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Weaker Forms of Soft Regular and Soft T_2 Soft Topological Spaces

Samer Al Ghour

Department of Mathematics, Jordan University of Science and Technology, Irbid 22110, Jordan; algore@just.edu.jo

Abstract: Soft ω-local indiscreetness as a weaker form of both soft local countability and soft local indiscreetness is introduced. Then soft ω-regularity as a weaker form of both soft regularity and soft ω-local indiscreetness is defined and investigated. Additionally, soft ω-T_2 as a new soft topological property that lies strictly between soft T_2 and soft T_1 is defined and investigated. It is proved that soft anti-local countability is a sufficient condition for equivalence between soft ω-locally indiscreetness (resp. soft ω-regularity) and soft locally indiscreetness (resp. soft ω-regularity). Additionally, it is proved that the induced topological spaces of a soft ω-locally indiscrete (resp. soft ω-regular, soft ω-T_2) soft topological space are (resp. ω-regular, ω-T_2) topological spaces. Additionally, it is proved that the generated soft topological space of a family of ω-locally indiscrete (resp. ω-regular, ω-T_2) topological spaces is soft ω-locally indiscrete and vice versa. In addition to these, soft product theorems regarding soft ω-regular and soft ω-T_2 soft topological spaces are obtained. Moreover, it is proved that soft ω-regular and soft ω-T_2 are hereditarily under soft subspaces.

Keywords: soft local indiscreetness; soft regularity; soft T_2 soft topological spaces; soft product; soft subspace; soft generated soft topological space; soft induced topological spaces

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1. Introduction and Preliminaries

Throughout this paper, we follow the concepts and terminologies as appeared in [1,2]. In this paper, we will denote topological space and soft topological space by TS and STS, respectively. Molodtsov defined soft sets [3] in 1999. The soft set theory offers a general mathematical tool for dealing with uncertain objects. Let X be a universal set and A be a set of parameters. A soft set over X relative to A is a function $G : A \longrightarrow \mathcal{P}(X)$. The family of all soft sets over X relative to A will be denoted by $SS(X, A)$. Throughout this paper, 0_A and 1_A will denote the null soft set and the absolute soft set, respectively. STSs as a contemporary structure of mathematics was defined in [4] as follows: An STS is a triplet (X, τ, A), where $\tau \subseteq SS(X, A)$, τ contains 0_A and 1_A, τ is closed under finite soft intersection, and τ is closed under arbitrary soft union. Let (X, τ, A) be an STS and $F \in SS(X, A)$, then F is said to be a soft open set in (X, τ, A) if $F \in \tau$ and F is said to be a soft closed set in (X, τ, A) if $1_A - F$ is a soft open set in (X, τ, A). The family of all soft closed sets in (X, τ, A) will be denoted by τ^c. Soft topological concepts and their applications are still a hot area of research ([1,2,5–22]). The notion of ω-regular TSs was introduced in [23], then the study of ω-regular TSs continued in [24] in which the authors defined and investigated ω-T_2 TSs. Recently, some results related to ω-regular TSs and ω-T_2 TSs appeared in [25,26].

Non-Hausdorff separation axioms are among the most widespread, significant, and motivating concepts via classical topology. For example, Alexandroff TSs as non-Hausdorff TSs have several applications, especially in digital topology and theoretical computer science [27,28]. This matter applies to them via soft topology as well. Therefore, many research studies about non-Hausdorff soft separation axioms and their properties have been carried out. In the work, we extend the notions of ω-regular and ω-T_2 as two known non-Hausdorff topological notions to define soft ω-regular and soft ω-T_2 as two new non-Hausdorff soft separation axioms. Distinguishing between two STSs (soft homeomorphic or not) is a very important area of research. Like other soft topological properties, the new separation axioms will play a role in distinguishing between two STSs. A link between a parametrized family of ω-regular (resp. ω-T_2) TSs and their generated STS is given, this introduces a correlation between the structures of TSs and STSs. we hope that this will open the door for several future related studies.

In this paper, we introduce soft ω-local indiscreetness as a weaker form of both soft local countability and soft local indiscreetness. Then we introduce soft ω-regularity as a weaker form of both soft regularity and soft ω-local indiscreetness. Additionally, we introduce soft ω-T_2 as a new soft topological property that lies strictly between soft T_2 and soft T_1. We prove that soft anti-local countability is sufficient for equivalence between soft ω-locally indiscreetness (resp. soft ω-regularity) and soft locally indiscreetness (resp. soft ω-regularity). We prove that the induced topological spaces of a soft ω-locally indiscrete (resp. soft ω-regular, soft ω-T_2) soft topological space are (resp. ω-regular, ω-T_2) topological spaces. Additionally, we prove that the generated soft topological space of a family of ω-locally indiscrete (resp. ω-regular, ω-T_2) topological spaces is soft ω-locally indiscrete and vice versa. In addition to these, we give soft product theorems regarding soft ω-regular and soft ω-T_2 soft topological spaces. Moreover, we prove that soft ω-regular and soft ω-T_2 are hereditarily under soft subspaces. Finally, we raise two open questions. In the next work, we hope to find an application for our new soft separation axioms in a decision-making problem.

The following definitions and results will be used throughout this work:

Definition 1. *Let $(X, \Im)$ be a TS and let $B \subseteq X$. Then*

(a) The set of all closed subsets of $(X, \Im)$ will be denoted by $\Im^c$.

(b) The closure of B in $(X, \Im)$ will be denoted by $Cl_{\Im}(B)$.

(c) a point $x \in X$ is said to be a condensation point of B if for any $U \in \Im$ with $x \in U$, $U \cap B \neq \emptyset$.

(d) ref. [29] B is said to be an ω-closed subset of $(X, \Im)$ if B contains all its condensation points.

(e) ref. [29]B is said to be an ω-open subset of $(X, \Im)$ if $X - B$ is an ω-closed subset of $(X, \Im)$.

(f) ref. [29] The set of all ω-open subsets of $(X, \Im)$ is denoted by $\Im_\omega$.

Theorem 1 ([29])**.** *For any TS $(X, \Im)$ and any $B \subseteq X$, we have the following:*

(a) $(X, \Im_\omega)$ is a TS with $\Im \subseteq \Im_\omega$, and $\Im \neq \Im_\omega$ in general.

(b) $B \in \Im_\omega$ if and only if for any $x \in B$ there exist $U \in \Im$ and a countable subset $C \subseteq X$ such that $x \in U - C \subseteq B$.

Definition 2. *A TS $(X, \Im)$ is said to be*

(a) Ref. [23] ω-regular if for any $B \in \Im^c$ and $x \in X - B$, there exist $U \in \Im$ and $V \in \Im_\omega$ such that $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$.

(b) Ref. [24] ω-T_2 if for any $x, y \in X$ with $x \neq y$, there exist $U \in \Im$ and $V \in \Im_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(c) Ref. [30] anti-locally countable if each $U \in \tau - \{\emptyset\}$ is uncountable.

