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On the Oscillation of Solutions of Differential Equations with Neutral Term

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Abstract: In this work, new criteria for the oscillatory behavior of even-order delay differential equations with neutral term are established by comparison technique, Riccati transformation and integral averaging method. The presented results essentially extend and simplify known conditions in the literature. To prove the validity of our results, we give some examples.

Keywords: oscillation; even order; neutral coefficients; differential equation

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1. Introduction

Neutral/delay differential equations are used in a variety of problems in economics, biology, medicine, engineering and physics, including lossless transmission lines, vibration of bridges, as well as vibrational motion in flight, and as the Euler equation in some variational problems, see [1–3].

Nowadays, there is an ongoing interest in obtaining several sufficient conditions for the oscillatory properties of the solutions of different kinds of differential equations, especially their the oscillation and asymptotic, see Agarwal et al. [4] and Saker [5].

Baculikova [6], Dzrina and Jadlovska [7], and Bohner et al. [8] developed approaches and techniques for studying oscillation criteria in order to improve the oscillation criteria of second-order differential equations with delay/advanced terms. Xing et al. [9] and Moaaz et al. [10] also extended this evolution to differential equations of the neutral type. Therefore, there are many studies on the oscillatory and asymptotic behavior of different orders of some differential equations, see [11–25].

Xing et al. [9] discussed the oscillation and asymptotic properties for equation

$$\left(\gamma(t)\left(z^{(r-1)}(t)\right)^{\alpha}\right)' + a(t)\varphi(x(w(t))) = 0,$$

where $z(t) = x(t) + h(t)x(\beta(t))$ and $0 \leq h(t) \leq h_0 < \infty$. They used comparison technique.

In [26], Zhang et al. studied the equation

$$\left(\gamma(t)\left(z^{(r-1)}(t)\right)^{\alpha}\right)' + a(t)x^{\beta}(\beta(t)) = 0,$$

under condition $\int_{t_0}^{\infty} \gamma^{-1/\alpha}(s) ds < \infty$ and they used comparison and Riccati techniques.

In case $\gamma(t) = 1$ and $\alpha = 1$, the authors in [27,28] studied the oscillatory properties for equation

$$z^{(r)}(t) + a(t)x(w(t)) = 0, \tag{1}$$

where r is an even and under the condition $0 \le h(t) < 1$.

In [29,30], authors investigated the oscillatory solutions of (1) where $h(t) \in [0, h_0]$ and $h_0 < \infty$.

Agarwal et al. [31] studied the oscillation conditions of the equation

$$\left[\left|z^{(r-1)}(t)\right|^{\alpha-1} z^{(r-1)}(t)\right]' + a(t)|x(\beta(t))|^{\alpha-1} x(\beta(t)) = 0,$$

where $\alpha > 1$. The authors used comparison method to find this conditions.

Elabbasy et al. [32] were interested in discussing the oscillatory properties of the equation

$$\left[\gamma(t)\left|\left(z^{(r-1)}(t)\right)\right|^{p-2} z^{(r-1)}(t)\right]' + a(t)\varphi(x(\beta(t))) = 0,\ p > 1,$$

under the assumption that

$$\int_{t_0}^{\infty} \frac{1}{\gamma^{1/(p-1)}(s)} ds = \infty$$

and r is an even positive integer.

Based on the above results of previous scholars, in this work, we are concerned with the following differential equations with neutral term of the form

$$\left(\gamma(t) z^{(r-1)}(t)\right)' + \sum_{i=1}^{j} a_i(t)\varphi(x(w_i(t))) = 0, \tag{2}$$

where $j \ge 1$, and

$$z(t) = |x(t)|^{p-2} x(t) + h(t)x(\beta(t)). \tag{3}$$

Throughout this work, we suppose the following hypotheses:

$$\begin{cases} \gamma, h \in C([t_0,\infty),[0,\infty)),\ a_i \in C([t_0,\infty),\mathbb{R}^+), \gamma(t) > 0, \gamma'(t) \ge 0, 0 \le h(t) < 1; \\ \beta \in C([t_0,\infty),(0,\infty)),\ \beta(t) \le t,\ \lim_{t\to\infty} \beta(t) = \infty; \\ \varphi \in C(\mathbb{R},\mathbb{R}), \varphi(x) \ge |x|^{p-2}x \text{ for } x \ne 0; \\ w_i \in C([t_0,\infty),\mathbb{R}), w_i(t) \le t, w_i'(t) > 0, \lim_{t\to\infty} w_i(t) = \infty,\ i = 1,2,...,j; \\ r \text{ and } p \text{ are positive integers, } r \text{ is even, } r \ge 2,\ p > 1. \end{cases}$$

