribution with parameter $\gamma(>0)$. We assume that the item does not perish when it is in service.

The MAP is a rich class of point processes that include many well-known processes such as the Poisson process. As is notable, the Poisson measure is the least complex and most manageable one, which is utilized widely in stochastic modeling.The possibility of the MAP is to fundamentally sum up the Poisson process and still save the manageability for modeling purposes. Hence, the MAP is a convenient tool for modeling both renewal and non-renewal arrivals. While MAP is defined for both discrete and continuous times, here we use only the continuous time case. For the description of the arrival process, we use the MAP's description as given in Lucantoni et al. [23]. Consider a continuous-time Markov chain on the state space 1, 2,. . ., x. When the chain is in state i, $1 \le i \le$ x, it remains for an exponential time with parameter v_i. When the sojourn time ends, the chain may transition in two ways. First, if the transition is with a customer arrival, then the chain enters state j with probability $c_{ij}, 1 \le j \le$ x. Second, if the transition is without a customer arrival, then the chain enters state j with probability $d_{ij}, 1 \le j \le$ x, $i \neq j$. Note that the chain can remain in the same state (i.e., from state i to state i) when an arrival occurs. Consider the matrices $F_f, f = 0, 1$ of size x as $[F_0]_{ii} = -v_i$ and $[F_0]_{ij} = v_i d_{ij}, i \neq j$, $[F_1]_{ij} = v_i c_{ij}$, $1 \le i, j \le$ x. Clearly, $F = F_0 + F_1$ is an infinitesimal generator of a continuous-time Markov chain. We assume that F is irreducible and $F_0 e \neq 0$.

Let $\boldsymbol{\varphi}$ be the stationary probability vector of a continuous-time Markov chain with generator F. Then, $\boldsymbol{\varphi}$ is the unique probability vector satisfying $\boldsymbol{\varphi} F = 0$, $\boldsymbol{\varphi}\boldsymbol{e} = 1$.

Suppose ω is the primary probability vector of the hidden Markov chain dependent on the MAP. Then we can obtain the time epochs by picking an appropriate ω, such as an independent arrival point, the end of the interval of at least k arrivals, and where the system is in a particular state such as the beginning or end of a busy period.

Setting $\omega = \boldsymbol{\varphi}$, we obtain the stationary distribution of the MAP. The constant $\lambda = \boldsymbol{\varphi} F_1 e$ is the fundamental rate, which provides the mean of the customer arrivals in unit time.

For more details on the MAP, we refer the interested reader to Latouche and Ramaswami [24], Lee and Jeon [25], and Chakravarthy and Dudin [26].

3. Analysis

Let *L(t)*,*T(t)*,*C(t)* and *J(t)*, respectively, denote the inventory level, server status, number of customer waiting and being served and phase of the arrival process at time t. From the assumptions made on the input and output processes, it can be shown that the quadruple $\{(I(t), T(t), C(t), J(t)), t \geq 0\}$ is a Markov process whose state space is

$\mathbb{E} = E_1 \cup E_2 \cup E_3 \cup E_4$, with
$E_1 = \{(i, 0, 0, r) : 1 \leq i \leq S; 1 \leq r \leq \mathrm{x}\}$
$E_2 = \{(i, k, m, r) : 1 \leq i \leq S; k = 0, 1; 1 \leq m \leq N-1; 1 \leq r \leq \mathrm{x}\}$
$E_3 = \{(i, k, m, r) : 1 \leq i \leq S; k = 1; N \leq m \leq H; 1 \leq r \leq \mathrm{x}\}$
$E_4 = \{(0, 0, m, r) : 0 \leq m \leq H; 1 \leq r \leq \mathrm{x}\}$

We order the elements of $\mathbb{E}$lexicographically. Then the infinitesimal generator $\mathbb{P}$ of the Markov process $\{(I(t), T(t), C(t), J(t)), t \geq 0\}$ has the following block partitioned form:

$$[\mathbb{P}]_{ij} = \begin{cases} \mathbb{Y}_i, j = i-1, i = 1, 2, \ldots S \\ \mathbb{X}_i, j = i, \quad i = 0, 1, \ldots S \\ \mathbb{Z}, j = i + \mathrm{Q}, i = 1, 2, \ldots s \\ \mathbb{Z}' \; j = i + \mathrm{Q}, \quad i = 0 \\ 0, \text{otherwise} \; . \end{cases}$$

where

$$\mathbb{Z}' = \begin{array}{c} \\ 0 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} F_{2_{((H+1)x \times Nx)}} & \mathbf{0}_{((H+1)x \times Hx)} \end{pmatrix} \end{array}$$

Submatrix F_2 is

$$F_2 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ N-1 \\ N \\ \vdots \\ H \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & N-1 \end{array} \\ \begin{pmatrix} \beta \boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \beta \boldsymbol{I_x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \beta \boldsymbol{I_x} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \end{array}$$

$$Z = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} F_{3_{(Nx \times Nx)}} & \mathbf{0}_{(Nx \times Hx)} \\ \mathbf{0}_{(Hx \times Nx)} & F_{4_{(Hx \times Hx)}} \end{pmatrix} \end{array}$$

