mathematics

MDPI

Article

An Economic Model for OECD Economies with Truncated *M*-Derivatives: Exact Solutions and Simulations

Luis A. Quezada-Téllez [1], Guillermo Fernández-Anaya [1], Dominique Brun-Battistini [1], Benjamín Nuñez-Zavala [1] and Jorge E. Macías-Díaz [2,3,*]

[1] Departamento de Física y Matemáticas, Universidad Iberoamericana, Mexico City 01219, Mexico; alquezada@ciencias.unam.mx (L.A.Q.-T.); guillermo.fernandez@ibero.mx (G.F.-A.); dominique.brun@ibero.mx (D.B.-B.); benuza9@hotmail.com (B.N.-Z.)
[2] Department of Mathematics and Didactics of Mathematics, Tallinn University, 10120 Tallinn, Estonia
[3] Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Aguascalientes 20131, Mexico
* Correspondence: jemacias@correo.uaa.mx; Tel.: +52-449-9108400

Abstract: This article proposes two conformal Solow models (with and without migration), accompanied by simulations for six Organisation for Economic Co-operation and Development economies. The models are proposed by employing suitable Inada conditions on the Cobb–Douglas function and making use of the truncated *M*-derivative for the Mittag–Leffler function. In the exact solutions derived in this manuscript, two new parameters play an important role in the convergence towards, or the divergence from, the steady state of capital and per capita product. The economical dynamics of these nations are influenced by the intensity of the capital and labor factors, as well as the level of depreciation, the labor force rate and the level of saving.

Keywords: Solow growth model; truncated *M*-derivative; fractional operators; Inada conditions

MSC: 34Axx; 91Bxx

Citation: Quezada-Téllez, L.A.; Fernández-Anaya, G.; Brun-Battistini, D.; Nuñez-Zavala, B.; Macías-Díaz, J.E. An Economic Model for OECD Economies with Truncated *M*-Derivatives: Exact Solutions and Simulations. *Mathematics* **2021**, *9*, 1780. https://doi.org/10.3390/math9151780

Academic Editors: Theodore E. Simos and Charampos Tsitouras

Received: 30 June 2021
Accepted: 21 July 2021
Published: 28 July 2021

1. Introduction

The Solow model (SM) [1,2] is a macroeconomic model that explains the behavior of national production through productive factors (capital and labor) and the technological level [3,4]. The SM works under various economic assumptions and considers the classic Cobb-Douglas production function. Multiple analyses of the Solow (also called Solow–Swan) model are presented in the specialized literature for various developed and developing nations. Differences in growth levels are reflected in population growth rate, labor productivity and savings levels in each country [5,6]. Recent reports on the SM have focused on the use of new production functions or, when appropriate, more realistic population growth functions. For example, the authors of [7] assumed a von Bertalanffy-style exponential growth, while a generalized logistic function (Richards' law) is used in [8]. Both studies show that the rate of population growth does not have an impact on the long-term per capita capital balance. Meanwhile, a discrete-time growth model assuming the Kadiyala production function is studied in [9]. In that work, it is shown that workers save more than shareholders, so that the growth path for developing countries is affected only by the shareholder saving rate, while the level of capital per capita in developed economies is modified by the workers' saving rate.

In addition to the classical SM, factors such as migration have been considered to determine the dynamics of capital and production. In [10], a workforce governed by a Malthusian law adding a constant migration is proposed. Under a Cobb–Douglas production function and depending on the migration value, it can be observed that the economy can collapse, stabilize or grow more slowly than in the traditional SM. Outside the economics approach, there is currently a great interest on fractional calculus and its applications. This approach allows real or complex orders in derivatives and integrals, and extends the integer-order calculus [11]. It is worth pointing out there that some research on economic growth models has been carried out using the fractional calculus approach via Caputo–Liouville, Grünwald–Letnikov, and Caputo fractional operators [12–14].

Some of the many applications of fractional calculus are circumscribed to fractional mathematical economics, where fractional methods are used to provide solutions to economic and financial systems [15,16]. Many fractional derivatives and integrals have been introduced in the literature, and one of the most popular differential operators in the last years has been the conformable derivative, which maintains certain properties that resemble those from integer-order calculus. Among the properties which are common to traditional calculus, we can quote the properties on the derivative of products and quotients of two functions, the chain rule, the formula for integration by parts, the Taylor series expansions, and the Laplace transform of some functions [17,18]. In fact, the study of economic equilibrium models are among the most interesting applications of the conformal derivative [19].

The aim of the present article is to analyze the Solow economic growth model with and without migration for six Organisation for Economic Co-operation and Development (OECD) member countries. The M-truncated derivative for the Mittag–Leffler function [20,21] is the cornerstone in the development of this study, where the Inada conditions for the classic Cobb–Douglas production function are employed. Solutions are obtained for the model with and without migration, where the new parameters ρ and β allow smoothing of the steady-state adjustment of capital and per capita product. More precisely, our study is organized as follows. Economics and mathematical fundamentals are presented in Section 2. We recall therein the definition of the M-fractional integral and the Inada conditions for the SM. Section 3 is devoted to obtain solutions in exact form and simulations for the Solow growth model without migration. Meanwhile, Section 4 investigates the same model with positive and negative migration. Finally, we close this work with a section of concluding remarks.

2. Economics and Mathematical Fundamentals

2.1. Preliminaries

The present subsection will be devoted to recall the main definitions and properties of M-fractional derivatives. The Mittag-Leffler function is then defined for a parameter and then used to define the M-truncated derivative.

Definition 1 (Acay et al. [20])**.** *Assume that* $\gamma > 0$*. Then the* truncated Mittag–Leffler function with one parameter *is defined by*

$$ {}_iE_\gamma(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\gamma k+1)}, \quad \forall z \in \mathbb{C}. \tag{1}$$

Definition 2 (Acay et al. [20])**.** *Let* $f : [0,\infty) \to \mathbb{R}$ *and* $\gamma > 0$*, and suppose that* $0 < \alpha < 1$*. When it exists, the* truncated M-derivative *of order* α *of the function* f *at the point* t*, is given by*

$$ {}_iD_M^{\alpha,\gamma} f(t) = lim_{\epsilon\to 0} \frac{f(t\,{}_iE_\gamma(\epsilon t^{-\alpha})) - f(t)}{\epsilon}, \quad \forall t > 0. \tag{2}$$

Here, ${}_iE_\gamma$ *is the truncated Mittag–Leffler function introduced in Definition 1. If the limit at the right-hand side of* (2) *exists, then we say that* f *is* α-differentiable *at* t*.*

2.2. Inada Conditions

This stage presents the Inada conditions, which are the hypotheses on the production function that guarantee the stability of economic growth in the neoclassical growth model. To start with, the following theorem recalls some important properties on the M-truncated derivative which will be used later on together with the Inada conditions.

