mathematics

Article

Some Results of Extended Beta Function and Hypergeometric Functions by Using Wiman's Function

Shilpi Jain [1], Rahul Goyal [2], Praveen Agarwal [2,3,4,5], Antonella Lupica [6] and Clemente Cesarano [7,*]

[1] Department of Mathematics, Poornima College of Engineering, Jaipur 302021, India; shilpijain1310@gmail.com or shilpi.jain@poornima.org
[2] Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India; rahul.goyal01@anandice.ac.in (R.G.); praveen.agarwal@anandice.ac.in or goyal.praveen2011@gmail.com (P.A.)
[3] Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211019, India
[4] International Center for Basic and Applied Sciences, Jaipur 302029, India
[5] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
[6] Department of Economics, Engineering, Society and Business Organization-DEIM Department, Tuscia University, 01100 Viterbo, Italy; alupica@unitus.it
[7] Section of Mathematics, International Telematic University Uninettuno, 00186 Roma, Italy
* Correspondence: c.cesarano@uninettunouniversity.net

Abstract: The main aim of this research paper is to introduce a new extension of the Gauss hypergeometric function and confluent hypergeometric function by using an extended beta function. Some functional relations, summation relations, integral representations, linear transformation formulas, and derivative formulas for these extended functions are derived. We also introduce the logarithmic convexity and some important inequalities for extended beta function.

Keywords: classical Euler beta function; gamma function; Gauss hypergeometric function; confluent hypergeometric function; Mittag-Leffler function

MSC: 33B15; 33C05; 33C15; 33E12

Citation: Jain, S.; Goyal, R.; Agarwal, P.; Lupica, A.; Cesarano, C. Some Results of Extended Beta Function and Hypergeometric Functions by Using Wiman's Function. *Mathematics* **2021**, *9*, 2944. https://doi.org/10.3390/math9222944

Academic Editors: Alberto Cabada, Manuel Manas and Sitnik Sergey

Received: 21 September 2021
Accepted: 15 November 2021
Published: 18 November 2021

1. Introduction and Preliminaries

The theory of special functions, especially extensions of beta function, gamma function, hypergeometric functions has been one of the fastest rising investigate a topic in mathematical science because scientific researchers feel it is important to study the behavior of special functions with extended domains. Special functions are naturally generalizations of the elementary functions and they play a vital role in the solution of the differential equations due to which study of the generalizations of the special functions always important. In the past years, many extensions, properties, and applications of special functions have been discussed by many researchers and authors [1,2]. Here, we aim to study some results of the extended beta function and hypergeometric functions by using Wiman's function.

To obtain our main results, we require prior knowledge of some special functions. First of all, throughout this present paper, there is a need to note the following notations: $\Re(z)$ represents the real part of any complex number z, C represents a set of complex numbers, and R^+ represents a set of positive real numbers.

Classical Euler beta function and gamma function are defined as [3]:

$$B(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1}dt, \tag{1}$$

where $\Re(y_1)$ and $\Re(y_2) > 0$;

$$\Gamma(y_1) = \int_0^\infty t^{y_1-1}e^{-t}dt, \tag{2}$$

where $\Re(y_1) > 0$.

Gauss hypergeometric function ${}_2F_1$ is defined as [4]:

$$ {}_2F_1(p_0, p_1, p_2; z) = F(p_0, p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{(p_0)_k (p_1)_k}{(p_2)_k} \frac{z^k}{k!}, \tag{3} $$

where $(u)_k$ represents the Pochhammer symbol defined below:

$$ (u)_k := \frac{\Gamma(u+k)}{\Gamma(u)} = \begin{cases} 1 & k = 0; u \in \mathbb{C} \setminus \{0\}, \\ u(u+1)\cdots(u+k-1) & k \in \mathbb{N}; u \in \mathbb{C}. \end{cases} $$