Submatrices F_3 and F_4 are

$$F_3 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & N-1 \end{array} \\ \begin{pmatrix} \beta \boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \beta \boldsymbol{I_x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \beta \boldsymbol{I_x} \end{pmatrix} \end{array}$$

$$F_4 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ H \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & H \end{array} \\ \begin{pmatrix} \beta \boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \beta \boldsymbol{I_x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \beta \boldsymbol{I_x} \end{pmatrix} \end{array}$$

For $i = 0$

$$\mathbb{X}_i = \begin{array}{c} \\ 0 \end{array} \begin{array}{c} 0 \\ \begin{pmatrix} F_{5_{((H+1)\mathrm{x} \times (H+1)\mathrm{x})}} \end{pmatrix} \end{array}$$

Submatrix F_5 is

$$F_5 = \begin{array}{cc} & \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & N-2 & N-1 & N & \cdots & H-2 & H-1 & H \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ N-1 \\ N \\ \vdots \\ H-1 \\ H \end{array} & \begin{pmatrix} F_0-\beta\boldsymbol{I_x} & F_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \alpha\boldsymbol{I_x} & F_0-(\alpha+\beta)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \alpha\boldsymbol{I_x} & F_0-(\alpha+\beta)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \alpha\boldsymbol{I_x} & F_0-\alpha\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \alpha\boldsymbol{I_x} & F_0-\alpha\boldsymbol{I_x} & F_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \alpha\boldsymbol{I_x} & F_0-\alpha\boldsymbol{I_x} \end{pmatrix} \end{array}$$

For i = 1,2, ... s

$$\mathbb{X}_i = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \mathrm{F}_{6_{(Nx\times Nx)}} & \mathbf{0}_{(Nx\times Hx)} \\ \mathbf{0}_{(Hx\times Nx)} & \mathrm{F}_{7_{(Hx\times Hx)}} \end{pmatrix} \end{array}$$

Submatrices F_6 and F_7 are

$$F_6 = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N-2 & N-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ N-2 \\ N-1 \end{array} & \begin{pmatrix} F_0-(i\gamma+\beta)\boldsymbol{I_x} & F_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \alpha\boldsymbol{I_x} & F_0-(\alpha+i\gamma+\beta)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_0-(\alpha+i\gamma+\beta)\boldsymbol{I_x} & F_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \alpha\boldsymbol{I_x} & F-(\alpha+i\gamma+\beta)\boldsymbol{I_x} \end{pmatrix} \end{array}$$

$$F_7 = \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & H-1 & H \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ H-1 \\ H \end{array} & \begin{pmatrix} F_0-(i\gamma+\mu+\beta)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_0-(i\gamma+\mu+\beta)\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & F_0-(i\gamma+\mu+\beta)\boldsymbol{I_x} & F_1 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F-(i\gamma+\mu+\beta)\boldsymbol{I_x} \end{pmatrix} \end{array}$$

For $i = s + 1, \ldots S$

$$\mathbb{X}_i = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} F_{8_{(Nx\times Nx)}} & \mathbf{0}_{(Nx\times Hx)} \\ \mathbf{0}_{(Hx\times Nx)} & F_{9_{(Hx\times Hx)}} \end{pmatrix} \end{array}$$

Submatrices F_8 and F_9 are

$$F_8 = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N-2 & N-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ N-2 \\ N-1 \end{array} & \begin{pmatrix} F_0-(i\gamma)\boldsymbol{I_x} & F_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \alpha\boldsymbol{I_x} & F_0-(\alpha+i\gamma)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_0-(\alpha+i\gamma)\boldsymbol{I_x} & F_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \alpha\boldsymbol{I_x} & F-(\alpha+i\gamma)\boldsymbol{I_x} \end{pmatrix} \end{array}$$

$$F_9 = \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & H-1 & H \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ H-1 \\ H \end{array} & \begin{pmatrix} F_0-(i\gamma+\mu)\boldsymbol{I_x} & F_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_0-(i\gamma+\mu)\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & F_0-(i\gamma+\mu)\boldsymbol{I_x} & F_1 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F-(i\gamma+\mu)\boldsymbol{I_x} \end{pmatrix} \end{array}$$

For $i = 1$

$$\mathbb{Y}_i = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} 0 \\ \begin{pmatrix} F_{10_{(N\mathrm{x}\times(H+1)\mathrm{x})}} \\ F_{11_{(H\mathrm{x}\times(H+1)\mathrm{x})}} \end{pmatrix} \end{array}$$