Definition 1. *The function $x \in C^{r-1}[t_x,\infty)$, $t \ge t_x \ge t_0$, is called a solution of (2), if $\gamma(t)z^{(r-1)}(t) \in C^1[t_x,\infty)$, and $x(t)$ satisfies (2) on $[t_x,\infty)$.*

Definition 2. *A solution of (2) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.*

The motivation for this article is to continue the previous works [33].

The authors in [34] used the comparison technique that differs from the one we used in this article. Our approach is based on using integral averaging method and the Riccati technique to reduce the main equation into a first-order inequality to obtain more effective oscillation conditions for Equation (2). Therefore, in order to highlight the novelty of the results that we obtained in this work, we presented a comparison between the previous results and our main results, represented in the Example 2.

Motivated by these reasons mentioned above, in this paper, we extend the results using integral averaging method and Riccati transformation under

$$\int_{t_0}^{\infty} \frac{1}{\gamma(s)} \mathrm{d}s = \infty. \tag{4}$$

These results contribute to adding some important conditions that were previously studied in the subject of oscillation of differential equations with neutral term. To prove our main results, we give some examples.

2. Oscillation Results

Now, we mention some important lemmas.

Lemma 1 ([34]). *Let $z(t)$ be an r times differentiable function on $[t_0, \infty)$ of constant sign and $z^{(r)}(t) \neq 0$ on $[t_0, \infty)$ which satisfies $z(t)z^{(r)}(t) \leq 0$. Then:*

(I) there exists $t_1 \geq t_0$ such that the functions $z^{(i)}(t),\ i = 1, 2, ..., r-1$, are of constant sign on $[t_0, \infty)$;

(II) there exists a number $l \in \{1, 3, 5, ..., r-1\}$ when r is even, $l \in \{0, 2, 4, ..., r-1\}$ when r is odd, such that, for $t \geq t_1$,

$$z(t)z^{(i)}(t) > 0,$$

for all $i = 0, 1, ..., l$ and

$$(-1)^{r+i+1} z(t)z^{(i)}(t) > 0,$$

for all $i = l+1, ..., r$.

Lemma 2 ([34]). *If $z \in C^r([t_0, \infty), (0, \infty))$ and $z^{(r-1)}(t)z^{(r)}(t) \leq 0$ for $t \geq t_0$, then for every $\varepsilon \in (0,1)$ there exists a constant $\ell > 0$ such that*

$$z(\varepsilon t) \geq \ell t^{r-1} \left| z^{(r-1)}(t) \right|,$$

for all large t.

Lemma 3 ([32]). *Let $z \in C^r([t_0, \infty), (0, \infty))$ and $z^{(r-1)}(t)z^{(r)}(t) \leq 0$. If $\lim_{t\to\infty} z(t) \neq 0$, then for every $\mu \in (0,1)$ there exists a $t_\mu \geq t_0$ such that*

$$z(t) \geq \frac{\mu}{(n-1)!} t^{r-1} \left| z^{(r-1)}(t) \right| \text{ for } t \geq t_\mu.$$

Lemma 4. *Assume that $x(t)$ is a positive solution of Equation (2). Then*

$$z(t) > 0,\ z'(t) > 0,\ z^{(r-1)}(t) \geq 0 \ \text{ and } \ z^{(r)}(t) \leq 0, \tag{5}$$

for $t \geq t_1 \geq t_0$.

Proof. Suppose that $x(t)$ is a positive solution of Equation (2). Then, we can assume that $x(t) > 0$, $x(\beta(t)) > 0$ and $x(w(t)) > 0$ for $t \geq t_1$. Hence, we deduce $z(t) > 0$ and

$$\left(\gamma z^{(r-1)} \right)'(t) = -\sum_{i=1}^{j} a_i(t) \varphi(x(w_i(t))) \leq 0. \tag{6}$$

Which means that $\gamma(t)z^{(r-1)}(t)$ is decreasing and $z^{(r-1)}(t)$ is eventually of one sign.