(d) Ref. [24] ω-locally indiscrete if $\Im \subseteq (\Im_\omega)^c$.

Theorem 2 ([23])**.** *A TS $(X, \Im)$ is ω-regular if and only if for any $V \in \Im$ and any $y \in V$ there exists $W \in \Im$ such that $y \in W \subseteq Cl_{\Im_\omega}(W) \subseteq V$.*

Definition 3. *Let X be a universal set and A is a set of parameters. Then $G \in SS(X, A)$ defined by*

(a) Ref. [1] $G(a) = \begin{cases} Y & \text{if } a = e \\ \emptyset & \text{if } a \neq e \end{cases}$ will be denoted by e_Y.

(b) Ref. [1] $G(a) = Y$ for all $a \in A$ will be denoted by C_Y.

(c) Ref. [31] $G(a) = \begin{cases} \{x\} & \text{if } a = e \\ \emptyset & \text{if } a \neq e \end{cases}$ will be denoted by e_x and will be called a soft point.

The set of all soft points in $SS(X, A)$ will be denoted $SP(X, A)$.

Definition 4 ([31]). *Let $G \in SS(X,A)$ and $a_x \in SP(X,A)$. Then a_x is said to belong to F (notation: $a_x \tilde{\in} G$) if $a_x \tilde{\subseteq} G$ or equivalently: $a_x \tilde{\in} G$ if and only if $x \in G(a)$.*

Theorem 3 ([4]). *Let (X,τ,A) be a STS. Then the collection $\{F(a) : F \in \tau\}$ defines a topology on X for every $a \in A$. This topology will be denoted by τ_a .*

Theorem 4 ([32]). *Let $(X,\Im)$ be a TS. Then the collection*

$$\{F \in SS(X,A) : F(a) \in \Im \text{ for all } a \in A\}$$

defines a soft topology on X relative to A. This soft topology will be denoted by $\tau(\Im)$.

Theorem 5 ([1]). *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X and let*

$$\tau = \{F \in SS(X,A) : F(a) \in \Im_a \text{ for all } a \in A\}.$$

Then τ defines a soft topology on X relative to A. This soft topology will be denoted by $\bigoplus_{a\in A} \Im_a$.

Definition 5 ([4]). *Let (X,τ,A) be an STS and Y be a nonempty subset of X. Then the soft topology $\{F \tilde{\cap} C_Y : F \in \tau\}$ on Y relative to A is said to be the relative soft topology on Y relative to A and is denoted by τ_Y.*

Definition 6 ([33]). *Let $F \in SS(X,A)$ and $G \in SS(Y,B)$. Then the soft Cartesian product of F and G is a soft set denoted by $F \times G \in SS(X \times Y, A \times B)$ and defined by $(F \times G)((a,b)) = F(a) \times G(b)$ for each $(a,b) \in A \times B$.*

Definition 7 ([34]). *Let (X,τ) and (Y,σ,B) be two soft topological spaces and let $\mathcal{B} = \{F \times G : F \in \tau \text{ and } G \in \sigma\}$. Then the soft topology over $X \times Y$ relative to $A \times B$ with $\mathcal{B}$ as a soft base is called the product soft topology and denoted by $\tau * \sigma$.*

Definition 8 ([35]). *An STS (X,τ,A) is said to be*

(a) soft T_1 if for any $a_x, b_y \in SP(X,A)$ with $a_x \neq b_y$, there exist $F,G \in \tau$ such that $a_x \tilde{\in} F - G$, $b_y \tilde{\in} G - F$.

(b) soft T_2 if for any $a_x, b_y \in SP(X,A)$ with $a_x \neq b_y$, there exist $F,G \in \tau$ such that $a_x \tilde{\in} F$, $b_y \tilde{\in} G$, and $F \tilde{\cap} G = 0_A$.

(c) soft regular if whenever $M \in \tau^c$ and $a_x \tilde{\in} 1_A - M$, then there exist $F,G \in \tau$ such that $a_x \tilde{\in} F$, $M \tilde{\subseteq} G$, and $F \tilde{\cap} G = 0_A$.

2. Soft ω-Locally Indiscrete STSs

Recall that a TS $(X,\Im)$ is said to be locally countable if for each $x \in X$, there is $U \in \Im$ such that $x \in U$ and U is countable. Additionally, recall that a TS $(X,\Im)$ is said to be locally indiscrete if $\Im \subseteq \Im^c$.

Definition 9. *An STS (X,τ,A) is said to be*

(a) Ref. [2] soft locally countable if for each $a_x \in SP(X,A)$, there exists $G \in CSS(X,A) \cap \tau$ such that $a_x \tilde{\in} G$.

(b) Ref. [36] soft locally indiscrete if $\tau \subseteq \tau^c$.

(c) soft ω-locally indiscrete if $\tau \subseteq (\tau_\omega)^c$.

Theorem 6. *Every soft locally countable STS is soft ω-locally indiscrete.*

Proof. Let (X,τ,A) be soft locally countable, then by Corollary 5 of [2] , (X,τ_ω,A) is a discrete STS. Thus, $(\tau_\omega)^c = \tau_\omega = SS(X,A)$, and hence (X,τ,A) is soft ω-locally indiscrete. □

The implication in Theorem 6 is not reversible in general as the following example shows:

Example 1. *Let $X = \mathbb{R}$, $A = \mathbb{Q}$, $F \in SS(X, A)$ defined by $F(a) = \mathbb{N}$ for every $a \in A$, and let $\tau = \{0_A, 1_A, F\}$. Consider the STS (X, τ, A). Since $F \in CSS(X, A)$, then by Theorem 2 (d) of [2] , $F \in (\tau_\omega)^c$. Thus, we have $\tau \subseteq (\tau_\omega)^c$ and hence, (X, τ, A) is soft ω-locally indiscrete. On the other hand, it is clear that (X, τ, A) is not soft locally countable.*

Theorem 7. *Every soft locally indiscrete STS is soft ω-locally indiscrete.*

Proof. Let (X, τ, A) be soft locally indiscrete, then $\tau \subseteq \tau^c$. Since $\tau \subseteq \tau_\omega$, then $\tau^c \subseteq (\tau_\omega)^c$. Therefore, we have $\tau \subseteq (\tau_\omega)^c$, and hence (X, τ, A) is soft ω-locally indiscrete. □

The implication in Theorem 7 is not reversible in general as the following example shows:

Example 2. *Let $X = \mathbb{Z}$, $A = \mathbb{N}$, $F \in SS(X, A)$ defined by $F(a) = \mathbb{N}$ for every $a \in A$ and let $\tau = \{0_A, 1_A, F\}$. Consider the STS (X, τ, A). It is clear that (X, τ, A) is soft locally countable, so by Corollary 5 of [2], (X, τ_ω, A) is a discrete STS. Thus, $(\tau_\omega)^c = \tau_\omega = SS(X, A)$, and hence (X, τ, A) is soft ω-locally indiscrete. On the other hand, since $F \in \tau - \tau^c$, then (X, τ, A) is not soft locally indiscrete.*