Submatrices F_{10} and F_{11} are

$$F_{10} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & \cdots & N-1 & N & \cdots & H \end{array} \\ \begin{pmatrix} i\gamma\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & i\gamma\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & i\gamma\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \end{array}$$

$$F_{11} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ H-1 \\ H \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & H-1 & H \end{array} \\ \begin{pmatrix} \mu\boldsymbol{I_x} & i\gamma\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu\boldsymbol{I_x} & i\gamma\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & i\gamma\boldsymbol{I_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mu\boldsymbol{I_x} & i\gamma\boldsymbol{I_x} \end{pmatrix} \end{array}$$

For $i = 2, \ldots S$

$$\mathbb{Y}_i = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} F_{12_{(Nx\times Nx)}} & \mathbf{0}_{(Nx\times Hx)} \\ F_{13(Hx\times Nx)} & F_{14_{(Hx\times Hx)}} \end{pmatrix} \end{array}$$

Submatrices F_{12}, F_{13}, and F_{14} are

$$F_{12} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & N-1 \end{array} \\ \begin{pmatrix} i\gamma\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & i\gamma\boldsymbol{I_x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & i\gamma\boldsymbol{I_x} \end{pmatrix} \end{array}$$

$$F_{13} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ H \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & H \end{array} \\ \begin{pmatrix} \mu\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \end{array}$$

$$F_{14} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ H \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & \mathrm{H}-1 & H \end{array} \\ \begin{pmatrix} i\gamma\boldsymbol{I_x} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mu\boldsymbol{I_x} & i\gamma\boldsymbol{I_x} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mu\boldsymbol{I_x} & i\gamma\boldsymbol{I_x} \end{pmatrix} \end{array}$$

It is noted that matrix $\mathbb{Z}'$ is of the order $((H+1)\mathrm{x}) \times (N\mathrm{x} + H\mathrm{x})$, matrix $\mathbb{Z}$ is of the order $(N\mathrm{x} + H\mathrm{x}) \times (N\mathrm{x} + H\mathrm{x})$, matrices $\mathbb{X}_i, i = 1,2,\ldots S$ are of order $(N\mathrm{x} + H\mathrm{x}) \times (N\mathrm{x} + H\mathrm{x})$, matrix $\mathbb{X}_0$ is of the order $((H+1)\mathrm{x}) \times ((H+1)\mathrm{x})$, matrices $\mathbb{Y}_i, i = 2,3,\ldots S$ are of order $(N\mathrm{x} + H\mathrm{x}) \times (N\mathrm{x} + H\mathrm{x})$, and matrix $\mathbb{Y}_1$ is of the order $(N\mathrm{x} + H\mathrm{x}) \times ((H+1)\mathrm{x})$, respectively.

4. Study of Steady-State Vector

The process $\{I(t), T(t), C(t), J(t); t \geq 0\}$ is a continuous-time Markov chain (CTMC) having the state space E. Hence, the steady-state vector

$$\Xi(i,k,m,r) = \lim_{t\to\infty} Pr[I(t) = i, T(t) = k, C(t) = m, J(t) = r / I(0), T(0), C(0), J(0)]$$

exists, and is independent of the initial state.

Let $\Xi = (\Xi(0), \Xi(1), \ldots, \Xi(S))$,
where $\Xi(i) = (\Xi(i,0), \Xi(i,1)), i = 0, 1, \ldots S$
with $\Xi(i,k) = (\Xi(i,k,0), \Xi(i,k,1), \ldots \Xi(i,k,H)), k = 0, 1$
with $\Xi(i,k,m) = (\Xi(i,k,m,1), \Xi(i,k,m,2), \ldots \Xi(i,k,m,\mathrm{x})), m = 0, 1, \ldots H$

Then, the steady state vector Ξ satisfies $\Xi\mathbb{P} = 0$, $\Xi\boldsymbol{e} = 1$.

Lemma 1. *For the Markov process, the steady-state vector Ξ whose rate matrix is $\mathbb{P}$ is defined by*

$$\Xi(i) = \Xi(Q)\nabla_i,\ i = 0, 1, \ldots S$$

where

$$\nabla_i = \begin{cases} (-1)^{Q-i}\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}, & i = 0, 1, \ldots Q-1; \\ \boldsymbol{I}, & i = Q; \\ (-1)^{2Q-i+1}\sum\limits_{j=1}^{S-i}\left\{ \begin{array}{c} \left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right)\mathbb{Z}\mathbb{X}_{S-j}^{-1} \\ \left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right) \end{array} \right\}, & i = Q+1, \ldots S; \end{cases}$$

and $\Xi(Q)$ can be attained by workout the following two equations:

$$\Xi(Q)\left(\left\{(-1)^Q\textstyle\sum_{j=0}^{S-1}\left\{ \begin{array}{c} \left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right) \\ \mathbb{Z}\mathbb{X}_{S-j}^{-1}\left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{Q+2}\mathbb{X}_{Q+1}^{-1}\right) \end{array} \right\}\right\}\mathbb{Y}_{Q+1} + \mathbb{X}_Q + \left\{(-1)^Q\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_1\mathbb{X}_0^{-1}\right\}\mathbb{Z}'\right) = 0$$

and

$$\Xi(Q)\left(\sum_{i=0}^{Q-1}\left\{(-1)^{Q-i}\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right\} + \boldsymbol{I} + \sum_{i=Q+1}^{S}\{(-1)^{2Q-i+1}\sum_{j=0}^{S-i}\{\left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right)\mathbb{Z}\mathbb{X}_{S-j}^{-1}\right.$$
$$\left.\left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right)\}\}\right)\boldsymbol{e} = 1$$