Series representation and integral representation of Gauss hypergeometric function ${}_2F_1$ is defined as [4]:

$$ F(r_0, r_1, r_2; z) = \sum_{n=0}^{\infty} \frac{B(r_1 + n, r_2 - r_1)}{B(r_1, r_2 - r_1)} (r_0)_n \frac{z^n}{n!}, \tag{4} $$

where $\Re(r_2) > \Re(r_1) > 0$ and $\mid z \mid < 1$.

$$ F(r_0, r_1, r_2; z) = \frac{1}{B(r_1, r_2 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{r_2 - r_1 - 1} (1 - zt)^{-r_0} dt. \tag{5} $$

Confluent hypergeometric function is defined as [4]:

$$ {}_1F_1(p_1, p_2; z) = \Phi(p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{(p_1)_k}{(p_2)_k} \frac{z^k}{k!}. \tag{6} $$

Series representation and integral representation of confluent hypergeometric function is defined as [4]:

$$ \Phi(r_1, r_2; z) = \sum_{n=0}^{\infty} \frac{B(r_1 + n, r_2 - r_1)}{B(r_1, r_2 - r_1)} \frac{z^n}{n!}, \tag{7} $$

where $\Re(r_2) > \Re(r_1) > 0$;

$$ \Phi(r_1, r_2; z) = \frac{1}{B(r_1, r_2 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{r_2 - r_1 - 1} e^{zt} dt. \tag{8} $$

One parameter Mittag-Leffler function and two parameter Mittag-Leffler function (Wiman's function) is defined as [5–7]:

$$ E_{y_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k y_1 + 1)}, \tag{9} $$

where $\Re(y_1) \geq 0$ and $z \in C$;

$$ E_{y_1, y_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k y_1 + y_2)}, \tag{10} $$

where $\Re(y_1) \geq 0$, $\Re(y_2) \geq 0$ and $z \in C$.

Since 1900, many researchers have been working on extensions of special functions. Chaudhry is one of the mathematician who continuously worked on extensions of beta function and other special functions. In 1997, Chaudhry et al. extended the classical Euler beta function defined in Reference [8], and then, in 2004, he used this extended beta function to generalize the Gauss hypergeometric function and confluent hypergeometric function as [9]:

$$ B(y_1, y_2; y) = \int_0^1 t^{y_1 - 1} (1-t)^{y_2 - 1} e^{\frac{-y}{t(1-t)}} dt, \tag{11} $$

where $\Re(y_1) > 0$, $\Re(y_2) > 0$ and $\Re(y) > 0$;

$$F_r(r_0, r_1, r_2; z) = \sum_{n=0}^{\infty} \frac{B(r_1 + n, r_2 - r_1; r)}{B(r_1, r_2 - r_1)} (r_0)_n \frac{z^n}{n!}, \tag{12}$$

where $r \geq 0$, $\Re(r_2) > \Re(r_1) > 0$, $| z |< 1$;

$$\Phi_r(r_1, r_2; z) = \sum_{n=0}^{\infty} \frac{B(r_1 + n, r_2 - r_1; r)}{B(r_1, r_2 - r_1)} \frac{z^n}{n!}, \tag{13}$$

where $r \geq 0$ and $\Re(r_2) > \Re(r_1) > 0$.

Remark 1. *If we set $r = 0$ in (12) and (13), we get Gauss hypergeometric function and confluent hypergeometric function given by (4) and (7), respectively:*

$$F_0(r_0, r_1, r_2; z) = F(r_0, r_1, r_2; z), \tag{14}$$

and

$$\Phi_0(r_1, r_2; z) = \Phi(r_1, r_2; z). \tag{15}$$

Subsequently, Özergin et al. [10] have used confluent hypergeometric function to extend classic Euler beta function; moreover, the latter function was then exploited to generalize hypergeometric functions as follows:

$$B_u^{(u_1,u_2)}(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} {}_1F_1\left(u_1, u_2; \frac{-u}{t(1-t)}\right) dt, \tag{16}$$

where $\Re(y_1), \Re(y_2) > 0, \Re(u_1) > 0, \Re(u_2) > 0$ and $\Re(u) > 0$;