We see that $z^{(r-1)}(t) > 0$. Otherwise, if there exists a $t_2 \geq t_1$ such that $z^{(r-1)}(t) < 0$ for $t \geq t_2$, and

$$\left(\gamma z^{(r-1)} \right)(t) \leq \left(\gamma z^{(r-1)} \right)(t_2) = -L, L > 0. \tag{7}$$

Integrating (7) from t_2 to t we find

$$z^{(r-2)}(t) - z^{(r-2)}(t_2) \leq -L \int_{t_2}^{t} \frac{1}{\gamma(s)} \mathrm{d}s.$$

So, we get

$$z^{(r-2)}(t) \leq z^{(r-2)}(t_2) - L \int_{t_2}^{t} \frac{1}{\gamma(s)} \mathrm{d}s.$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} z^{(r-2)}(t) = -\infty$, which contradicts the fact that $z(t)$ is a positive solution by Lemma 1. Hence, we obtain $z^{(r-1)}(t) \geq 0$ for $t \geq t_1$.

From Equation (2), we obtain

$$\left(\gamma' t^{(r-1)}\right)(t) + \left(\gamma t^{(r)}\right)(t) - \sum_{i=1}^{j} a_i(t)\varphi(x(w_i(t))) \leq 0. \tag{8}$$

From Equations (4) and (8), we find

$$\left(\gamma z^{(r)}\right)(t) = -\left(\gamma' z^{(r-1)}\right)(t) - \sum_{i=1}^{j} a_i(t)\varphi(x(w_i(t))) \leq 0,$$

this implies that $z^{(r)}(t) \leq 0,\ t \geq t_1$. By using Lemma 1, we find that (5) holds. The proof is complete. □

Theorem 1. *If the equation*

$$x'(t) + \widehat{M}(t)x(w_i(t)) = 0 \tag{9}$$

is oscillatory, where

$$\widehat{M}(t) := \frac{\mu w_i^{r-1}(t)}{(r-1)!\gamma(w_i(t))} M(t)$$

and

$$M(t) := \sum_{i=1}^{j} a_i(t)(1 - h(w_i(t))),$$

then (2) is oscillatory.

Proof. Suppose that (2) has a nonoscillatory solution. Without loss of generality, we can assume that $x(t) > 0$. Using Lemma 4, we find that (5) holds. From (3), we see

$$z(t) = |x(t)|^{p-2}x(t) + h(t)x(\beta(t)),$$

we see that

$$\begin{aligned} x^{p-1}(t) &= z(t) - h(t)x(\beta(t)) \\ &\geq z(t) - h(t)z(\beta(t)) \\ &\geq z(t) - h(t)z(t) \\ &\geq (1 - h(t))z(t) \end{aligned}$$

and so

$$x^{p-1}(w_i(t)) \geq z(w_i(t))(1 - h(w_i(t))). \tag{10}$$

From (10), we see

$$\varphi(x(w_i(t))) \geq z(w_i(t))(1 - h(w_i(t))). \tag{11}$$

Combining (2) and (11), we find

$$\begin{aligned}\left(\gamma z^{(r-1)}\right)'(t) &\leq -\sum_{i=1}^{j} a_i(t)z(w_i(t))(1-h(w_i(t))) \\ &\leq -z(w(t))\sum_{i=1}^{j} a_i(t)(1-h(w_i(t))) \\ &= -M(t)z(w_i(t)). \end{aligned} \tag{12}$$

By Lemma 3, we get

$$z(t) \geq \frac{\mu}{(r-1)!}t^{r-1}z^{(r-1)}(t),$$

for all $t \geq t_2 \geq \max\{t_1, t_\mu\}$. Thus, by using (12), we see

$$\left(\gamma(t)z^{(r-1)}(t)\right)' + \frac{\mu w_i^{r-1}(t)M(t)}{(r-1)!\gamma(w_i(t))}\left(\gamma(w_i(t))z^{(r-1)}(w_i(t))\right) \leq 0.$$

Therefore, we get $x(t) = \gamma(t)z^{(r-1)}(t)$ is a positive solution of the inequality

$$x'(t) + \widehat{M}(t)x(w_i(t)) \leq 0.$$

From [23] (Corollary 1), we find Equation (9) also has a positive solution, a contradiction. Theorem 1 is proved. □

By using Theorem 2.1.1 in [35], we get the following corollary.