Proof. The well-known equations are,

$$\Xi\mathbb{P} = 0 \text{ and } \Xi\boldsymbol{e} = 1.$$

The equation $\Xi\mathbb{P} = 0$ can be written as

$$\begin{array}{c} \Xi(i+1)\mathbb{Y}_{i+1} + \Xi(i)\mathbb{X}_i = 0,\ i = 0, 1, \ldots Q-1 \\ \Xi(i+1)\mathbb{Y}_{i+1} + \Xi(i)\mathbb{X}_i + \Xi(i-Q)\mathbb{Z}' = 0,\ i = Q \\ \Xi(i+1)\mathbb{Y}_{i+1} + \Xi(i)\mathbb{X}_i + \Xi(i-Q)\mathbb{Z} = 0,\ i = Q+1, Q+2, \ldots S-1 \\ \Xi(i)\mathbb{X}_i + \Xi(i-Q)\mathbb{Z} = 0,\ i = S \end{array} \tag{1}$$

Except (1), the above equations can be solved recursively, yielding

$$\Xi(i) = \Xi(Q)\nabla_i,\ i = 0, 1, \ldots S.$$

where

$$\nabla_i = \begin{cases} (-1)^{Q-i}\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}, & i = 0, 1, \ldots Q-1; \\ \boldsymbol{I}, & i = Q; \\ (-1)^{2Q-i+1}\sum\limits_{j=0}^{S-i}\left\{ \begin{array}{c} \left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right)\mathbb{Z}\mathbb{X}_{S-j}^{-1} \\ \left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right) \end{array} \right\}, & i = Q+1, \ldots S; \end{cases}$$

Solving Equation (1) and normalizing the condition after putting the value of ∇_i in that equation, we obtain $\Xi(Q)$, i.e.,

$$\Xi(Q)\left(\left\{(-1)^Q\sum_{j=0}^{S-1}\left\{\begin{array}{c}\left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right)\\ \mathbb{Z}\mathbb{X}_{S-j}^{-1}\left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{Q+2}\mathbb{X}_{Q+1}^{-1}\right)\end{array}\right\}\right\}\mathbb{Y}_{Q+1}+\mathbb{X}_Q+\left\{(-1)^Q\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_1\mathbb{X}_0^{-1}\right\}\mathbb{Z}'\right)=0$$

and

$$\Xi(Q)(\sum_{i=0}^{Q-1}\left\{(-1)^{Q-i}\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right\}+\boldsymbol{I}+\sum_{i=Q+1}^{S}\left\{(-1)^{2Q-i+1}\sum_{j=0}^{S-i}\left\{\left(\mathbb{Y}_Q\mathbb{X}_{Q-1}^{-1}\mathbb{Y}_{Q-1}\ldots\mathbb{Y}_{s+1-j}\mathbb{X}_{s-j}^{-1}\right)\mathbb{Z}\mathbb{X}_{S-j}^{-1}\right.\right.$$

$$\left.\left.\left(\mathbb{Y}_{S-j}\mathbb{X}_{S-j-1}^{-1}\mathbb{Y}_{S-j-1}\ldots\mathbb{Y}_{i+1}\mathbb{X}_i^{-1}\right)\right\}\right\})\boldsymbol{e}=1$$

□

5. Derivation of System Performance Measures

We infer some performance measures of this system during a steady state. It is seen that $\Xi(i)$ is the steady-state probability vector for the inventory level being i with every constituent mentioned: server status in the system, the number of customers, waiting and being served, and the phase of the arrival process. Hence, $\Xi(i)\boldsymbol{e}$ provides the probability that the inventory level in a steadystate is i. Similarly, $\Xi(i,k,m)\boldsymbol{e}$ is the probability that the inventory level i, server status j, and customers waiting (including being served) k are in a steadystate.