Definition 10 ([2])**.** *An STS (X, τ, A) is called soft anti-locally countable if for every $F \in \tau - \{0_A\}$, $F \notin CSS(X, A)$.*

Proposition 1 ([2])**.** *Let (X, τ, A) be soft anti-locally countable. Then for all $G \in \tau_\omega$, $Cl_\tau(G) = Cl_{\tau_\omega}(G)$.*

Theorem 8. *If an STS is soft anti-locally countable, then soft ω-locally indiscrete is equivalent to soft locally indiscrete.*

Proof. Let (X, τ, A) be soft anti-locally countable soft ω-locally indiscrete STS. Let $G \in \tau$, then by soft ω-local indiscreetness of (X, τ, A), $G \in (\tau_\omega)^c$, and hence $Cl_{\tau_\omega}(G) = G$. Since (X, τ, A) is soft anti-locally countable, then by Proposition 1, $Cl_\tau(G) = Cl_{\tau_\omega}(G)$. Therefore, $Cl_\tau(G) = G$ and hence $G \in \tau^c$. It follows that (X, τ, A) is soft locally indiscrete. □

Example 2 shows that soft locally countable STSs are not soft locally indiscrete STSs, in general. The following is an example of soft locally indiscrete STS that is not soft locally countable:

Example 3. *Let $X = \mathbb{R}$, $A = \mathbb{Q}$, and $F \in SS(X, A)$ defined by $F(a) = \mathbb{N}$ for every $a \in A$. Let $\tau = \{0_A, 1_A, F, 1_A - F\}$. Consider the STS (X, τ, A). Then $\tau = \tau^c$, and so (X, τ, A) is soft locally indiscrete. On the other hand, it is clear that (X, τ, A) is not soft locally countable.*

Theorem 9. *If (X, τ, A) is soft locally indiscrete, then (X, τ_a) is locally indiscrete for all $a \in A$.*

Proof. Suppose that (X, τ, A) is soft locally indiscrete, then $\tau \subseteq \tau^c$. Let $U \in \tau_a$, then there exists $G \in \tau$ such that $G(a) = U$. Therefore, we have $G \in \tau^c$ and hence $U = G(a) \in (\tau_a)^c$. This shows that $\tau_a \subseteq (\tau_a)^c$, and hence (X, τ_a) is locally indiscrete. □

Theorem 10. *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X. Then $(X, \bigoplus_{a \in A} \Im_a, A)$ is soft locally indiscrete if and only if $(X, \Im_a)$ is locally indiscrete for all $a \in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft locally indiscrete. Then by Theorem 9, $(X, \left(\underset{a\in A}{\oplus} \Im_a\right)_a)$ is locally indiscrete for all $a \in A$. However, by Theorem 3.7 of [1], $\left(\underset{a\in A}{\oplus} \Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is locally indiscrete for all $a \in A$. Let $G \in \underset{a\in A}{\oplus} \Im_a$, then $G(a) \in \Im_a$ for all $a \in A$. Since $(X, \Im_a)$ is locally indiscrete for all $a \in A$, then $G(a) \in (\Im_a)^c$ for all $a \in A$. Thus, $G \in \left(\underset{a\in A}{\oplus} \Im_a\right)^c$. Hence, $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft locally indiscrete. □

Corollary 1. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft locally indiscrete if and only if $(X, \Im)$ is locally indiscrete.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus} \Im_a$. So, by Theorem 10, we obtain the result. □

Theorem 11. *If (X, τ, A) is soft ω-locally indiscrete, then (X, τ_a) is ω-locally indiscrete for all $a \in A$.*

Proof. Suppose that (X, τ, A) is soft ω-locally indiscrete, then $\tau \subseteq (\tau_\omega)^c$. Let $U \in \tau_a$, then there exists $G \in \tau$ such that $G(a) = U$. Therefore, we have $G \in (\tau_\omega)^c$, and hence $U = G(a) \in ((\tau_\omega)_a)^c$. However, by Theorem 7 of [2], $(\tau_\omega)_a = (\tau_a)_\omega$. Therefore, $U \in ((\tau_a)_\omega)^c$. This shows that $\tau_a \subseteq ((\tau_a)_\omega)^c$, and hence (X, τ_a) is ω-locally indiscrete. □

Theorem 12. *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X. Then $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-locally indiscrete if and only if $(X, \Im_a)$ is ω-locally indiscrete for all $a \in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-locally indiscrete. Then by Theorem 11, $(X, \left(\underset{a\in A}{\oplus} \Im_a\right)_a)$ is ω-locally indiscrete for all $a \in A$. However, by Theorem 3.7 of [1], $\left(\underset{a\in A}{\oplus} \Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is ω-locally indiscrete for all $a \in A$. Let $G \in \underset{a\in A}{\oplus} \Im_a$, then $G(a) \in \Im_a$ for all $a \in A$. Since $(X, \Im_a)$ is ω-locally indiscrete for all $a \in A$, then $G(a) \in ((\Im_a)_\omega)^c$ for all $a \in A$. Thus, $G \in \left(\underset{a\in A}{\oplus} (\Im_a)_\omega\right)^c$. So, by Theorem 8 of [2], $G \in \left(\left(\underset{a\in A}{\oplus} \Im_a\right)_\omega\right)^c$. Therefore, $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-locally indiscrete. □

Corollary 2. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft ω-locally indiscrete if and only if $(X, \Im)$ is ω-locally indiscrete.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus} \Im_a$. So, by Theorem 12, we obtain the result. □

The following example shows that the converse of each of the implications in Theorems 9 and 11 is not true, in general:

Example 4. *Let $X = \mathbb{R}$ and $A = \{a, b\}$. Let*
$F = \{(a, (-\infty, 0)), (b, (-\infty, 1))\}$,
$G = \{(a, [0, 1)), (b, (-\infty, 1))\}$,

$H = \{(a, [1, 2)), (b, [1, \infty))\}$,
$K = \{(a, [2, \infty)), (b, [1, \infty))\}$,
$L = \{(a, \emptyset), (b, (-\infty, 1))\}$,
$M = \{(a, \emptyset), (b, [1, \infty))\}$.

Let $\mathcal{M} = \{F, G, H, K, L, M\}$. Consider the STS (X, τ, A), where τ with $\mathcal{M}$ as a soft base. Then τ_a is the topology on X with $\{(-\infty, 0), [0, 1), [1, 2), [2, \infty)\}$ as a base, and τ_b is the topology on X with $\{(-\infty, 1), [1, \infty)\}$ as a base. Hence, (X, τ_a) and (X, τ_b) are both locally indiscrete. Since (X, τ, A) is soft anti-locally countable and $1_A - F \notin \tau$, then by Theorem 8 $1_A - F \notin \tau_\omega$. It follows that (X, τ, A) is not soft ω-locally indiscrete.

3. Soft ω-Regularity

We start by the main definition of this section.