Theorem 1 (Acay et al. [20]). *Assume that $0 < \alpha < 1$, $\gamma > 0$ and $a, b \in \mathbb{R}$. Suppose that $f, g : [0, \infty) \to \mathbb{R}$ are α-differentiable functions at $t > 0$. Then the following properties are satisfied:*

(a) $\ {}_iD_M^{\alpha,\gamma}(af + bg)(t) = a\ {}_iD_M^{\alpha,\gamma} f(t) + b\ {}_iD_M^{\alpha,\gamma} g(t)$.

(b) $\ {}_iD_M^{\alpha,\gamma}(f \cdot g)(t) = f(t)\ {}_iD_M^{\alpha,\gamma} g(t) + g(t)\ {}_iD_M^{\alpha,\gamma} f(t)$.

(c) $\ {}_iD_M^{\alpha,\gamma}\left(\frac{f}{g}\right)(t) = \frac{g(t)\ {}_iD_M^{\alpha,\gamma} f(t) - f(t)\ {}_iD_M^{\alpha,\gamma} g(t)}{[g(t)]^2}$.

(d) $\ {}_iD_M^{\alpha,\gamma} c = 0$, *for any constant* $c \in \mathbb{R}$.

(e) *If f is α-differentiable at $g(t)$, then* $\ {}_iD_M^{\alpha,\gamma}(f \circ g)(t) = f'(g(t))\ {}_iD_M^{\alpha,\gamma} g(t)$.

(f) $\ {}_iD_M^{\alpha,\gamma} f(t) = \frac{t^{1-\alpha}}{\Gamma(\gamma+1)} \frac{df(t)}{dt}$.

The following is the definition of the M-fractional integral.

Definition 3 (Acay et al. [20]). *Let $a \geq 0$, $0 < \alpha < 1$ and $\beta > 0$. Suppose that $t \geq a$, and let $f : (a, t] \to \mathbb{R}$ be a function. The M-*fractional integral *of order α of f is defined by*

$$ {}_M I_a^{\alpha,\beta} f(t) = \Gamma(\beta+1) \int_a^t \frac{f(x)}{x^{1-\alpha}} dx. \tag{3}$$

Suitable Inada conditions for the growth model will be derived from the previous definitions and theorem. In general, an economic growth model is described by the system

$$Y(t) = F(A, K(t), L(t)), \quad \forall t > 0. \tag{4}$$

Here, t represents time, Y is the national production, K and L are state variables that denote, respectively, the quantities of capital and labour factors used in production (both measured with appropriate units). Meanwhile, A represents a technological constant that is usually interpreted as the total productivity of all factors [10]. Obviously, the SM indicates that national production is the result of combining capital, labor and certain technological constant, for any period of time t. Moreover, the function Y must satisfy properties which are typical in a production function, as required in the following definition.

Definition 4. *A function $Y : (0, \infty) \to \mathbb{R}$ is a* production function *if it satisfies the following properties [10], which are known in Economics as the* Inada conditions*:*

1. *It is* increasing *for both capital and labor force, that is,* ${}_iD_{K,M}^{\rho,\beta} Y > 0$ *and* ${}_iD_{L,M}^{\rho,\beta} Y > 0$.
2. *It has* constant returns to scaling, *meaning that $Y(\lambda K, \lambda L) = \lambda Y(K, L)$, for all $\lambda > 0$.*
3. *It satisfies* $\lim_{K \to 0} {}_iD_{K,M}^{\rho,\beta} Y = \lim_{L \to 0} {}_iD_{L,M}^{\rho,\beta} Y = \infty$ *and* $\lim_{K \to \infty} {}_iD_{K,M}^{\rho,\beta} Y = \lim_{L \to \infty} {}_iD_{L,M}^{\rho,\beta} Y = 0$.
4. *Finally,* ${}_iD_{K,M}^{2\rho,\beta} Y < 0$ *and that* ${}_iD_{L,M}^{2\rho,\beta} Y < 0$.

Here, ${}_iD_{K,M}^{\rho,\beta} Y$ and ${}_iD_{L,M}^{\rho,\beta} Y$ are the partial truncated M-derivatives of Y of order ρ with respect to the variables K and L, respectively.

It is worth noting that we recover the usual Inada conditions of integer order when ρ is equal to 1. Moreover, the following identities are satisfied whenever $A, K, L > 0$:

$$ {}_iD^{\rho,\beta}_{K,M}Y(K,L) = \frac{K^{1-\rho}}{\Gamma(\beta+1)}D_K Y(K,L), \tag{5} $$

$$ {}_iD^{\rho,\beta}_{L,M}Y(K,L) = \frac{L^{1-\rho}}{\Gamma(\beta+1)}D_L Y(K,L), \tag{6} $$

$$ {}_iD^{2\rho,\beta}_{K,M}Y(K,L) = {}_iD^{\rho,\beta}_{K,M}({}_iD^{\rho,\beta}_{K,M}Y(K,L)) = \frac{K^{1-2\rho}}{\Gamma^2(\beta+1)}[(1-\rho)D_{K,M}Y + KD^2_K Y], \tag{7} $$

$$ {}_iD^{2\rho,\beta}_{L,M}Y(K,L) = {}_iD^{\rho,\beta}_{L,M}({}_iD^{\rho,\beta}_{L,M}Y(K,L)) = \frac{L^{1-2\rho}}{\Gamma^2(\beta+1)}[(1-\rho)D_{L,M}Y + LD^2_L Y]. \tag{8} $$

In the following result, we do not assume an explicit form for the function $Y(K,L)$.

Theorem 2. *If the Inada conditions for the truncated M-derivative are satisfied for some $\rho \in (0,1)$, then the Inada conditions of integer order are satisfied.*

Proof. Suppose that the Inada conditions hold for the truncated M-derivative of some order $\rho_0 \in (0,1)$. Then the Inada conditions of integer order are also fulfilled in view of the following:

1. Since $D^{\rho_0,\beta}_{K,M}Y = \frac{K^{1-\rho}}{\Gamma(\beta+1)}D_K Y > 0$ is satisfied, then $D_K Y > 0$. Likewise, $D_L Y > 0$ holds in view of the fact that $D^{\rho_0,\beta}_{L,M}Y = \frac{L^{1-\rho}}{\Gamma(\beta+1)}D_L Y > 0$.
2. This Inada condition readily follows letting $\rho = 1$.
3. If
$$ \lim_{K\to 0} D^{\rho_0,\beta}_{K,M}Y = \lim_{K\to 0}\frac{K^{1-\rho_0}}{\Gamma(\beta+1)}D_K Y = \infty, \tag{9} $$
then $D_K Y$ is of order $\mathcal{O}(K^{(1-\rho_0+\epsilon)}/\Gamma(\beta+1))$, for some $\epsilon > 0$. Consequently, $\lim_{K\to 0} D_K Y = \infty$. In similar fashion, $\lim_{L\to 0} D^{\rho_0,\beta}_{L,M}Y = \infty$ implies $\lim_{L\to 0} D_L Y = \infty$. On the other hand, if
$$ \lim_{K\to\infty} D^{\rho_0,\beta}_{K,M}Y = \lim_{K\to\infty}\frac{K^{1-\rho_0}}{\Gamma(\beta+1)}D_K Y = 0, \tag{10} $$
then $D_K Y$ is of order $\mathcal{O}(K^{(1-\rho_0+\epsilon)}/\Gamma(\beta+1))$, which implies that $\lim_{K\to\infty} D_K Y = 0$. Finally, we must point out that the case of $\lim_{L\to\infty} D_L Y = 0$ is handled similarly.
4. Observe that $D^{2\rho_0,\beta}_{K,M}Y < 0$ holds if and only if $\frac{K^{1-2\rho_0}}{\Gamma^2(\beta+1)}[(1-\rho_0)D_{K,M}Y + KD^2_K Y] < 0$, which implied that $(1-\rho_0)D_{K,M}Y + KD^2_K Y < 0$. But $(1-\rho_0)$, K and $D_K Y$ are positive, which yields that $D^2_K Y < 0$. Similarly, $D^{2\rho_0,\beta}_{L,M}Y < 0$ implies that $D^2_L Y < 0$, as desired.

□

Theorem 3. *Consider the SM with production function $Y = AK^\alpha L^{1-\alpha}$. The Inada conditions for the MD are satisfied if and only if $\rho > \max\{\alpha, 1-\alpha\}$.*

Proof. First let's note that $Y = AK^\alpha L^{1-\alpha}$ satisfies the Inada condition of integer order. Indeed, observe that $D^{\rho,\beta}_{K,M}(Y) = \frac{K^{1-\rho}}{\Gamma(\beta+1)}D_K(Y) > 0$ and $D^{\rho,\beta}_{L,M}(Y) = \frac{L^{1-\rho}}{\Gamma(\beta+1)}D_L(Y) > 0$, in view that K, L, $D_K(Y)$ and $D_L(Y)$ are positive for $Y = AK^\alpha L^{1-\alpha}$. This establishes the first condition of Definition 4, the second condition being trivial. To establish condition 3, observe that

$$ \lim_{K\to 0} D^{\rho,\beta}_{K,M}(Y) = \lim_{K\to 0}\frac{K^{1-\rho}}{\Gamma(\beta+1)}D_K(AK^\alpha L^{1-\alpha}) = \frac{\alpha A L^{1-\alpha}}{\Gamma(\beta+1)}\lim_{K\to 0} K^{\alpha-\rho} = \infty, \quad \text{if } \rho > \alpha, \tag{11} $$

and

$$\lim_{L\to 0} D^{\rho,\beta}_{L,M}(Y) = \lim_{L\to 0} \frac{L^{1-\rho}}{\Gamma(\beta+1)} D_L(Y) = \frac{AK^{\alpha}(1-\alpha)}{\Gamma(\beta+1)} \lim_{L\to 0} L^{1-(\rho+\alpha)} = \infty, \quad \text{if } \rho > 1-\alpha. \tag{12}$$

It follows that $\rho > \max\{\alpha, 1-\alpha\}$. Analogously, we can check that $\lim_{K\to\infty} D^{\rho,\beta}_{K,M}(Y) = 0$, if $\rho > \alpha$, and $\lim_{L\to\infty} D^{\rho,\beta}_{L,M}(Y) = 0$, if $\rho > 1-\alpha$. Again, we obtain that $\rho > \max\{\alpha, 1-\alpha\}$. Finally, observe that

$$D^{2\rho,\beta}_{K,M}(Y) = \frac{K^{1-2\rho}}{\Gamma^2(\beta+1)}[(1-\rho)D_K(Y) + KD^2_K(Y)] < 0, \tag{13}$$

is satisfied if and only if

$$|D^2_K(Y)| > \frac{1-\rho}{K}|D_K(Y)|. \tag{14}$$

Substituting $Y = AK^{\alpha}L^{(1-\alpha)}$ in Equation (14) and taking the absolute values, we readily notice that $A\alpha(1-\alpha)K^{\alpha-2}L^{1-\alpha} > (1-\rho)A\alpha K^{\alpha-2}L^{1-\alpha}$ if and only if $\rho > \alpha$. Similarly, $D^{2\rho,\beta}_{L,M}(Y) < 0$ if $|D^2_L(Y)| > \frac{(1-\rho)}{L}|D_L(Y)|$. Moreover, in view that $\rho > 1-\alpha$, the last inequality holds if and only if $AK^{\alpha}(1-\alpha)L^{-\alpha-1}(1-\rho) < AK^{\alpha}(1-\alpha)\alpha L^{-\alpha-1}$. Therefore both conditions are fulfilled if and only if $\rho > \max\{\alpha, 1-\alpha\}$. Notice that $\alpha \in (0,1)$ implies that $\rho \in (1/2, 1]$ since ρ cannot be greater than 1. □

Theorem 2 indicates that the first units of capital and labor are very productive, and that when the units of capital or labor are large, its marginal products are close to zero. On the other hand, Theorem 3 assures that the Inada conditions for a *Cobb–Douglas production function*

$$Y = AK^{\alpha}(t)L(t)^{1-\alpha}, \quad \alpha \in (0,1), \tag{15}$$

are preserved if and only if $\rho > \max\{\alpha, 1-\alpha\}$. Note that $\alpha \in (0,1)$, which implies that $\rho \in (1/2, 1]$, guarantees that the Inada conditions for truncated M-derivatives are satisfied. As opposed to the Inada conditions of integer order, the value of α is restricted when using truncated M-derivatives.

3. Solow Model without Migration

In this section, we analyse the case without migration and with a Malthusian law, considering the population as labor force. Our approach will be similar to that followed in [10].

Definition 5 (Solow [1])**.** *Suppose that a production function is of the Cobb–Douglas form. If α is close to* 0*, then we say that the economy is* work intensive*. Meanwhile, if α is approximately equal to* 1*, then we say that the economy is* capital intensive*.*

According to (15), the capital stock dynamics is governed by the ordinary differential equation

$$\dot{K} = sY - \delta K = sAK^{\alpha}L^{1-\alpha} - \delta K, \tag{16}$$

where the parameters s and δ are the savings constant and the rate of depreciation of capital, respectively. Neoclassically, sY can be taken as the gross investment and δK is the capital depreciation of the entire economy [10]. For the remainder of this work, we will use the notation $L^{(\rho)}_M = D^{\rho,\beta}_M(L)$, where β is a fixed real parameter. Observe that $L^{(\rho)}_M = \gamma L$ implies that

$$L(t) = L_0 e^{\gamma\Gamma(\beta+1)\frac{t^{\rho}}{\rho}}, \quad \forall t > 0. \tag{17}$$