Corollary 1. *If*

$$\liminf_{t\to\infty}\int_{w_i(t)}^{t}\frac{w_i^{r-1}(s)}{\gamma(w_i(s))}M(s)\mathrm{d}s > \frac{(r-1)!}{\mu \mathrm{e}},$$

for some constant $\mu \in (0,1)$, then (2) is oscillatory.

Theorem 2. *If $\varpi \in C^1([t_0,\infty),\mathbb{R}^+)$ and $\ell > 0$ such that*

$$\int_{t_0}^{\infty}\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}\left(\frac{\varpi'(u)}{\varpi(u)}\right)^2 A(u)\right)\mathrm{d}u = \infty, \tag{13}$$

for $\varepsilon \in (0,1)$, then (2) is oscillatory, where

$$A(t) := \frac{\gamma(t)\varpi(t)}{\ell w_i^{r-2}(t)w_i'(t)}.$$

Proof. Assume on the contrary that (2) has a nonoscillatory, say positive solution x. From Lemma 2 with $x = z'$, there exists a $\ell > 0$ and $w_i(t) \leq t$ such that

$$\begin{aligned} z'(\varepsilon w_i(t)) &\geq \ell w_i^{r-2}(t)z^{(r-1)}(w_i(t)) \\ &\geq \ell w_i^{r-2}(t)z^{(r-1)}(t). \end{aligned} \tag{14}$$

Defining

$$B(t) := \varpi(t)\frac{\gamma(t)z^{(r-1)}(t)}{z(\varepsilon w_i(t))} > 0,$$

we have

$$B'(t) = \frac{\varpi'(t)}{\varpi(t)}B(t) + \varpi(t)\frac{\left(\gamma(t)z^{(r-1)}(t)\right)'}{z(\varepsilon w_i(t))} - \varepsilon\varpi(t)\frac{\gamma(t)z^{(r-1)}(t)z'(\varepsilon w_i(t))w_i'(t)}{(z(\varepsilon w(t)))^2}.$$

From (12), we obtain

$$B'(t) \leq \frac{\varpi'(t)}{\varpi(t)}B(t) - \varpi(t)M(t) - \varepsilon\frac{z'(w_i(t))w_i'(t)}{z(\varepsilon w_i(t))}B(t).$$

By using (14), we have

$$\begin{aligned} B'(t) &\leq \frac{\varpi'(t)}{\varpi(t)}B(t) - \varpi(t)M(t) - \varepsilon\frac{\ell w_i^{r-2}(t)z^{(r-1)}(t)w_i'(t)}{z(\varepsilon w_i(t))}B(t) \\ &\leq \frac{\varpi'(t)}{\varpi(t)}B(t) - \varpi(t)M(t) - \varepsilon\frac{\ell w_i^{r-2}(t)w_i'(t)}{\gamma(t)\varpi(t)}\frac{\varpi(t)\gamma(t)z^{(r-1)}(t)}{z(\varepsilon w_i(t))}B(t) \\ &\leq \frac{\varpi'(t)}{\varpi(t)}B(t) - \varpi(t)M(t) - \frac{\varepsilon}{A(t)}B^2(t). \end{aligned} \tag{15}$$

Using the inequality

$$xz - uz^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{x^{\gamma+1}}{u^\gamma},$$

with $x = \varpi'/\varpi$, $u = \varepsilon\ell w_i^{r-2}(t)w_i'(t)/(\gamma(t)\varpi(t))$ and $z = B(t)$, we find

$$B'(t) \leq -\varpi(t)M(t) + \frac{1}{4\varepsilon}\left(\frac{\varpi'(t)}{\varpi(t)}\right)^2\frac{\gamma(t)\varpi(t)}{\ell w_i^{r-2}(t)w_i'(t)}. \tag{16}$$

Integrating (16) from t_1 to t we find

$$\begin{aligned} \int_{t_1}^{t}\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}\left(\frac{\varpi'(u)}{\varpi(u)}\right)^2 A(u)\right)\mathrm{d}u &\leq B(t_1) - B(t) \\ &\leq B(t_1), \end{aligned}$$

which contradicts (13). Theorem 2 is proved. □

3. Philos-Type Oscillation Results

Definition 3. *Let*

$$D_0 = \{(t,s) : t > s > t_0\} \text{ and } D = \{(t,s) : t \geq s \geq t_0\}.$$