Definition 11. *An STS (X, τ, A) is said to be soft ω-regular if whenever $M \in \tau^c$ and $a_x \widetilde{\in} 1_A - M$, then there exist $F \in \tau$ and $G \in \tau_\omega$ such that $a_x \widetilde{\in} F$, $M \widetilde{\subseteq} G$, and $F \widetilde{\cap} G = 0_A$.*

Theorem 13. *An STS (X, τ, A) is soft ω-regular if whenever $S \in \tau$ and $a_x \widetilde{\in} S$, then there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} S$.*

Proof. *Necessity.* Suppose that (X, τ, A) is soft ω-regular. Let $S \in \tau$ and $a_x \widetilde{\in} S$. Then we have $1_A - S \in \tau^c$ and $a_x \widetilde{\in} 1_A - (1_A - S)$, and by soft ω-regular of (X, τ, A), there exist $F \in \tau$ and $G \in \tau_\omega$ such that $a_x \widetilde{\in} F$, $1_A - S \widetilde{\subseteq} G$ and $F \widetilde{\cap} G = 0_A$. Since $1_A - S \widetilde{\subseteq} G$, then $1_A - G \widetilde{\subseteq} S$. Since $F \widetilde{\cap} G = 0_A$, then $F \widetilde{\subseteq} 1_A - G$ and so, $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_{\tau_\omega}(1_A - G) = 1_A - G \widetilde{\subseteq} S$.

Sufficiency. Suppose that whenever $S \in \tau$ and $a_x \widetilde{\in} S$, then there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} S$. Let $M \in \tau^c$ and $a_x \widetilde{\in} 1_A - M$. By assumption, there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} 1_A - M$. Put $G = 1_A - Cl_{\tau_\omega}(F)$, then $G \in (\tau_\omega)^c$ and $M \widetilde{\subseteq} G$. Therefore, (X, τ, A) is soft ω-regular. □

Theorem 14. *Every soft ω-locally indiscrete STS is soft ω-regular.*

Proof. Let (X, τ, A) be soft ω-locally indiscrete. Let $S \in \tau$ and $a_x \widetilde{\in} S$. By soft ω-local indiscreetness of (X, τ, A), $S \in (\tau_\omega)^c$ and so $S = Cl_{\tau_\omega}(S)$. Hence, we have $S \in \tau$ and $a_x \widetilde{\in} S \widetilde{\subseteq} Cl_{\tau_\omega}(S) \widetilde{\subseteq} S$. Therefore, by Theorem 13, it follows that (X, τ, A) is soft ω-regular. □

Corollary 3. *Every soft locally countable STS is soft ω-regular.*

Proof. Follows from Theorems 6 and 14. □

Theorem 15. *Every soft regular STS is soft ω-regular.*

Proof. Let (X, τ, A) be a soft regular STS. Let $S \in \tau$ and $a_x \widetilde{\in} S$. By soft regularity of (X, τ, A), there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_\tau(F) \widetilde{\subseteq} S$. Since $Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_\tau(F)$, then we have $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_\tau(F) \widetilde{\subseteq} S$. Hence, (X, τ, A) is soft ω-regular. □

Lemma 1 ([4]). *Let (X, τ, A) be an STS and $F \in SS(X, A)$. Then for all $a \in A$, $Cl_{\tau_a}(F(a)) \subseteq (Cl_\tau(F))(a)$.*

Theorem 16. *If (X, τ, A) is soft regular, then (X, τ_a) is regular for all $a \in A$.*

Proof. Suppose that (X, τ, A) is soft regular and let $a \in A$. Let $U \in \tau_a$ and $x \in U$. Choose $S \in \tau$ such that $S(a) = U$. Then we have $a_x \widetilde{\in} S \in \tau$, and by soft regularity of (X, τ, A), there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_\tau(F) \widetilde{\subseteq} S$. Thus, we have $F(a) \in \tau_a$ and by Lemma 1, $x \in F(a) \subseteq Cl_{\tau_a}(F(a)) \subseteq (Cl_\tau(F))(a) \subseteq S(a) = U$. It follows that (X, τ_a) is regular. □

Proposition 2. *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X and $\sigma = \underset{a\in A}{\oplus} \Im_a$. Then for any $b \in A$ and $S \subseteq X$, $Cl_\sigma(b_S) = b_{Cl_{\Im_b}(S)}$.*

Proof. Let $b \in A$ and $S \subseteq X$. Let $a \in A$. If $a \neq b$, $b_{Cl_{\Im_b}(S)}(a) = \emptyset \subseteq (Cl_\sigma(b_S))(a)$. On the other hand, by Theorem 3.7 of [1], $\sigma_b = \Im_b$ and by Lemma 1, $Cl_{\sigma_b}(b_S(b)) \subseteq (Cl_\sigma(b_S))(b)$. Therefore, $\left(b_{Cl_{\Im_b}(S)}\right)(b) = Cl_{\Im_b}(S) = Cl_{\Im_b}(b_S(b)) = Cl_{\sigma_b}(b_S(b)) \subseteq (Cl_\sigma(b_S))(b)$. This shows that $b_{Cl_{\Im_b}(S)} \widetilde{\subseteq} Cl_\sigma(b_S)$. To show that $(Cl_\sigma(b_S))(b) \subseteq b_{Cl_{\Im_b}(S)}$, we show that $(Cl_\sigma(b_S))(b) \subseteq Cl_{\Im_b}(S)$ and $(Cl_\sigma(b_S))(a) = \emptyset$ for every $a \in A - \{b\}$. Let $x \in (Cl_\sigma(b_S))(b)$ and $U \in \Im_b$ such that $x \in U$. Then we have $b_x \widetilde{\in} Cl_\sigma(b_S)$ with $b_x \widetilde{\in} b_U \in \sigma$, and so $b_S \widetilde{\cap} b_U = b_{S\cap U} \neq 0_A$. Therefore, $S \cap U \neq \emptyset$. Hence $x \in b_{Cl_{\Im_b}(S)}$. This shows that $(Cl_\sigma(b_S))(b) \subseteq b_{Cl_{\Im_b}(S)}$. Let $a \in A - \{b\}$ and suppose that there exists $x \in (Cl_\sigma(b_S))(a)$, then $a_x \widetilde{\in} Cl_\sigma(b_S)$. Since $a_x \widetilde{\in} a_X \in \sigma$, then $b_S \widetilde{\cap} a_X \neq 0_A$ which is impossible. It follows that $(Cl_\sigma(b_S))(a) = \emptyset$. □