Here, $L_0 > 0$ is the initial population of workers, and γ is the inter-temporal rate of growth or Malthusian parameter. Substituting (17) into (15), we obtain

$$Y = AK^{\alpha}\left(L_0 e^{\gamma\Gamma(\beta+1)\frac{t^{\rho}}{\rho}}\right)^{1-\alpha}. \tag{18}$$

If we define the *per capita capital* and the *labor growth rate*, respectively, as

$$k(t) = \frac{K(t)}{L(t)}, \tag{19}$$

$$n(t) = \frac{L_M^{(\rho)}(t)}{L(t)}. \tag{20}$$

Taking the derivative of (19) with respect to t and letting $n(t) = \gamma$, we obtain that $k_M^{(\rho)} = \frac{K_M^{(\rho)}}{L} - \gamma k(t)$. Using now Equation (16), replacing the capital stock and using the truncated M-derivative, we reach

$$k_m^{(\rho)} + (\delta+\gamma)k = sAk^{\alpha}. \tag{21}$$

Notice that (21) is a Bernoulli equation. Using a well-known variable change and letting $w = k^{1-\alpha}$, we readily reach the linear system

$$w_M^{(\rho)} + (1-\alpha)(\gamma+\delta)w = (1-\alpha)sA. \tag{22}$$

From (21) and (22), it follows that the solution for $k(t)$ is given by

$$k(t) = \left[c_1 e^{-\Gamma(\beta+1)(1-\alpha)(\gamma+\delta)\frac{t^{\rho}}{\rho}} + \frac{sA}{\gamma+\delta}\right]^{\frac{1}{1-\alpha}}, \tag{23}$$

where c_1 is a constant with suitable units.

It is important to notice here that the steady state of per capita capital k_{∞} is given by

$$k_{\infty} = \lim_{t\to\infty} k(t) = \left(\frac{sA}{\gamma+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{24}$$

In addition, notice that the limit when $t \to \infty$ coincides with the integer-order limit. Let us define y, the *per capita product*, as the total production ratio with respect to work, that is,

$$y(t) = \frac{Y(t)}{L(t)} = Ak^{\alpha}(t), \quad \forall t > 0. \tag{25}$$

Using Equation (15), in the long-term the per capita production tends to

$$y_{\infty} = \lim_{t\to\infty} Ak^{\alpha}(t) = A\left(\frac{sA}{\gamma+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{26}$$

Notice that the asymptotic solutions obtained in this work when $t \to +\infty$ and using truncated M-derivatives, present the same behavior as those solutions corresponding to the classical SM.

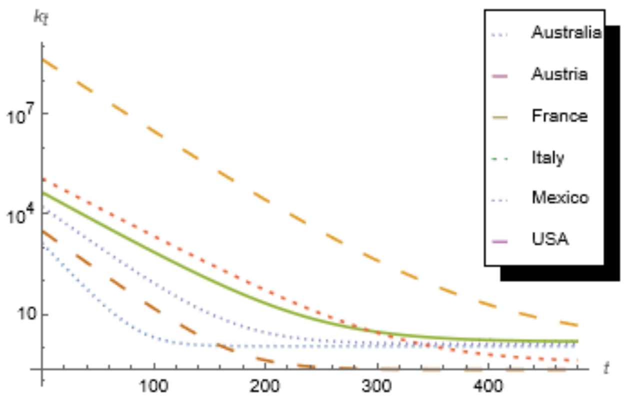
The selection of countries in the following examples was based exclusively on the availability of information from OECD for the parameters of labor, capital, depreciation, labor force, savings and levels of migration. It is worth pointing out that, unfortunately, we were not able to access the complete information on all these parameters for other countries.

Example 1. *We produce now some simulations related to six randomly chosen countries which are members of the OECD, namely, Australia, Austria, France, Italy, Mexico and the United States of*

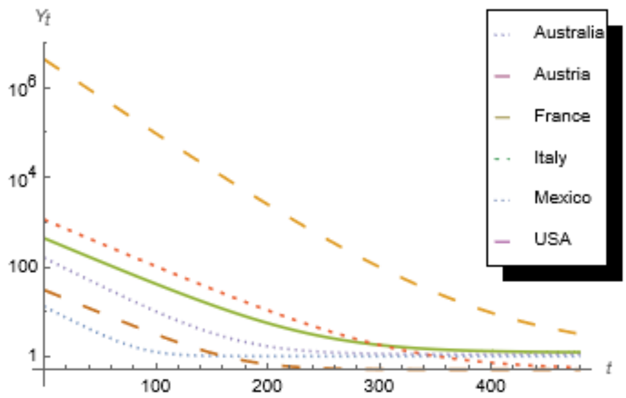
America. Databases for Gross Domestic Product (2000–2019), FDI Stocks (2005–2019), Gross Domestic Spending on R&D (2000–2018), Labor Force (2005–2019), Permanent Immigration Inflows (2005–2016) and Save Rate (2000–2019) were taken from the OECD website. The adjustment of the α parameter in the Cobb–Douglas function (15) was obtained by taking logarithm on both sides of the equation and solving. As a consequence,

$$\alpha_{WM} = \frac{ln\left(\frac{Y}{AL}\right)}{ln\left(\frac{K}{L}\right)}. \tag{27}$$

SM simulations without migration were generated with the help of Mathematica *software, using the values from Table 1 and the parameters $A = 1$, $k_0 = 100$, $L_0 = 100$, $\rho = 1.0$ and $\beta = 0.9$. Under these circumstances, Figure 1 shows the trajectories of (a) per capita capital and (b) per capita product in the integer-order case. For a small value of ρ and β, we observe that capital and per capita product decrease at a slower rate towards the steady state, depending of each country. It is worth noting that this behavior is not observed when β is greater than 1. On the other hand, the speed of convergence in each nation depends on the value of α, the level of depreciation δ, the labor force rate γ and the level of saving s.*



(**a**)



(**b**)

Figure 1. Value of (**a**) per capita capital and (**b**) per capita production in the SM model of integer order without migration for several countries.

Table 1. Percentage average for several countries of: without migration (α_{WM}), with Migration (α_M), depreciation rate δ, Laboral Force Rate (γ) and Save Rate (s).