A function $G \in C(D,\mathbb{R})$ is said to belong to the function class ψ, written by $G \in \psi$, if

(i) *$G(t,s) > 0$ on D_0 and $G(t,s) = 0$ for $t \geq t_0$ with $(t,s) \notin D_0$;*

(ii) *$G(t,s)$ has a continuous and nonpositive partial derivative $\partial G/\partial s$ on D_0 and $g \in C(D_0,\mathbb{R})$ such that*

$$\frac{\partial G(t,s)}{\partial s} = -g(t,s)\sqrt{G(t,s)}.$$

Theorem 3. *If $\varpi \in C^1([t_0,\infty),\mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty}\frac{1}{G(t,t_0)}\int_{t_0}^{t}G(t,u)\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}A(u)\psi^2(t,u)\right)\mathrm{d}u = \infty, \tag{17}$$

where

$$\psi(t,s) = \frac{\varpi'(s)}{\varpi(s)} - \frac{g(t,s)}{\sqrt{G(t,s)}},$$

for $\varepsilon \in (0,1)$*, then* (2) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1. By Theorem 2, we see that (15) holds. Multiplying (15) by $G(t,s)$ and integrating both sides from t_2 to t, we obtain

$$\begin{aligned}\int_{t_2}^{t} G(t,u)\varpi(u)M(u)\mathrm{d}u \quad &\leq \quad -\int_{t_2}^{t} G(t,u)B'(u)\mathrm{d}u - \int_{t_2}^{t} G(t,u)\frac{\varepsilon}{A(u)}B^2(u)\mathrm{d}u \\ &\quad + \int_{t_2}^{t} G(t,u)\frac{\varpi'(u)}{\varpi(u)}B(u)\mathrm{d}u \\ &\leq \quad G(t,t_2)B(t_2) - \int_{t_2}^{t} G(t,u)\frac{\varepsilon}{A(u)}B^2(u)\mathrm{d}u \\ &\quad + \int_{t_2}^{t} G(t,u)B(u)\psi(t,u)\mathrm{d}u\end{aligned}$$

which implies that

$$\begin{aligned}\int_{t_2}^{t} G(t,u)\varpi(u)M(u)\mathrm{d}u \quad &\leq \quad G(t,t_2)B(t_2) \\ &\quad - \int_{t_2}^{t} G(t,u)\frac{\varepsilon}{A(u)}\left(B^2(u) - \frac{A(u)}{\varepsilon}\psi(t,u)B(u)\right)\mathrm{d}u.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\frac{1}{G(t,t_2)}\int_{t_2}^{t} G(t,u)\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}A(u)\psi^2(t,u)\right)\mathrm{d}u \\ &\leq B(t_2) - \frac{1}{G(t,t_2)}\int_{t_2}^{t} G(t,u)\frac{\varepsilon}{A(u)}\left(B(u) - \frac{1}{2\varepsilon}A(u)\psi(t,u)\right)^2\mathrm{d}u,\end{aligned}$$

which implies

$$\limsup_{t\to\infty}\frac{1}{G(t,t_2)}\int_{t_2}^{t} G(t,u)\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}A(u)\psi^2(t,u)\right)\mathrm{d}u \leq B(t_2).$$

From (17), we have a contradiction. Theorem 3 is proved. □

Corollary 2. *Suppose that*

$$0 < \inf_{s\geq t}\left(\liminf_{t\to\infty}\frac{G(t,s)}{G(t,t_0)}\right) \leq \infty$$

and

$$\limsup_{t\to\infty}\frac{1}{G(t,t_0)}\int_{t_0}^{t} G(t,u)A(u)\psi^2(t,u)\mathrm{d}u < \infty.$$

If there exists a function $\varphi \in C([t_0,\infty),\mathbb{R})$ *satisfying for* $t \geq t_0$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\varphi_+^2(s)}{A(s)}\mathrm{d}s = \infty$$

where $\varphi_+(t) = \max\{\varphi(t),0\}$*, and also*

$$\limsup_{t\to\infty}\frac{1}{G(t,t_0)}\int_{t_0}^{t} G(t,u)\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}A(u)\psi^2(t,u)\right)\mathrm{d}u \geq \sup_{t\geq t_0}\varphi(t),$$

then (2) is oscillatory.