Theorem 17. *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X. Then $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft regular if and only if $(X, \Im_a)$ is regular for all $a \in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft regular. Then by Theorem 16, $(X, \left(\underset{a\in A}{\oplus} \Im_a\right)_a)$ is regular for all $a \in A$. However, by Theorem 3.7 of [1], $\left(\underset{a\in A}{\oplus} \Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is regular for all $a \in A$. Put $\sigma = \underset{a\in A}{\oplus} \Im_a$. Let $G \in \sigma$ and let $b_x \widetilde{\in} G$. Then $x \in G(b) \in \Im_b$ and by regularity of $(X, \Im_b)$, there exists $V \in \Im_b$ such that $x \in V \subseteq Cl_{\Im_b}(V) \subseteq G(b)$. Thus, we have $b_x \widetilde{\in} b_V \widetilde{\subseteq} b_{Cl_{\Im_b}(V)} \widetilde{\subseteq} G$ with $b_V \in \sigma$. Moreover, by Proposition 2, $b_{Cl_{\Im_b}(V)} = Cl_\sigma(b_V)$. Therefore, $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft regular. □

Corollary 4. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft regular if and only if $(X, \Im)$ is regular.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus} \Im_a$. So, by Theorem 17, we obtain the result. □

Theorem 18. *If (X, τ, A) is soft ω-regular, then (X, τ_a) is ω-regular for all $a \in A$.*

Proof. Suppose that (X, τ, A) is soft ω-regular and let $a \in A$. Let $U \in \tau_a$ and $x \in U$. Choose $S \in \tau$ such that $S(a) = U$. Then we have $a_x \widetilde{\in} S \in \tau$ and by soft ω-regularity of (X, τ, A) and Theorem 13, then there exists $F \in \tau$ such that $a_x \widetilde{\in} F \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} S$. Thus, we have $F(a) \in \tau_a$ and by Lemma 3.6, $x \in F(a) \subseteq Cl_{\tau_\omega}(F(a)) \subseteq (Cl_{\tau_\omega}(F))(a) \subseteq S(a) = U$. Therefore, (X, τ_a) is ω-regular. □

Theorem 19. *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X. Then $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-regular if and only if $(X, \Im_a)$ is ω-regular for all $a \in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-regular. Then by Theorem 18, $(X, \left(\underset{a\in A}{\oplus} \Im_a\right)_a)$ is ω-regular for all $a \in A$. However, by Theorem 3.7 of [1], $\left(\underset{a\in A}{\oplus} \Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is ω-regular for all $a \in A$. Put $\sigma = \underset{a\in A}{\oplus} \Im_a$. Let $G \in \sigma$ and let $b_x \widetilde{\in} G$. Then $x \in G(b) \in \Im_b$, and by ω-regularity of $(X, \Im_b)$, there exists $V \in \Im_b$ such that $x \in V \subseteq Cl_{(\Im_b)_\omega}(V) \subseteq G(b)$. Thus, we have $b_x \widetilde{\in} b_V \widetilde{\subseteq} b_{Cl_{(\Im_b)_\omega}(V)} \widetilde{\subseteq} G$ with $b_V \in \sigma$. Please note that by Theorem 8 of [2] , $\sigma_\omega = \left(\underset{a\in A}{\oplus} \Im_a\right)_\omega = \underset{a\in A}{\oplus} (\Im_a)_\omega$. So, by Proposition 2, $b_{Cl_{(\Im_b)_\omega}(V)} = Cl_{\sigma_\omega}(b_V)$. Therefore, $(X, \underset{a\in A}{\oplus} \Im_a, A)$ is soft ω-regular. □

Corollary 5. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft ω-regular if and only if $(X, \Im)$ is ω-regular.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus} \Im_a$. So, by Theorem 19, we obtain the result. □

The following example will show that the converse of each of Theorem 14 and Corollary 3 is not true in general:

Example 5. *Let $X = \mathbb{R}$, $A = \{a, b\}$, $\Im_a$ is the usual topology on $\mathbb{R}$, and $\Im_b$ is the discrete topology on $\mathbb{R}$. Then the TSs $(X, \Im_a)$ and $(X, \Im_b)$ are regular, so by Theorem 17, $(X, \underset{c\in A}{\oplus} \Im_c, A)$ is soft regular. Hence, by Theorem 15, $(X, \underset{c\in A}{\oplus} \Im_c, A)$ is soft ω-regular. On the other hand, since $(-\infty, 0) \in \Im_a - ((\Im_a)_\omega)^c$, then $(X, \Im_a)$ is not ω-locally indiscrete. So, by Theorem 14, $(X, \underset{c\in A}{\oplus} \Im_c, A)$ is not soft ω-locally indiscrete. Moreover, by Theorem 6, $(X, \underset{c\in A}{\oplus} \Im_c, A)$ is not soft locally countable.*

The following example will show that the converse of Theorem 15 is not true in general:

Example 6. *Let $X = \mathbb{N}$, A be any nonempty set of parameters, and $\Im = \{\emptyset\} \cup \{U \subseteq X : X - U$ is finite$\}$. It is well known that the TS $(X, \Im)$ is not regular. So, by Corollary 4, $(X, \tau(\Im), A)$ is not soft regular. On the other hand, clearly that $(X, \tau(\Im), A)$ is soft locally countable, so by Corollary 3, $(X, \tau(\Im), A)$ is soft ω-regular.*

The following example will show that the converse of each of Theorems 16 and 18 is not true, in general:

Example 7. *Let (X, τ, A) be as in Example 4. We proved in Example 4 that (X, τ_a) and (X, τ_b) are both locally indiscrete and so both are regular, and hence both are ω-regular. Suppose that (X, τ, A) is soft ω-regular. Let $x = -1$, then $a_x \widetilde{\in} 1_A - (1_A - F)$ with $1_A - F \in \tau^c$. Therefore, there exist $S \in \tau$ and $T \in \tau_\omega$ such that $a_x \widetilde{\in} S$, $1_A - F \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_A$. It is not difficult to see that we must have $a_x \widetilde{\in} F \widetilde{\subseteq} S$, and so $F \widetilde{\cap} T = 0_A$. Thus, $T \widetilde{\subseteq} 1_A - F$ which implies that $T = 1_A - F$. However, we have shown in Example 4 that $1_A - F \notin (\tau_\omega)^c$. Therefore, (X, τ, A) is not soft ω-regular, and by Theorem 15 it is also not soft regular.*

Theorem 20. *If (X, τ, A) is soft anti-locally countable and soft ω-regular, then (X, τ, A) is soft regular.*

Proof. Follows directly from the definitions and Proposition 1. □

Proposition 3. *Let (X, τ, A) and (Y, σ, B) be two STSs. Then*

(a) $(\tau * \sigma)_\omega \subseteq \tau_\omega * \sigma_\omega$.

(b) For any $S \in SS(X, A)$ and $T \in SS(Y, B)$, $Cl_{\tau_\omega}(S) \times Cl_{\sigma_\omega}(T) \widetilde{\subseteq} Cl_{(\tau*\sigma)_\omega}(S \times T)$.