5.1. Mean Inventory Level

Let M_L be the mean inventory level in a steadystate, which can be expressed as

$$M_L=\sum_{i=1}^{S}i\left(\sum_{k=0}^{1}\sum_{m=1}^{N-1}\Xi(i,k,m)\right)\boldsymbol{e}+\sum_{i=1}^{S}i\left(\sum_{m=N}^{H}\Xi(i,1,m)\right)\boldsymbol{e}+\sum_{i=1}^{S}i(\Xi(i,0,0))\boldsymbol{e}.$$

5.2. Mean Reorder Rate

Let M_{RO} be the mean reorder rate in a steady state. If a demand service is completed or any of the (s + 1) items fails, then the inventory level drops to s from level (s + 1), a stock reorder is triggered. This then leads to

$$M_{RO}=\mu\sum_{m=1}^{H}\Xi(s+1,1,m)\boldsymbol{e}+(s+1)\gamma\sum_{k=0}^{1}\sum_{m=1}^{N-1}\Xi(s+1,k,m)\boldsymbol{e}$$
$$+(s+1)\gamma\sum_{m=N}^{H}\Xi(s+1,1,m)\boldsymbol{e}+(s+1)\gamma\Xi(s+1,0,0)\boldsymbol{e}.$$

5.3. Mean Perishable Rate

Let M_P be the mean perishable rate in a steadystate, which is given by

$$M_P=\sum_{i=1}^{S}\sum_{k=0}^{1}\sum_{m=1}^{N-1}i\gamma\Xi(i,k,m)\boldsymbol{e}+\sum_{i=1}^{S}\sum_{m=N}^{H}i\gamma\Xi(i,1,m)\boldsymbol{e}+\sum_{i=1}^{S}i\gamma\Xi(i,0,0)\boldsymbol{e}.$$

5.4. Mean Balking Rate

Let M_B be the mean balking rate in a steadystate, which can be stated as

$$M_B=\frac{1}{\lambda}\sum_{i=1}^{S}\Xi(i,1,H)F_1\boldsymbol{e}+\frac{1}{\lambda}\Xi(0,0,H)F_1\boldsymbol{e}.$$

5.5. Mean Reneging Rate

Let M_R be the mean reneging rate in a steadystate, which is given by

$$M_R=\sum_{i=0}^{S}\sum_{m=1}^{N-1}m\alpha\Xi(i,0,m)\boldsymbol{e}+\sum_{m=1}^{H}m\alpha\Xi(0,0,m)\boldsymbol{e}.$$

5.6. Mean Waiting Time

Let M_W be the mean waiting time of the customers in the waiting hall in a steady state. Then, by Little's formula,

$$M_W = \frac{L}{\lambda_a}$$

where $L = \sum_{m=1}^{N-1} m\left(\sum_{i=1}^{S}\sum_{k=0}^{1}\Xi(i,k,m)\right)\boldsymbol{e} + \sum_{m=N}^{H} m\left(\sum_{i=1}^{S}\Xi(i,1,m)\right)\boldsymbol{e} + \sum_{m=1}^{H} m\Xi(0,0,m)\boldsymbol{e}$

and the effective arrival rate (Ross [27]) λ_a is given by

$$\lambda_a = \frac{1}{\lambda}\sum_{i=1}^{S}\sum_{k=0}^{1}\sum_{m=1}^{N-1}\Xi(i,k,m)F_1\boldsymbol{e} + \frac{1}{\lambda}\sum_{i=1}^{S}\sum_{m=N}^{H-1}\Xi(i,1,m)F_1\boldsymbol{e} + \frac{1}{\lambda}\sum_{i=1}^{S}\Xi(i,0,0)F_1\boldsymbol{e} + \frac{1}{\lambda}\sum_{m=0}^{H=1}\Xi(0,0,m)F_1\boldsymbol{e}.$$

6. Cost Analysis

In order to calculate the total expected cost per unit time, we consider the following cost components.

C_C: Unit inventory carrying cost per unit time
C_S: Setup cost per order
C_B: Balking cost per customer per unit time
C_P: Perishable cost per item per unit time
C_R: Reneging cost per customer per unit time

Using the system performance measures from Section 5, the long-run expected system cost rate is given by

$$TC(S,s,H) = C_C M_L + C_S M_{RO} + C_B M_B + C_P M_P + C_R M_R + C_W M_W$$

where $M_L, M_{RO}, M_P,\ M_R$, and M_W are given in Section 5.

7. Numerical Illustration

This section presents some numerical experimentations that feature the convexity of the total expected system cost rate. In particular, we show the calculability of the outcomes inferred in our work and uncover the presence of local optima when the total cost function is a bivariate function. It is difficult to show convexity as the computations of Ξ's are recursive. The arrival process is Erlang, and as an MAP, its parameters are given by (F_0, F_1), with

$$F_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In Tables 1–3, each row has a value in bold, and each column has a value that is underlined to represent the minima of the row and column, respectively. The value that is bold and underlined is then the least cost rate of the inventory system. Therefore, we have a (local) optimum for the related cost function of the table.

Table 1. Total expected cost rate interms of S and s.

S/*s*	8	9	10	11	12	13	14
30	3.2593	2.5439	2.1758	**2.0335**	2.0755	2.3231	2.8720
31	2.0682	1.5162	1.2608	**1.1835**	1.2480	1.4639	1.8954
32	<u>1.4396</u>	0.9747	<u>0.8044</u>	**<u>0.7869</u>**	<u>0.8831</u>	<u>1.1048</u>	<u>1.5060</u>
33	1.4966	<u>0.9729</u>	**0.8102**	0.8291	0.9707	1.2480	1.7235
34	2.6559	1.7741	**1.4900**	1.5060	1.7365	2.1750	2.8915

Table 2. Total expected cost rate interms of s and H.