Country	α_{WM}	α_M	δ	γ	s
Australia	35.91	68.12	3.72	4.82	9.01
Austria	76.78	10.82	4.44	0.80	5.59
France	56.86	17.47	3.83	0.59	5.23
Italy	60.42	18.22	3.68	0.51	2.73
Mexico	52.42	53.20	3.82	1.93	6.43
USA	42.68	18.61	4.98	0.73	2.38

4. Solow Model with Migration

Throughout this stage, we will consider the same assumptions as in the previous section. However, in the present section, we will investigate the model (17) with an additive and constant migration rate, which will be represented by I. This constant will determine the labor force of the economy. As a consequence, we will consider herein the equation

$$L_m^{(\rho)} = \gamma L + I, \tag{28}$$

whose solution in closed form is given by

$$L(t) = \left[\left(L_0 + \frac{I}{\gamma}\right)e^{\gamma\Gamma(\beta+1)\frac{t^\rho}{\rho}} - \frac{I}{\gamma}\right]. \tag{29}$$

Note that, if $\rho = 1$, then we retrieve the solution of the integer-order migration model. Thus, if a labor growth rate is taken as (20) and we employ truncated M-derivatives, then

$$\bar{n}(t) = \frac{L_m^{(\rho)}}{L(t)} = \frac{\gamma(\gamma L + I)}{(\gamma L_0 + I)e^{\gamma\Gamma(\beta+1)\frac{t^\rho}{\rho}} - I}. \tag{30}$$

Using (16) and replacing now the stock capital with migration, we notice that the per capita capital satisfies the following equation

$$\bar{k}_m^{(\rho)} + (\bar{n}(t) + \delta)\bar{k} = sA\bar{k}^\alpha, \tag{31}$$

where (31) was obtained analogously to Equation (21), but considering now migration. Notice that this is also a Bernoulli equation with the truncated M-derivative. If we take now $Z = \bar{k}^{1-\alpha}$ and $Z(0) = Z_0 = \bar{k}_0^{1-\alpha}$, where $Z_m^{(\rho)} = (1-\alpha)\bar{k}^{-\alpha}\bar{k}_m^{(\rho)}$, we readily reach that

$$Z_m^{(\rho)} - (\alpha - 1)(\delta + \bar{n}(t))Z = (1-\alpha)sA. \tag{32}$$

Using now Equation (30) and letting $\gamma L(t) = (\gamma L_0 + I)e^{\frac{\gamma\Gamma(\beta+1)t^\rho}{\rho}} - I$ yield

$$\begin{aligned}\int(\delta + \bar{n}(t))t^{\rho-1}dt &= \delta\Gamma(\beta+1)\frac{t^\rho}{\rho} + \int\frac{dt\gamma L(t)}{\gamma L_0} = \delta\Gamma(\beta+1)\frac{t^\rho}{\rho} \\ &\quad + \ln\left[(\gamma L_0 + I)e^{\gamma\Gamma(\beta+1)\frac{t^\rho}{\rho}} - I\right] - \ln(\gamma L_0).\end{aligned} \tag{33}$$

A direct substitution shows then that

$$e^{\pm\int_0^y(1-\alpha)(\delta+\bar{n}(t))t^{\rho-1}dt} = e^{\pm\Gamma(\beta+1)(1-\alpha)\delta\frac{y^\rho}{\rho}}\left[\left(\frac{\gamma L_0 + I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\gamma\frac{y^\rho}{\rho}} - \frac{I}{\gamma L_0}\right]^{\pm(1-\alpha)}. \tag{34}$$

As a consequence, the solution of Equation (32) is given by

$$\begin{aligned}Z(t) &= Z_0 e^{-\int(1-\alpha)(\delta+\bar{n}(t))t^{\rho-1}dt} + e^{-\int(1-\alpha)(\delta+\bar{n}(t))t^{\rho-1}dt} \\ &\quad\times\int e^{\Gamma(\beta+1)(1-\alpha)\delta\frac{y^\rho}{\rho}}\left[\left(\frac{\gamma L_0 + I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\gamma\frac{y^\rho}{\rho}} - \frac{I}{\gamma L_0}\right]^{(1-\alpha)}(1-\alpha)sAy^{\rho-1}dy \\ &= Z_0 e^{\Gamma(\beta+1)(\alpha-1)\delta\frac{t^\rho}{\rho}}\left[\left(\frac{\gamma L_0 + I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma t^\rho}{\rho}} - \frac{I}{\gamma L_0}\right]^{(\alpha-1)} \\ &\quad + (1-\alpha)sAe^{(\alpha-1)\delta\frac{t^\rho}{\rho}}\left[\left(\frac{\gamma L_0 + I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma t^\rho}{\rho}} - \frac{I}{\gamma L_0}\right]^{(\alpha-1)} \\ &\quad\times\int_0^t\left(e^{-\Gamma(\beta+1)(\alpha-1)\delta\frac{y^\rho}{\rho}}\left[\left(\frac{\gamma L_0 + I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma y^\rho}{\rho}} - \frac{I}{\gamma L_0}\right]^{(1-\alpha)} y^{\rho-1}dy\right).\end{aligned} \tag{35}$$

It is easy to see that the per capita capital $\bar{k}$ and the per capita production are given, respectively, by

$$\bar{k}(t) = [Z(t)]^{\frac{1}{1-\alpha}}, \tag{36}$$

$$\bar{y}(t) = A[Z(t)]^{\frac{\alpha}{1-\alpha}}. \tag{37}$$

Obviously, the last two equations provide closed-form solutions for capital and per capita production in the SM with migration. Next, we will derive solution for the case of negative migration by explicitly solving the integral of (35) in terms of hypergeometric functions.

4.1. Closed Analytic Solutions

In this subsection, we provide an exact analytical solution for the SM model with migration. To that end, we will employ an approach similar to that used in [10] in order to calculate the integral of the second term on the right-end of Equation (35). In the next subsection, we will derive an exact analytical solution for the SM model with negative migration.

This integral will be denoted by J in the sequel, and we employ the change of variable $u = e^{\frac{\Gamma(\beta+1)\gamma t^{\rho}}{\rho}}$. Substituting and simplifying algebraically, we obtain that

$$\begin{aligned} J &= \int_0^t e^{\Gamma(\beta+1)(1-\alpha)\frac{\delta\tau^{\rho}}{\rho}} \left[\left(\frac{\gamma L_0 + I}{\gamma L_0} \right) e^{\Gamma(\beta+1)\frac{\gamma\tau^{\rho}}{\rho}} - \frac{I}{\gamma L_0} \right]^{(1-\alpha)} \tau^{(\rho-1)} d\tau \\ &= \frac{1}{\Gamma(\beta+1)\gamma} \int_1^{e^{\Gamma(\beta+1)\frac{\gamma\tau^{\rho}}{\rho}}} u^{\frac{(1-\alpha)\delta}{\gamma}-1} \left[-\frac{I}{\gamma L_0} + \left(\frac{\gamma L_0 + I}{\gamma L_0} \right) u \right]^{(1-\alpha)} du \\ &= \frac{1}{\Gamma(\beta+1)\gamma} \left(-\frac{I}{\gamma L_0} \right)^{(1-\alpha)} \int_1^{e^{\Gamma(\beta+1)\frac{\gamma t^{\rho}}{\rho}}} u^{\frac{(1-\alpha)\delta}{\gamma}-1} \left[1 - \left(1 + \frac{\gamma L_0}{I} \right) u \right]^{(1-\alpha)} du. \end{aligned} \tag{38}$$