Proof. (a) Let $K \in (\tau * \sigma)_\omega$ and let $(a, b)_{(x,y)} \widetilde{\in} K$, then there exists $L \in \tau * \sigma$ and $H \in CSS(X \times Y, A \times B)$ such that $(a, b)_{(x,y)} \widetilde{\in} L - H \widetilde{\subseteq} K$. Choose $F \in \tau$ and $G \in \sigma$ such that

$(a,b)_{(x,y)}\widetilde{\in}F\times G\widetilde{\subseteq}L$. Put $M=\left(\widetilde{\cup}\left\{c_z:(c,d)_{(z,w)}\widetilde{\in}H \text{ for some } d_w\widetilde{\in}SP(Y,B)\right\}\right)-a_x$ and $N=\left(\widetilde{\cup}\left\{d_w:(c,d)_{(z,w)}\widetilde{\in}H \text{ for some } c_z\widetilde{\in}SP(X,A)\right\}\right)-b_y$. Then $M\in CSS(X,A)$ and $N\in CSS(Y,B)$. Thus, we have $F-M\in\tau_\omega$, $G-N\in\sigma_\omega$, and $(a,b)_{(x,y)}\widetilde{\in}(F-M)\times(G-N)\widetilde{\subseteq}(F\times G)-(M\times N)\widetilde{\subseteq}L-H\widetilde{\subseteq}K$. Hence, $K\in\tau_\omega*\sigma_\omega$.

(b) Let $(a,b)_{(x,y)}\widetilde{\in}Cl_{\tau_\omega}(S)\times Cl_{\sigma_\omega}(T)$ and let $K\in(\tau*\sigma)_\omega$ such that $(a,b)_{(x,y)}\widetilde{\in}K$. By (a), $K\in\tau_\omega*\sigma_\omega$ and so there exist $W\in\tau_\omega$ and $E\in\sigma_\omega$ such that $(a,b)_{(x,y)}\widetilde{\in}W\times E\widetilde{\subseteq}K$. Since $a_x\widetilde{\in}W\widetilde{\cap}Cl_{\tau_\omega}(S)$ and $b_y\widetilde{\in}E\widetilde{\cap}Cl_{\sigma_\omega}(T)$, then $W\widetilde{\cap}S\neq 0_A$ and $E\widetilde{\cap}T\neq 0_B$. Therefore, $(W\times E)\widetilde{\cap}(S\times T)\neq 0_{A\times B}$ and hence $K\widetilde{\cap}(S\times T)\neq 0_{A\times B}$. It follows that $(a,b)_{(x,y)}\widetilde{\in}Cl_{(\tau*\sigma)_\omega}(S\times T)$. □

Theorem 21. *Let (X,τ,A) and (Y,σ,B) be two STSs. If $(X\times Y,\tau*\sigma,A\times B)$ is soft ω-regular, then (X,τ,A) and (Y,σ,B) are soft ω-regular.*

Proof. Let $F\in\tau$, $G\in\sigma$, $a_x\widetilde{\in}F$, and $b_y\widetilde{\in}G$, then $(a,b)_{(x,y)}\widetilde{\in}F\times G\in\tau*\sigma$ and by soft ω-regularity of $(X\times Y,\tau*\sigma,A\times B)$, there exists $K\in\tau*\sigma$ such that $(a,b)_{(x,y)}\widetilde{\in}K\widetilde{\subseteq}Cl_{(\tau*\sigma)_\omega}(K)\widetilde{\subseteq}F\times G$. Choose $S\in\tau$ and $T\in\sigma$ such that $(a,b)_{(x,y)}\widetilde{\in}S\times T\widetilde{\subseteq}K$. Then by Proposition 3 (b), $(a,b)_{(x,y)}\widetilde{\in}S\times T\widetilde{\subseteq}Cl_{\tau_\omega}(S)\times Cl_{\sigma_\omega}(T)\widetilde{\subseteq}Cl_{(\tau*\sigma)_\omega}(T\times S)\widetilde{\subseteq}Cl_{(\tau*\sigma)_\omega}(K)\widetilde{\subseteq}F\times G$. Therefore, we have $a_x\widetilde{\in}S\widetilde{\subseteq}Cl_{\tau_\omega}(S)\widetilde{\subseteq}F$ and $b_y\widetilde{\in}T\widetilde{\subseteq}Cl_{\tau_\omega}(T)\widetilde{\subseteq}G$. It follows that (X,τ,A) and (Y,σ,B) soft ω-regular. □

Question 1. *Let (X,τ,A) and (Y,σ,B) be two soft ω-regular STSs. Is $(X\times Y,\tau*\sigma,A\times B)$ soft ω-regular?*

Theorem 22. *If (X,τ,A) is a soft ω-regular STS, then for any nonempty subset $Y\subseteq X$, the soft subspace STS (Y,τ_Y,A) is soft ω-regular.*

Proof. Suppose that (X,τ,A) is soft ω-regular and let $\emptyset\neq Y\subseteq X$. Let $M\in(\tau_Y)^c$ and $a_y\widetilde{\in}C_Y-M$. Choose $N\in\tau^c$ such that

$M=N\widetilde{\cap}C_Y$. Since (X,τ,A) is a soft ω-regular, and we have $N\in\tau^c$ and $a_y\widetilde{\in}1_A-N$, then there exist $F\in\tau$ and $G\in\tau_\omega$ such that $a_y\widetilde{\in}F$, $N\widetilde{\subseteq}G$ and $F\widetilde{\cap}G=0_A$. Then $a_y\widetilde{\in}F\widetilde{\cap}C_Y\in\tau_Y$, $M=N\widetilde{\cap}C_Y\widetilde{\subseteq}G\widetilde{\cap}C_Y$ with $G\widetilde{\cap}C_Y\in(\tau_\omega)_Y$, and $\left(F\widetilde{\cap}C_Y\right)\widetilde{\cap}\left(G\widetilde{\cap}C_Y\right)=\left(F\widetilde{\cap}G\right)\widetilde{\cap}C_Y=0_A\widetilde{\cap}C_Y=0_A$. Moreover, by Theorem 15 of [2], $G\widetilde{\cap}C_Y\in(\tau_Y)_\omega$. This ends the proof. □

4. Soft ω-T_2 STSs

The following is the main definition of this section:

Definition 12. *An STS (X,τ,A) is said to be soft ω-T_2 if for any $a_x,b_y\in SP(X,A)$ with $a_x\neq b_y$, there exist $F\in\tau$ and $G\in\tau_\omega$ such that $a_x\widetilde{\in}F$, $b_y\widetilde{\in}G$, and $F\widetilde{\cap}G=0_A$.*

Theorem 23. *If (X,τ,A) is soft ω-T_2, then (X,τ_a) is ω-T_2 for all $a\in A$.*

Proof. Suppose that (X,τ,A) is soft ω-T_2 and let $a\in A$. Let $x,y\in X$ with $x\neq y$, then $a_x\neq a_y$. Since (X,τ,A) is soft ω-T_2, then there exist $F\in\tau$ and $G\in\tau_\omega$ such that $a_x\widetilde{\in}F$, $a_y\widetilde{\in}G$, and $F\cap G=0_A$. Then we have $x\in F(a)\in\tau_a$, $y\in G(a)\in(\tau_\omega)_a$ and $F(a)\cap G(a)=(F\cap G)(a)=\emptyset$. Moreover, by Theorem 7 of [2] we have $(\tau_\omega)_a=(\tau_a)_\omega$. This shows that (X,τ_a) is ω-T_2. □

Theorem 24. *Let X be an initial universe and A be a set of parameters. Let $\{\Im_a:a\in A\}$ be an indexed family of topologies on X. Then $(X,\underset{a\in A}{\oplus}\Im_a,A)$ is soft ω-T_2 if and only if $(X,\Im_a)$ is ω-T_2 for all $a\in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus}\Im_a, A)$ is soft ω-T_2. Then by Theorem 4.2, $(X, \left(\underset{a\in A}{\oplus}\Im_a\right)_a)$ is ω-T_2 for all $a \in A$. However, by Theorem 3.7 of [1] , $\left(\underset{a\in A}{\oplus}\Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is ω-T_2 for all $a \in A$. Let $a_x, b_y \in SP(X, A)$ with $a_x \neq b_y$.