H/*s*	3	4	5	6	7
3	0.9623	**0.9560**	0.9772	1.1089	1.3626
4	0.8967	**0.8933**	0.9065	1.0349	1.2932
5	3.8915	3.7048	**3.5771**	3.5779	3.7260
6	6.6794	6.4398	6.2803	**6.2629**	6.3866
7	9.7410	9.5087	9.3401	**9.2999**	9.3926

Table 3. Total expected cost rate in terms of S and H.

S/*H*	6	7	8	9	10
9	32.0697	**30.3781**	32.8875	38.2844	44.1008
10	31.8089	**29.1228**	32.8407	37.5364	42.5232
11	29.5793	**28.5009**	33.1264	37.1302	41.1433
12	30.5683	**28.1685**	33.3080	36.8679	39.9816
13	31.3798	**28.3862**	33.4184	36.7971	39.1530
14	32.0893	**28.6924**	33.7491	36.6116	38.6069
15	32.6321	**29.2989**	34.2411	36.8371	38.0025
16	33.1071	**29.8624**	34.7161	37.2047	37.9498
17	33.5074	**30.6399**	35.2495	37.5077	37.8875
18	33.9234	**31.3228**	35.7665	37.9689	38.0269

Let $H = 8$, $N = 5$, $\beta = 0.95$, $\mu = 1.04$, $\gamma = 0.6$, $\alpha = 0.35$, $\lambda = 0.8$ and $C_C = 0.1$, $C_S = 0.8$, $C_B = 0.07$, $C_P = 0.05$, $C_R = 0.1$, $C_W = 0.1$.

In Table 1, the values of $TC(S,s,8)$ are shown.

The numerical example suggests that $TC(S,s,8)$ in (S,s) is convex and that the local optimum occurs at $(S,s) = (32,11)$, as displayed in Table 1 and Figure 1.

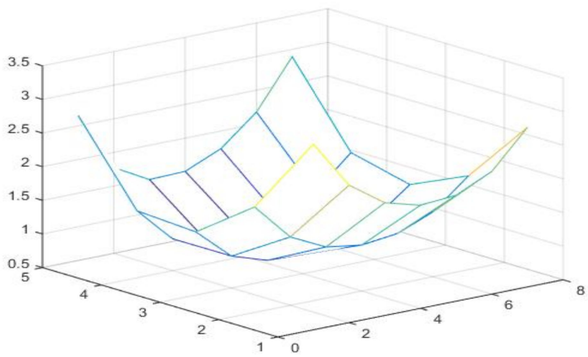

Figure 1. Total expected cost rate of S and s.

Let $S = 40$, $N = 2$, $\beta = 0.11$, $\mu = 1$, $\gamma = 0.235$, $\alpha = 0.59$, $\lambda = 0.93$ and $C_C = 0.011$, $C_S = 0.001$, $C_B = 0.03$, $C_P = 0.01$, $C_R = 0.4$, $C_W = 0.05$.

From Table 2, the numerical example suggests that *TC(40,s,H)* in (*s*,*H*) is convex and that the local optimum occurs at (*s*,*H*) = (4,4).

Let $s = 2$, $N = 2$, $\beta = 0.46$, $\mu = 1.25$, $\gamma = 0.14$, $\alpha = 0.1$, $\lambda = 0.24$ and $C_C = 0.17$, $C_S = 0.005$, $C_B = 0.97$, $C_P = 0.03$, $C_R = 0.08$, $C_W = 0.06$.

TC(S,2,H) values are displayed in Table 3.

The numerical example suggests that $TC(S,2,H)$ in (S,H) is convex and that the local optimum occurs at $(S,H) = (12,7)$.

Figure 2 grants the impact of the impatient customer rates(α), on the total expected cost rate TC via five curves thatrelate to N = 2,3,4,5,6. The acquired values for the remaining parameters and costs are displayed in the actual figure. Because ofFigure 2, we perceive that the total cost value decreases when the customer requirements for service begin $(i.e., N)$ increases and the impatient customers' rate(α) increases.

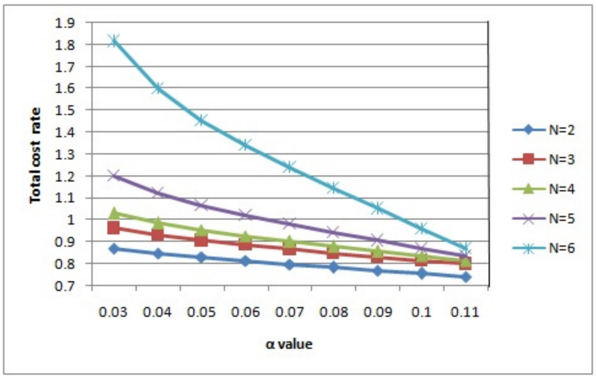

Figure 2. TC(32,11) vs. α, H = 8, $\beta = 0.95$, $\mu = 1.04$, $\gamma = 0.6$, C_C = 0.1, C_S = 0.8, C_B = 0.07, C_P = 0.05, C_R = 0.1, C_W = 0.1.