It is important to point out here that the last integral is related to Euler's integral representation of the Gaussian hypergeometric Function ${}_2F_1$. More precisely,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z_1 \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{39}$$

where $(\cdot)_n = \Gamma(\cdot + n)/\Gamma(\cdot)$ is the Pochhammer symbol [22,23]. The series is convergent for any a, b, c if $|z| < 1$, and for $Re(a+b-c) < 0$ if $|z| = 1$. For the integral representation, the conditions $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ are required. Thus,

$$J = \frac{1}{\Gamma(\beta+1)\gamma} \left(-\frac{I}{\gamma L_0} \right)^{(1-\alpha)} (J_t - J_0), \tag{40}$$

where

$$J_0 = \frac{\Gamma(\beta+1)\gamma}{(1-\alpha)\delta} \, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z_1 \right), \tag{41}$$

$$J_t = \frac{\Gamma(\beta+1)\gamma e^{\Gamma(\beta+1)(1-\alpha)\delta\frac{t^{\rho}}{\rho}}}{(1-\alpha)\delta} \, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z_2(t) \right). \tag{42}$$

Moreover, here $a = \alpha - 1$,

$$b = \frac{(1-\alpha)\delta}{\gamma}, \tag{43}$$

$$c = \frac{(1-\alpha)\delta}{\gamma} + 1, \tag{44}$$

$$z_1 = \left(1 + \frac{\gamma L_0}{I} \right), \tag{45}$$

$$z_2(t) = \left(1 + \frac{\gamma L_0}{I} \right) e^{\Gamma(\beta+1)\gamma\frac{t^{\rho}}{\rho}}. \tag{46}$$

In conclusion, we have obtained the following explicit formulas for the function Z, and the capital and per capita production with negative migration:

$$
\begin{aligned}
Z(t) = {} & Z_0 e^{\Gamma(\beta+1)(\alpha-1)\delta\frac{t^\rho}{\rho}}\left[\left(\frac{\gamma L_0+I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma t^\rho}{\rho}}-\frac{I}{\gamma L_0}\right]^{(\alpha-1)} \\
& + (1-\alpha)sAe^{\Gamma(\beta+1)(\alpha-1)\delta\frac{t^\rho}{\rho}}\left[\left(\frac{\gamma L_0+I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma t^\rho}{\rho}}-\frac{I}{\gamma L_0}\right]^{(\alpha-1)} \\
& \times\left(\frac{1}{\Gamma(\beta+1)\gamma}\left(-\frac{I}{\gamma L_0}\right)^{(1-\alpha)}(J_t-J_0)\right),
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\bar{k}(t) = {} & e^{-\Gamma(\beta+1)\delta\frac{t^\rho}{\rho}}\left[\left(\frac{\gamma L_0+I}{\gamma L_0}\right)e^{\Gamma(\beta+1)\frac{\gamma t^\rho}{\rho}}-\frac{I}{\gamma L_0}\right]^{-1} \\
& \times\left[\bar{k}_0^{1-\alpha}+\left(\frac{(1-\alpha)sA}{\Gamma(\beta+1)\gamma}\left(-\frac{I}{\gamma L_0}\right)^{(1-\alpha)}(J_t-J_0)\right)\right]^{\frac{1}{1-\alpha}},
\end{aligned} \tag{48}
$$

and

$$\bar{y}(t) = A\bar{k}(t)^\alpha. \tag{49}$$

Notice that the presence of the factor $\left(-\frac{I}{\gamma L_0}\right)^{(1-\alpha)}$ yields some easy consequences. For example, if $I \le 0$, then $\gamma > 0$, and if $I > 0$, then $\gamma < 0$.

4.2. Analytical Results

We will derive now some restrictions on the capital and per capita production, when $I < 0$ and $\gamma > 0$. The most important results are summarized in the following propositions.

Proposition 1. *Assume that $\gamma > 0$, $I < 0$ and $L_0 > 0$, and let $t_f = \left[\ln\left(1+\frac{\gamma L_0}{I}\right)\right]^{\frac{-1}{\Gamma(\beta+1)\gamma}}$.*

(i) *If $I \in [-\gamma L_0, 0]$, then $L(t) > 0$ with $0 < \frac{I}{\gamma} + L_0$.*
(ii) *If $I \in (-\infty, -\gamma L_0)$ and $t < t_f$, then $L(t) > 0$ with $\frac{I}{\gamma} + L_0 < 0$.*

Proof.

(i) Notice that $\gamma > 0$, $0 < \frac{I}{\gamma} + L_0$ and $I < 0$ are satisfied if and only if $I \in [-\gamma L_0, 0]$. In turn, this implies that $L(t) = \left(\frac{I}{\gamma} + L_0\right)e^{\Gamma(\beta+1)\gamma\frac{t^\rho}{\rho}} - \frac{I}{\gamma} > 0$.
(ii) Observe now that $\gamma > 0$, $\frac{I}{\gamma} + L_0 < 0$ and $I < 0$ hold if and only if $I \in (-\infty, -\gamma L_0)$. Moreover, observe that $\left(\frac{I}{\gamma} + L_0\right)e^{\Gamma(\beta+1)\gamma\frac{t^\rho}{\rho}} > \frac{I}{\gamma}$ if and only if $t < t_f$, where t_f is as in the hypotheses. □

Proposition 2. *The inequalities $\gamma > 0$, $|z_1| < 1$ and $|z_2| < 1$ are satisfied if and only if $I \in (-\infty, -\gamma L_0)$ and $t < t_f$, where t_f is as in Proposition 1.*

Proof. To start with, observe that $|z_1| < 1$ is equivalent to $-2 < \frac{\gamma L_0}{I} < 0$. On the other hand, the condition $|z_2| < 1$ holds if and only if

$$\frac{-1}{1+\frac{\gamma L_0}{I}} < e^{\Gamma(\beta+1)\gamma\frac{t^\rho}{\rho}} < \frac{1}{1+\frac{\gamma L_0}{I}} \tag{50}$$

is satisfied. But the first inequality of (50) holds if $-1 < \frac{\gamma L_0}{I}$. If $\gamma > 0$ and $I < 0$, or $\gamma < 0$ and $I > 0$, then $-(1+\frac{\gamma L_0}{I})^{-1} < 1 = e^0$, which implies that $-2 < \frac{\gamma L_0}{I} < 0$. Also, notice that $-1 < \frac{\gamma L_0}{I} < 0$ if and only if $I \in (-\infty, -\gamma L_0)$. On the another hand, $e^{\Gamma(\beta+1)\gamma\frac{t^\rho}{\rho}} < (1+\frac{\gamma L_0}{I})^{-1}$ is fulfilled if $t < t_f$, whence the statement of this result follows. □

If the conditions of Proposition 2 are met, then the conditions of (ii) in Proposition 1 also hold. As a corollary, we can only consider the case $I \in (-\infty, -\gamma L_0)$ and $t < t_f$ for our analysis.