Example 1. *Let second-order equation:*

$$\left[t\left(x(t)+\frac{1}{2}x\left(\frac{t}{3}\right)\right)'\right]' + \frac{a_0}{t}\left(x^2+x\right)\left(\frac{t}{2}\right) = 0, t \geq 1, \tag{18}$$

where $a_0 > 0$ *is a constant. Let* $r = p = 2$, $\gamma(t) = t$, $h(t) = 1/2$, $\beta(t) = t/3$, $a(t) = a_0/t$, $w_i(t) = t/2$, $\varphi(x) = x^2 + x$.

Thus, we find

$$M(t) = a(t)(1 - h(w_i(t))) = \frac{a_0}{2t}.$$

If we set $\varpi = t$*, then* $A(t) = \frac{\gamma(t)\varpi(t)}{\ell w_i^{r-2}(t) w_i'(t)} = \frac{2t^2}{\ell}$ *and for any constants* $\ell > 0$, $0 < \varepsilon < 1$, *we have*

$$\begin{aligned} &\int_{t_0}^{\infty}\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}\left(\frac{\varpi'(u)}{\varpi(u)}\right)^2 A(u)\right)\mathrm{d}u \\ = &\int_{t_0}^{\infty}\left(\frac{a_0}{2} - \frac{1}{2\varepsilon\ell}\right)\mathrm{d}u \\ = &\ \infty \quad \text{if } a_0 > 1. \end{aligned}$$

Using Theorem 2, Equation (18) is oscillatory if $a_0 > 1$.

Example 2. *Consider the fourth-order equation:*

$$\left[t z'''(t)\right]' + \frac{b}{t}x\left(\frac{t}{3}\right) = 0, t \geq 1, \tag{19}$$

where $z(t) = x(t) + \frac{1}{3}x\left(\frac{t}{2}\right)$ *and* $b > 0$ *is a constant. Let* $r = 4$, $p = 2$, $\gamma(t) = t$, $h(t) = 1/3$, $\beta(t) = t/2$, $a(t) = b/t$, $w_i(t) = t/3$, $\varphi(x) = x$.

Thus, we see that

$$\int^{\infty} \frac{1}{\gamma(t)} dz = \infty.$$

If we set $G(t,s) = (t-s)^2$, $g(t,s) = 2$ *and* $\varpi = 1$*, then*

$$A(t) = \frac{\gamma(t)\varpi(t)}{\ell w_i^{r-2}(t) w_i'(t)} = \frac{27}{\ell t}$$

and

$$\psi(t,s) = \frac{\varpi'(s)}{\varpi(s)} - \frac{g(t,s)}{\sqrt{G(t,s)}} = -\frac{2}{t-s}.$$

So, it can be easily verified that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{G(t,t_0)} \int_{t_0}^{t} G(t,u)\left(\varpi(u)M(u) - \frac{1}{4\varepsilon}A(u)\psi^2(t,u)\right)\mathrm{d}u \\ = &\ \infty. \end{aligned}$$

Using Theorem 3, Equation (19) is oscillatory.

Remark 1. *The results of [33] cannot solve (19) because of* $\gamma(t) = t$*. Thus, our results extend and complement upon the results of previous papers on this topic.*

4. Conclusions

In this work, a large amount of attention has been focused on the oscillation problem of Equation (2). By Riccati transformation, comparison technique and integral averages method, we establish some new oscillation conditions. These results contribute to adding some important criteria that were previously studied in the literature. For future consideration, it will be of a great importance to study the oscillation of

$$\left[\gamma(t)\left|\left(z^{(r-1)}(t)\right)\right|^{p-2}z^{(r-1)}(t)\right]' + a(t)\varphi(x(\beta(t))) = 0,$$

under the assumption that

$$\int_{t_0}^{\infty} \frac{1}{\gamma^{1/(p-1)}(s)} ds < \infty,$$

where $z(t) = |x(t)|^{p-2}x(t) + h(t)x(\beta(t))$ and $p > 1$ is a constant.

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