Figure 3 grants the impact of the impatience customer rates (α), on the total expected cost rate TC via three curves thatrelate to μ = 2,3,4. Because ofFigure 3, we perceive that the total cost value decreases when the service rate (μ) decreases, and the impatient customer rate(α) increases.

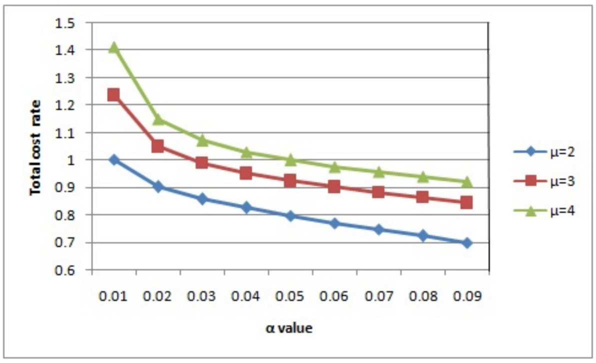

Figure 3. TC(32,11) vs. α, H = 8; β = 0.95; N = 3; γ = 0.6; C_C = 0.1; C_S = 0.8; C_B = 0.07; C_P = 0.05; C_R = 0.1; C_W = 0.1.

Figure 4 grants the impact of the service rates (μ) on the total expected cost rate TC via four curves thatrelate to γ = 0.03,0.04,0.05,0.06. Because ofFigure 4, we perceive that the total cost value increases when the service rate (μ) increases and the perishable rate (γ) increases.

In Tables 4–9, we show the impact of the setup cost C_S, the carrying cost C_C, the balking cost C_B, the reneging cost C_R, and the waiting time cost C_W on the optimal values (S^*, s^*) and the corresponding total expected cost rate TC^*. Towards this end, we first fix the parameters and cost value as H = 8, N = 5, $\beta = 0.95$, $\mu = 1.04$, $\gamma = 0.6$, $\alpha = 0.35$, $\lambda = 0.8$, and C_P = 0.05.

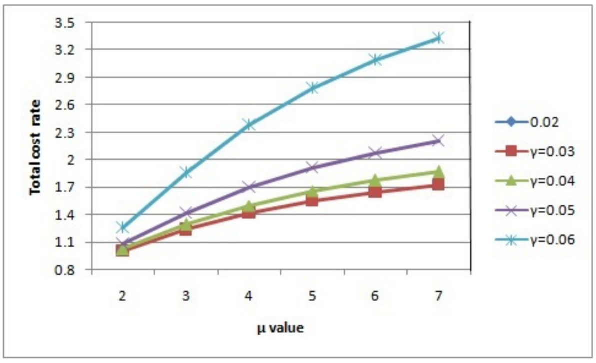

Figure 4. $TC(32{,}11)$ vs. μ, $H = 8$; $\beta = 0.95$; $N = 5$; $C_C = 0.1$; $C_S = 0.8$; $C_B = 0.07$; $C_P = 0.05$; $C_R = 0.1$; $C_W = 0.1$.

Table 4. Impact of C_C and C_S costs on the optimal value.

C_C/C_S	0.7	0.8	0.9	1.0	1.1
	32 11	32 11	32 11	32 11	32 11
0.09	0.664642	0.698524	0.732406	0.766289	0.800171
	32 11	32 11	32 11	32 11	32 11
0.10	0.694095	0.727977	0.761860	0.795742	0.829625
	32 11	32 11	32 11	32 11	32 11
0.11	0.723549	0.757431	0.791313	0.825196	0.859078
	32 11	32 11	32 11	32 11	32 11
0.12	0.753002	0.786885	0.820767	0.854649	0.888532
	32 11	32 11	32 11	31 10	31 10
0.13	0.782456	0.816338	0.850221	0.884103	0.917985

Table 5. Impact of C_W and C_B costs on the optimal value.

C_B/C_W	0.09	0.10	0.11	0.12	0.13
	32 11	32 11	32 11	32 11	32 11
0.07	0.719856	0.721000	0.722143	0.723287	0.724431
	32 11	32 11	32 11	32 11	32 11
0.08	0.786885	0.788028	0.789172	0.790316	0.791460
	32 11	32 11	32 11	32 11	32 11
0.09	0.853914	0.855057	0.856201	0.857345	0.858489
	32 11	32 11	32 11	32 11	32 11
0.10	0.920942	0.922086	0.923230	0.924374	0.925518
	32 12	32 12	32 12	33 12	33 12
0.11	0.980997	0.982192	0.983386	0.984581	0.985776

Table 6. Impact of C_W and C_R costs on the optimal value.