Example 2. *Consider the same problem as in Example 1, but now with migration. The adjustments of the parameter α and the migration are carried out as in that example. More precisely,*

$$\alpha_M = \frac{\ln\left(\frac{Y}{A(\gamma L+I)}\right)}{\ln\left(\frac{K}{\gamma L+I}\right)}. \tag{51}$$

We use now Table 1 and fix the parameter values $A = 1$, $k_0 = 100$, $L_0 = 100$, $I = -10$ and $\beta = 0.9$. Different values of ρ are used in our experiments, along with negative values for the migration. Figure 2 shows the value of per capita capital (left column) and per capita production (right column) for the SM with migration of fractional order for several countries, using the parameter $\beta = 0.9$. Various values of the parameter ρ were used, namely, $\rho = 0.90$ (top row), $\rho = 0.95$ (middle row) and $\rho = 0.99$ (bottom row). In all cases, the dynamics of capital and product is increasing for all countries. As we mentioned earlier, this growth is not similar in all nations due to the values adopted by labor, capital, depreciation, labor force, savings and the level of migration. It is also important to note that growth is faster in capital per capita than in output, approaching its vertical asymptote without converging to a steady state. In the SM without migration it is clearly observed the existence of a convergence towards the steady state on the part of capital and the product. Nevertheless, in the case of negative migration, it diverges indefinitely in a determined time.

In the previous examples, solutions were obtained using various OECD member countries. It is worth mentioning that results corresponding to the integer-order case were previously presented in [4]. It is not the aim of the present work to study again the integer-order dynamics, but rather the dynamics under the derived M-truncated with parameters adjusted for each country. On the other hand, the articles [12–14] do not investigate the Solow growth model. They are based on other growth models and the mathematical approach differs substantially from the methodology followed in the present study. In addition, we work presently with a particular type of conformal derivative, while the articles mentioned above work with fractional operators of the Caputo–Liouville, Grünwald–Letnikov, and Caputo types, which are not equivalent to the conformal derivative.

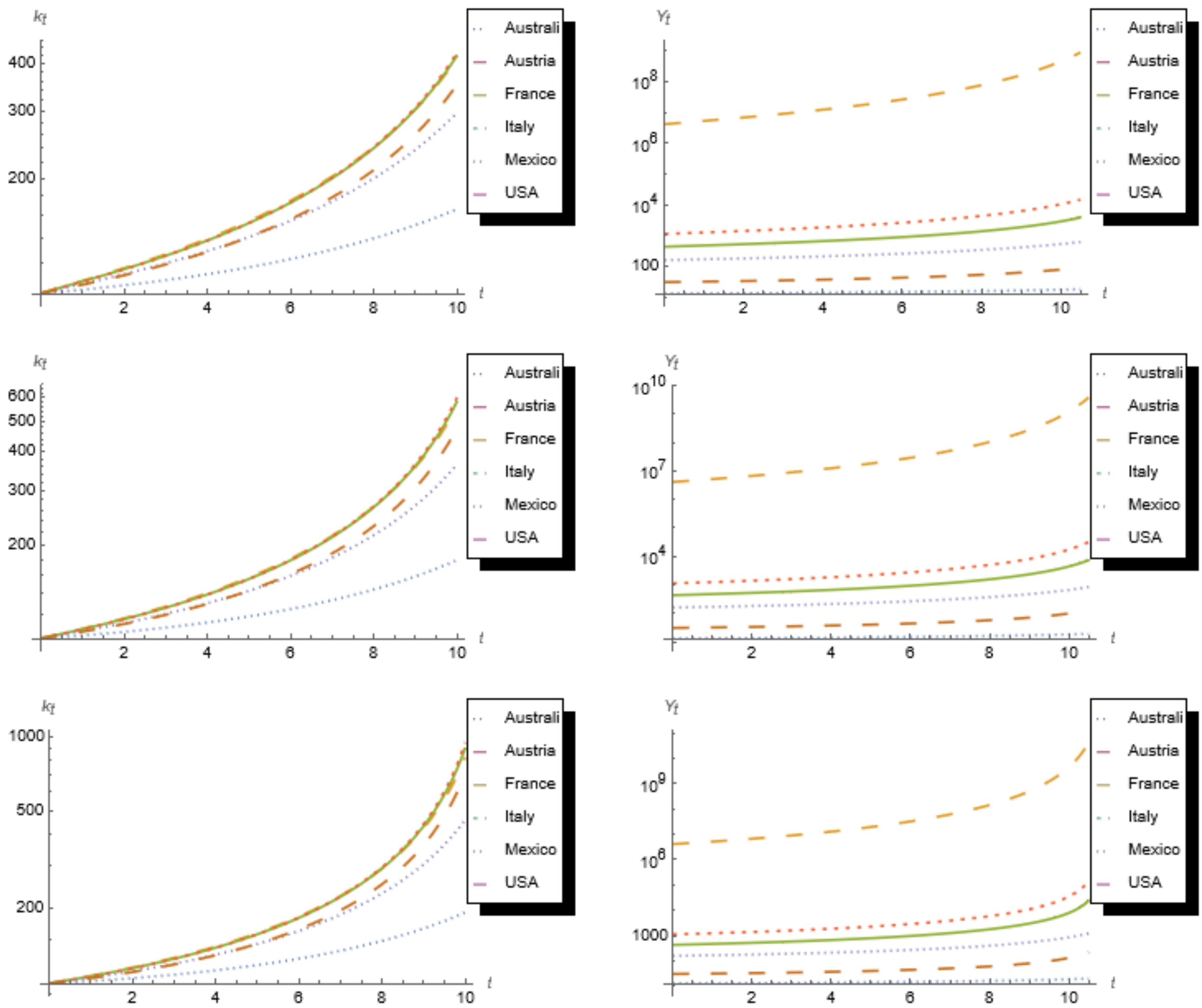


Figure 2. Value of per capita capital (**left column**) and production (**right column**) for the SM with migration of fractional order for several countries, using the parameter $\beta = 0.9$. Various values of the parameter ρ were used, namely, $\rho = 0.90$ (**top row**), $\rho = 0.95$ (**middle row**) and $\rho = 0.99$ (**bottom row**).

5. Conclusions

This work proposes a new Solow economic growth model with and without migration. Data from six OECD member countries were used for the simulations. For this analysis, Inada conditions are proposed in the traditional Cobb–Douglas function, and truncated M-derivatives is used for the Mittag–Leffler function. Solutions are obtained for the model with and without migration. In our derivations, the parameters ρ and β allow us to smooth the steady-state adjustment of capital and per capita product. The solutions obtained were similar and consistent with the classical model when $\rho = 1.0$, as expected. We observed that the convergence of capital and per capita product of each nation depends on the value of α as a function of the labor and capital factors, but also on the level of depreciation, the labor force rate and the savings level. In the model with negative migration, there is no convergence towards the steady state since capital and production per capita reach a vertical asymptote without stagnation. It is worth pointing out that the truncated M-derivative is a flexible operator since it has two parameters that can be adjusted. In view of this fact, we would expect that the model proposed in this work to be more realistic. The parameters ρ and β allow for rapid or slow convergence to, and divergence from, capital and product per capita depending on the values they take. For values close to zero, it requires a longer period of time, while the period of time is short for high values of this parameter.