Case 1. $x \neq y$ and $a = b$. Since $(X, \Im_a)$ is ω-T_2 TS, then there exist $U \in \Im_a$ and $V \in (\Im_a)_\omega$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then we have $a_x \widetilde{\in} a_U \in \underset{a\in A}{\oplus}\Im_a$, $a_y \widetilde{\in} b_V \in \underset{a\in A}{\oplus}(\Im_a)_\omega$ and $a_U \widetilde{\cap} a_V = a_{U\cap V} = a_\emptyset = 0_A$. Moreover, by Theorem 8 of [2] , $b_V \in \left(\underset{a\in A}{\oplus}\Im_a\right)_\omega$.

Case 2. $a \neq b$. Please note that $a_x \widetilde{\in} a_X \in \underset{a\in A}{\oplus}\Im_a$, $b_x \widetilde{\in} b_X \in \left(\underset{a\in A}{\oplus}\Im_a\right)_\omega$, and $a_X \widetilde{\cap} b_X = 0_A$.

It follows that $(X, \underset{a\in A}{\oplus}\Im_a, A)$ is soft ω-T_2. □

Corollary 6. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft ω-T_2 if and only if $(X, \Im)$ is ω-T_2.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus}\Im_a$. So, by Theorem 24, we obtain the result. □

Theorem 25. *Let X be an initial universe and A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X. Then $(X, \underset{a\in A}{\oplus}\Im_a, A)$ is soft T_2 if and only if $(X, \Im_a)$ is T_2 for all $a \in A$.*

Proof. *Necessity.* Suppose that $(X, \underset{a\in A}{\oplus}\Im_a, A)$ is soft T_2. Then by Proposition 10 of [35] , $(X, \left(\underset{a\in A}{\oplus}\Im_a\right)_a)$ is T_2 for all $a \in A$. However, by Theorem 3.7 of [1], $\left(\underset{a\in A}{\oplus}\Im_a\right)_a = \Im_a$ for all $a \in A$. This ends the proof.

Sufficiency. Suppose that $(X, \Im_a)$ is T_2 for all $a \in A$. Let $a_x, b_y \in SP(X, A)$ with $a_x \neq b_y$.

Case 1. $x \neq y$ and $a = b$. Since $(X, \Im_a)$ is T_2 TS, then there exist $U, V \in \tau_a$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then we have $a_x \widetilde{\in} a_U \in \underset{a\in A}{\oplus}\Im_a$, $a_y \widetilde{\in} b_V \in \underset{a\in A}{\oplus}\Im_a$ and $a_U \widetilde{\cap} a_V = a_{U\cap V} = a_\emptyset = 0_A$.

Case 2. $a \neq b$. Please note that $a_x \widetilde{\in} a_X \in \underset{a\in A}{\oplus}\Im_a$, $b_x \widetilde{\in} b_X \in \underset{a\in A}{\oplus}\Im_a$, and $a_X \widetilde{\cap} b_X = 0_A$.

It follows that $(X, \underset{a\in A}{\oplus}\Im_a, A)$ is soft T_2. □

Corollary 7. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $(X, \tau(\Im), A)$ is soft T_2 if and only if $(X, \Im)$ is T_2.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus}\Im_a$. So, by Theorem 25, we obtain the result. □

Theorem 26. *Let (X, τ, A) and (Y, σ, B) be two STSs. If $(X \times Y, \tau * \sigma, A \times B)$ is soft ω-T_2, then (X, τ, A) and (Y, σ, B) are soft ω-T_2.*

Proof. *Necessity.* Suppose that $(X \times Y, \tau * \sigma, A \times B)$ is soft ω-T_2. To show that (X, τ, A) is soft ω-T_2, let $a_x, c_z \in SP(X, A)$ with $a_x \neq c_z$. Choose $b_y \widetilde{\in} 1_B$, then $(a, b)_{(x,y)}, (c, b)_{(z,y)} \in SP(X \times Y, A \times B)$ with $(a, b)_{(x,y)} \neq (c, b)_{(z,y)}$. Therefore, there exist $K \in \tau * \sigma$ and $W \in (\tau * \sigma)_\omega$ such that $(a, b)_{(x,y)} \widetilde{\in} K$, $(c, b)_{(z,y)} \widetilde{\in} W$ and $K \cap W = 0_A$. Choose $M \in \tau$

and $N \in \sigma$ such that $(a,b)_{(x,y)} \widetilde{\in} M \times N \widetilde{\subseteq} K$. By Proposition 3 (a), $W \in \tau_\omega * \sigma_\omega$, and so there exist $S \in \tau_\omega$ and $T \in \sigma_\omega$ such that $(c,b)_{(z,y)} \widetilde{\in} S \times T \widetilde{\subseteq} W$. Since $\left(M \widetilde{\cap} S\right) \times \left(N \widetilde{\cap} T\right) = (M \times N) \widetilde{\cap} (S \times T) \widetilde{\subseteq} K \widetilde{\cap} W = 0_A$ and $b_y \widetilde{\in} N \widetilde{\cap} T$, then $M \widetilde{\cap} S = 0_A$. Therefore, (X, τ, A) is soft ω-T_2. Similarly, we can show that (Y, σ, B) is soft ω-T_2. □

Question 2. *Let (X, τ, A) and (Y, σ, B) be two soft ω-T_2 STSs. Is $(X \times Y, \tau * \sigma, A \times B)$ soft ω-T_2?*

Theorem 27. *If (X, τ, A) is a soft ω-T_2 STS, then for any nonempty subset $Y \subseteq X$, the soft subspace STS (Y, τ_Y, A) is soft ω-T_2.*