C_W/C_R	**0.09**	**0.10**	**0.11**	**0.12**	**0.13**
	32 11	32 11	32 11	32 11	32 11
0.09	0.696894	0.719856	0.742817	0.765779	0.788741
	32 11	32 11	32 11	32 11	32 11
0.10	0.763923	0.786885	0.809846	0.832808	0.855770
	32 11	32 11	32 11	32 11	32 11
0.11	0.830952	0.853914	0.876875	0.899837	0.922799
	32 12	32 11	32 11	32 11	32 11
0.12	0.897962	0.920942	0.943904	0.966866	0.989828
	32 12	32 12	32 12	32 12	33 12
0.13	0.956834	0.980997	1.005160	1.029322	1.053485

Table 7. Impact of C_W and C_S costs on the optimal value.

C_W/C_S	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**
	32 11	32 11	32 11	32 11	32 11
0.09	0.685973	0.719856	0.753738	0.787620	0.821503
	32 11	32 11	32 11	32 11	32 11
0.10	0.753002	0.786885	0.820767	0.854649	0.888532
	32 11	32 11	32 11	32 11	32 11
0.11	0.820031	0.853914	0.887796	0.921678	0.955561
	32 11	32 11	32 11	32 11	32 11
0.12	0.885104	0.920942	0.954825	0.988707	1.022590
	33 11	33 11	33 11	33 11	33 11
0.13	0.943977	0.980997	1.018017	1.055037	1.089619

Table 8. Impact of C_W and C_C costs on the optimal value.

C_W/C_C	**0.09**	**0.10**	**0.11**	**0.12**	**0.13**
	32 11	32 11	32 11	32 11	32 11
0.09	0.631495	0.660949	0.690402	0.719856	0.749309
	32 11	32 11	32 11	32 11	32 11
0.10	0.698524	0.727977	0.757431	0.786885	0.816338
	32 11	32 11	32 11	32 11	32 11
0.11	0.765553	0.795006	0.824460	0.853914	0.883367
	32 11	32 11	32 11	32 11	32 11
0.12	0.832582	0.862035	0.891489	0.920942	0.946720
	31 11	31 11	31 11	31 10	31 10
0.13	0.899611	0.929064	0.956402	0.980997	1.005592

Table 9. Impact of C_S and C_R costs on the optimal value.

C_S/C_R	**0.09**	**0.10**	**0.11**	**0.12**	**0.13**
	32 11	32 11	32 11	32 11	32 11
0.7	0.730040	0.753002	0.775964	0.798926	0.821888
	32 11	32 11	32 11	32 11	32 11
0.8	0.763923	0.786885	0.809846	0.832808	0.855770
	32 11	32 11	32 11	32 11	32 11
0.9	0.797805	0.820767	0.843729	0.866691	0.889652
	32 11	32 11	32 11	32 11	32 11
1.0	0.831688	0.854649	0.877611	0.900573	0.923535
	32 11	32 11	32 11	31 11	31 11
1.1	0.865570	0.888532	0.911493	0.934455	0.957417

From Tables 4–9, we observe the below monotonic behavior of (S^*, s^*):

- The total expected cost rate increases when each of the setup cost C_S, the carrying cost C_C, the balking cost C_B, the reneging cost C_R, and the waiting time cost C_W increase.
- As is to be expected, (S^*, s^*) monotonically increase when C_W increases.
- (S^*, s^*) monotonically decrease when C_C and C_S increase.
- (S^*, s^*) monotonically increase when C_C and C_W increase.
- S^* increases with C_B and C_W increasing.

8. Conclusions

In this paper, we proposed a perishable inventory system model in which the demands arrive according to a MAP and the replenishment process is negatively exponential. The server provides service at least N number of customers in the system(i.e., N-policy). We investigated the effect of the N-policy on the arriving customers to the inventory system. The joint distribution is derived in the steady-state, and we analyzed some measures of system performance and obtained the total expected cost rate in the steady-state. Additionally, we presented the numerical illustration that describes the impact of impatient customers caused by the N-policy on the inventory system's total expected cost rate. From the sensitive analysis, we can see that the total expected cost value diminishes because of the impatient customer rate. The total expected cost value seriously diminishes when the customer requirements for service begin(i.e., N) with rate increments. The service rate building also did not assist with decreases in the effect on the total expected cost rate. Additionally, the total expected cost value decreases due to the customer loss cost by the impatient customer rate, which is greater than the total expected cost value decrease due to other cost and rate values. From these perceptions, we stated that the impatient customers due to N-policy have an enormous impact on the total expected cost of the system. Future work can investigate the way to reduce the increasing of impatient customers caused by the N-policy server in the inventory system by adding other concepts like vacation policy with the N-policy server.

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