Author Contributions: Conceptualization, L.A.Q.-T., G.F.-A. and J.E.M.-D.; methodology, L.A.Q.-T., G.F.-A. and J.E.M.-D.; software, L.A.Q.-T., G.F.-A., D.B.-B., B.N.-Z. and J.E.M.-D.; validation, L.A.Q.-T., G.F.-A., D.B.-B., B.N.-Z. and J.E.M.-D.; formal analysis, L.A.Q.-T., G.F.-A. and J.E.M.-D.; investigation, L.A.Q.-T., G.F.-A. and J.E.M.-D.; resources, L.A.Q.-T., G.F.-A. and J.E.M.-D.; data curation, L.A.Q.-T., G.F.-A., D.B.-B., B.N.-Z. and J.E.M.-D.; writing–original draft preparation, L.A.Q.-T., G.F.-A. and J.E.M.-D.; writing–review and editing, L.A.Q.-T., G.F.-A. and J.E.M.-D.; visualization, L.A.Q.-T., G.F.-A., D.B.-B., B.N.-Z. and J.E.M.-D.; supervision, L.A.Q.-T., G.F.-A. and J.E.M.-D.; project administration, L.A.Q.-T., G.F.-A. and J.E.M.-D.; funding acquisition, L.A.Q.-T., G.F.-A. and J.E.M.-D. All authors have read and agreed to the published version of the manuscript.

Funding: The corresponding author (J.E.M.-D.) wishes to acknowledge the financial support from the National Council for Science and Technology of Mexico (CONACYT) through grant A1-S-45928.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Dataset License: CC0.

Acknowledgments: The present manuscript is a submission to the special issue of *Mathematics MDPI* on "Fractional Calculus—Theory and Applications". They also wish to thank the anonymous reviewers for their comments and criticisms. All of their comments were taken into account in the revised version of the paper, resulting in a substantial improvement with respect to the original submission.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Solow, R.M. A contribution to the theory of economic growth. *Q. J. Econ.* **1956**, *70*, 65–94. [CrossRef]
2. Swan, T.W. Economic growth and capital accumulation. *Econ. Rec.* **1956**, *32*, 334–361. [CrossRef]
3. Fernández-Anaya, G.; Quezada-Téllez, L.A.; Nuñez-Zavala, B.; Brun-Battistini, D. Katugampola Generalized Conformal Derivative Approach to Inada Conditions and Solow-Swan Economic Growth Model. *arXiv* **2019**, arXiv:1907.00130.
4. Neto, J.P.J.; Claeyssen, J.C.R.; Ritelli, D.; Scarpello, G.M. Closed-form solution to an economic growth logistic model with constant migration. *Ciência E Nat.* **2016**, *38*, 764–770. [CrossRef]
5. Mankiw, N.G.; Romer, D.; Weil, D.N. A contribution to the empirics of economic growth. *Q. J. Econ.* **1992**, *107*, 407–437. [CrossRef]
6. Barossi-Filho, M.; Silva, R.G.; Diniz, E.M. The empirics of the Solow growth model: Long-term evidence. *J. Appl. Econ.* **2005**, *8*, 31–51. [CrossRef]
7. Brida, J.G.; Maldonado, E.L. Closed form solutions to a generalization of the Solow growth model. *Appl. Math. Sci.* **2007**, *1*, 1991–2000.
8. Accinelli, E.; Brida, J.G. Re-Formulation of the Solow Economic Growth Model Whit the Richards Population Growth Law. GE, Growth, Math Methods 0508006, EconWPA, 2005. Available online: https://econpapers.repec.org/paper/wpawuwpge/0508006.htm (accessed on 22 July 2021)
9. Grassetti, F.; Hunanyan, G. On the economic growth theory with Kadiyala production function. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *58*, 220–232. [CrossRef]
10. Juchem Neto, J.P.; Claeyssen, J.C.R.; Ritelli, D.; Scarpello, G.M. Closed-form solution for the solow model with constant migration. *TEMA* **2015**, *16*, 147–159. [CrossRef]
11. Hilfer, R. *Applications of Fractional Calculus in Physics*, 1st ed.; World Scientific: Singapore, 2000.
12. Luo, D.; Wang, J.; Feckan, M. Applying fractional calculus to analyze economic growth modelling. *J. Appl. Math. Stat. Inform.* **2018**, *14*, 25–36. [CrossRef]
13. Tejado, I.; Pérez, E.; Valério, D. Fractional derivatives for economic growth modelling of the Group of Twenty: Application to Prediction. *Mathematics* **2020**, *8*, 50. [CrossRef]
14. Traore, A.; Sene, N. Model of economic growth in the context of fractional derivative. *Alex. Eng. J.* **2020**, *59*, 4843–4850. [CrossRef]
15. Valentina, T.; Vasily, T. Fractional dynamics of natural growth and memory effect in economics. *Eur. Res.* **2016**, *23*, 12.
16. Tenreiro Machado, J.; Duarte, F.B.; Duarte, G.M. Fractional dynamics in financial indices. *Int. J. Bifurc. Chaos* **2012**, *22*, 1250249. [CrossRef]
17. Chaudhary, M.; Kumar, R.; Singh, M.K. Fractional convection-dispersion equation with conformable derivative approach. *Chaos Solitons Fractals* **2020**, *141*, 110426. [CrossRef]

18. Mayo-Maldonado, J.C.; Fernandez-Anaya, G.; Ruiz-Martinez, O.F. Stability of conformable linear differential systems: A behavioural framework with applications in fractional-order control. *IET Control Theory Appl.* **2020**, *14*, 2900–2913. [CrossRef]
19. Bas, E.; Acay, B.; Ozarslan, R. The price adjustment equation with different types of conformable derivatives in market equilibrium. *AIMS Math.* **2019**, *4*, 805–820. [CrossRef]
20. Acay, B.; Bas, E.; Abdeljawad, T. Non-local fractional calculus from different viewpoint generated by truncated M-derivative. *J. Comput. Appl. Math.* **2020**, *366*, 112410. [CrossRef]
21. Sousa, J.V.d.C.; de Oliveira, E.C. A New Truncated M-Fractional Derivative Type Unifying Some Fractional Derivative Types with Classical Properties. *Int. J. Anal. Appl.* **2018**, *16*, 83–96.
22. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Cambridge University Press: Cambridge, MA, USA, 1999; Volume 71.
23. Erdélyi, A. Higher transcendental functions. In *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA, 1953; p. 59.