Proof. Suppose that (X, τ, A) is soft ω-T_2 and let $\emptyset \neq Y \subseteq X$. Let $a_x, b_y \in SP(Y, A)$ with $a_x \neq b_y$. Then $a_x, b_y \in SP(X, A)$ with $a_x \neq b_y$. Since (X, τ, A) is soft ω-T_2, there exist $F \in \tau$ and $G \in \tau_\omega$ such that $a_x \widetilde{\in} F$, $b_y \widetilde{\in} G$ and $F \widetilde{\cap} G = 0_A$. Put $F_1 = F \widetilde{\cap} C_Y$ and $G_1 = G \widetilde{\cap} C_Y$, then $F_1 \in \tau_Y$, $G_1 \in (\tau_\omega)_Y$, and $F_1 \widetilde{\cap}\, G_1 = \left(F \widetilde{\cap} C_Y\right) \widetilde{\cap} \left(G \widetilde{\cap} C_Y\right) = \left(F \widetilde{\cap} G\right) \widetilde{\cap} C_Y = 0_A \widetilde{\cap} C_Y = 0_A$. Moreover, by Theorem 15 of [2] , $G_1 \in \left(\tau_Y\right)_\omega$. Therefore, (Y, τ_Y, A) is soft ω-T_2. □

Theorem 28. *If (X, τ, A) is a ω-T_2 soft STS, then (X, τ_ω, A) is soft T_2.*

Proof. Obvious. □

The converse of Theorem 28 is not true in general as the following example clarifies:

Example 8. *Let X be any countable set which contains at least two distinct points, $\Im$ be the indiscrete topology on X, and $\aleph$ be the discrete topology on X. Let $A = \{a, b\}$ and $\tau = \{F \in SS(X, A) : F(a) \in \Im$ and $F(b) \in \aleph\}$. It is clear that (X, τ_ω, A) is a discrete STS and so (X, τ_ω, A) is soft T_2. Choose $x, y \in X$ such that $x \neq y$, then $a_x, a_y \in SP(X, A)$ with $a_x \neq a_y$. Suppose that (X, τ, A) is soft ω-T_2, then there exist $F \in \tau$ and $G \in \tau_\omega$ such that $a_x \widetilde{\in} F$ and $a_y \widetilde{\in} G$ and so, $x \in F(a) \in \Im$ and $y \in G(a) \in \Im_\omega$. Therefore, we must have $F(a) = X$ and so, $y \in F(a) \cap G(a)$. Hence, $F \widetilde{\cap} G \neq 0_A$. Therefore, (X, τ, A) is not soft ω-T_2.*

Theorem 29. *Every soft ω-T_2 STS is soft T_1.*

Proof. Let (X, τ, A) be soft ω-T_2. We show that $1_A - a_x \in \tau$ for every $a_x \in SP(X, A)$. Let $a_x \in SP(X, A)$ and let $b_y \widetilde{\in} 1_A - a_x$. Since (X, τ, A) be soft ω-T_2, then there exist $F \in \tau_\omega$ and $G \in \tau$ such that $a_x \widetilde{\in} F$, $b_y \widetilde{\in} G$ and $F \widetilde{\cap} G = 0_A$. Then we have $G \in \tau$ and $b_y \widetilde{\in} G \widetilde{\subseteq} 1_A - F \widetilde{\subseteq} 1_A - a_x$. This shows that $1_A - a_x \in \tau$. □

The following example shows that the converse of Theorem 29 is not true in general:

Example 9. *Let X be any uncountable set, $A = \mathbb{Z}$, and $\tau = \{0_A\} \cup \{F \in SS(X, A) : X - F(a)$ is countable for all $a \in A\}$. Then (X, τ, A) is soft T_1. If (X, τ, A) is soft ω-T_2, then by Theorem 23, (X, τ_0) is ω-T_2. It is not difficult to see that τ_0 is the cocountable topology on X, and so $\left(\tau_0\right)_\omega = \tau_0$. Thus, (X, τ_0) is ω-T_2 implies that (X, τ_0) is T_2. However, it is well known that (X, τ_0) is not T_2.*

Theorem 30. *Every soft locally countable soft T_1 STS is soft ω-T_2.*

Proof. Let (X, τ, A) be soft locally countable T_1 STS. Let $a_x, b_y \in SP(X, A)$ with $a_x \neq b_y$. Since (X, τ, A) is soft locally countable, then by Corollary 5 of [2], (X, τ_ω, A) is a discrete STS and so $b_y \in \tau_\omega$. Since (X, τ, A) is soft T_1, then $b_y \in \tau^c$. Put $F = 1_A - b_y$ and $G = b_y$. Then $F \in \tau$ and $G \in \tau_\omega$, $a_x \widetilde{\in} F$, $b_y \widetilde{\in} G$, and $F \cap G = 0_A$. Hence, (X, τ, A) is soft ω-T_2. □

Theorem 31. *Soft T_2 STSs are Soft ω-T_2.*

Proof. Straightforward. □

Soft ω-T_2 STSs are not Soft T_2, in general, as it the following example shows:

Example 10. *Let $(X, \tau(\Im), A)$ be as in Example 6. Since $(X, \tau(\Im), A)$ is soft locally countable and soft T_1, then by Theorem 30, $(X, \tau(\Im), A)$ is soft ω-T_2. On the other hand, it is well known that $(X, \Im)$ is not a T_2 TS. So, by Corollary 7, $(X, \tau(\Im), A)$ is not soft T_2.*

The following example shows that soft ω-T_2 STSs are not soft ω-regular, in general:

Example 11. *Let $X = \mathbb{R}$, $A = \mathbb{N}$, $\Im$ is the usual topology on $\mathbb{R}$ and $\tau = \{F \in SS(X, A) : F(a) \in \Im_\omega \text{ for all } a \in A\}$. It is known that $(X, \Im_\omega)$ is T_2 but not regular. This implies that (X, τ, A) is soft T_2 but not soft regular. Since clearly that (X, τ, A) is soft anti-locally countable, then by Theorem 20, (X, τ, A) is not soft ω-regular.*

Theorem 32. *Every soft ω-regular soft T_1 STS is soft ω-T_2.*

Proof. Let (X, τ, A) be ω-regular T_1 STS. Let $a_x, b_y \in SP(X, A)$ with $a_x \neq b_y$. Since (X, τ, A) is T_1, then $b_y \in \tau^c$. Since (X, τ, A) is soft ω-regular, $b_y \in \tau^c$ and $a_x \widetilde{\in} 1_A - b_y$, then there exist $F \in \tau$ and $G \in \tau_\omega$ such that $a_x \widetilde{\in} F$, $b_y \widetilde{\subseteq} G$, and $F \cap G = 0_A$. Therefore, (X, τ, A) is soft ω-T_2. □

5. Conclusions

As three new notions of STSs, soft ω-local indiscrete, soft ω-regular, and soft ω-T_2 are introduced and investigated. The study deals with soft products and soft subspaces. Additionally, the study focuses on induced TSs and generated STSs. We expect that our results will be important for the forthcoming in STSs to build a good background for some practical applications and to answer the most intricate problems containing uncertainty in medical, engineering, economics, environment, and in general. The following topics could be considered in future studies: (1) To solve two open questions raised in this work; (2) To define separation soft ω-normality.

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