$$F_r^{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{B_r^{(r_1,r_2)}(p_1 + k, p_2 - p_1)}{B(p_1, p_2 - p_1)} (p_0)_k \frac{z^k}{k!}, \tag{17}$$

where $\Re(p_2) > \Re(p_1) > 0$, $| z |< 1$ and $r \geq 0$;

$$\Phi_r^{(r_1,r_2)}(p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{B_r^{(r_1,r_2)}(p_1 + k, p_2 - p_1)}{B(p_1, p_2 - p_1)} \frac{z^k}{k!}, \tag{18}$$

where $\Re(p_2) > \Re(p_1) > 0$ and $r \geq 0$.

Later, in 2014, Choi et al. [11] defined a generalization of extended beta function reported in (11), using it to broaden the definition of extended hypergeometric Gauss function and extended confluent hypergeometric as follows:

$$B(y_1, y_2; u; v) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} e^{\left(\frac{-u}{t} - \frac{v}{(1-t)}\right)} dt, \tag{19}$$

where $\Re(y_1), \Re(y_2) > 0, \Re(v) > 0$ and $\Re(u) > 0$;

$$F_{r,s}(r_0, r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1; r, s)}{B(r_1, r_2 - r_1)} (r_0)_k \frac{z^k}{k!}, \tag{20}$$

where $\Re(r_2) > \Re(r_1) > 0$, $r \geq 0$, $s \geq 0$ and $| z |< 1$;

$$\Phi_{r,s}(r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1; r, s)}{B(r_1, r_2 - r_1)} \frac{z^k}{k!}, \tag{21}$$

where $\Re(r_2) > \Re(r_1) > 0$, $s \geq 0$ and $r \geq 0$.

Remark 2. *If we set $r = 0$ and $s = 0$ in (20) and (21), we get Gauss hypergeometric function and confluent hypergeometric function given by (4) and (7), respectively:*

$$F_{0,0}(r_0, r_1, r_2; z) = F(r_0, r_1, r_2; z), \tag{22}$$

and

$$\Phi_{0,0}(r_1, r_2; z) = \Phi(r_1, r_2; z). \tag{23}$$

In 2018, Shadab et al. [12] extended classic Euler beta function using one parameter Mittag-Leffler function; this generalization was then exploited to expand hypergeometric function of the Gauss and confluent hypergeometric function, thus allowing to study various properties and relationships of these functions:

$$B_u^y(y_1, y_2) = \int_0^1 t^{y_1 - 1}(1 - t)^{y_2 - 1} E_u\left(\frac{-y}{t(1 - t)}\right) dt, \tag{24}$$

where $\Re(y_1), \Re(y_2) > 0$, $\Re(u) > 0$ and $\Re(y) > 0$;

$$F_{r,\alpha}(r_0, r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B_\alpha^r(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} (r_0)_k \frac{z^k}{k!}, \tag{25}$$

where $\Re(r_2) > \Re(r_1) > 0$, $\alpha \geq 0$, $r \geq 0$ and $\mid z \mid < 1$;

$$\Phi_{r,\alpha}(r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B_\alpha^r(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} \frac{z^k}{k!}, \tag{26}$$

where $\Re(r_2) > \Re(r_1) > 0$, $\alpha \geq 0$ and $r \geq 0$.

Remark 3. *(i) If we set $\alpha = 1$ in (25) and (26), we get extended Gauss hypergeometric function and extended confluent hypergeometric function given by (12) and (13), respectively:*

$$F_{r,1}(r_0, r_1, r_2; z) = F_r(r_0, r_1, r_2; z), \tag{27}$$

and

$$\Phi_{r,1}(r_1, r_2; z) = \Phi_r(r_1, r_2; z). \tag{28}$$

(ii) If we set $\alpha = 1$ and $r = 0$ in (25) and (26), we get Gauss hypergeometric function and confluent hypergeometric function given by (4) and (7), respectively:

$$F_{0,1}(r_0, r_1, r_2; z) = F(r_0, r_1, r_2; z), \tag{29}$$

and

$$\Phi_{0,1}(r_1, r_2; z) = \Phi(r_1, r_2; z). \tag{30}$$

Very recently, Goyal et al. [13] introduced an extension of the beta function using the Wiman function, thus studying various properties and relationships of that function:

$$B_{(u_1,u_2)}^{(u)}(y_1, y_2) = \int_0^1 t^{y_1 - 1}(1 - t)^{y_2 - 1} E_{u_1,u_2}\left(-u(t(1 - t))^{-1}\right) dt, \tag{31}$$

where $\min\{\Re(y_1), \Re(y_2)\} > 0, \Re(u_1) > 0, \Re(u_2) > 0$, $u \geq 0$ and $E_{u_1,u_2}(z)$ is 2-parameter Mittag-Leffler function given by (10).

In this paper, we discuss convexity and inequalities related to extended beta function given in (31). Now, we give some definitions about convex functions and show some results.

Definition 1. *(References [14,15]) Let Y be a convex set in a real vector space, and let $g : Y \longrightarrow (-\infty, +\infty)$ be a function. Then, g is called convex on Y if the inequality*

$$g(\beta x_1 + (1-\beta)x_2) \leq \beta g(x_1) + (1-\beta)g(x_2) \tag{32}$$

holds for any $x_1, x_2 \in Y$ and $\beta \in [0,1]$.

The function g is concave if the function $-g$ is convex.

The function g is called logarithmically convex (or logarithmically concave, respectively) on Y if $g > 0$ and $\log g$ ($-\log g$, respectively) is convex (or concave, respectively) on Y.

Lemma 1. *(References [16–18]) (**Chebyshev's integral inequality**)*

Let $f_1, f_2 : [c,d] \subseteq (-\infty,+\infty) \longrightarrow (-\infty,+\infty)$ be integrable functions. Assume that:

$$[f_1(x) - f_1(y)][f_2(x) - f_2(y)] \gtreqless 0, \quad \forall\, x, y \in [c,d].$$

Let $h(x) : [c,d] \subseteq (-\infty,+\infty) \longrightarrow (-\infty,+\infty)$ be a positive integrable function. Then:

$$\int_c^d h(x)f_1(x)dx \cdot \int_c^d h(x)f_2(x)dx \gtreqless \int_c^d h(x)dx \cdot \int_c^d h(x)f_1(x)f_2(x)dx. \tag{33}$$

Lemma 2. *(References [19,20]) (**Hölder Inequality**) Let θ_1 and θ_2 be positive numbers such that*

$$\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1.$$

Let $f_1, f_2 : [c,d] \longrightarrow (-\infty,+\infty)$ be integrable functions. Then

$$\int_c^d f_1(x)f_2(x)dx \leq \left(\int_c^d f_1(x)^{\theta_1}dx\right)^{\frac{1}{\theta_1}} \cdot \left(\int_c^d f_2(x)^{\theta_2}dx\right)^{\frac{1}{\theta_2}}. \tag{34}$$

2. Inequalities of Extended Beta function $B^{(u)}_{(u_1,u_2)}(y_1, y_2)$

Theorem 1. *Assume that:*

- *x, y, x_1, y_1 are non-zero and non-negative numbers such that $(x - x_1)(y - y_1) \geq 0$,*
- *$u_1 \in [0,1]$ and $u_2 \in [0,1]$.*

Then,

$$B^{(u)}_{(u_1,u_2)}(x, y_1) \cdot B^{(u)}_{(u_1,u_2)}(x_1, y) \leq B^{(u)}_{(u_1,u_2)}(x, y)B^{(u)}_{(u_1,u_2)}(x_1, y_1). \tag{35}$$

Proof. Let $f_1, f_2, f_3 : [0,1] \longrightarrow \mathbb{R}^+ \cup 0$ be functions such that

$$f_1(t) = t^{(x-x_1)}, f_2(t) = (1-t)^{(y-y_1)},$$

$$f_3(t) = t^{(x_1-1)}(1-t)^{(y_1-1)}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)$$

Since

$$f_1'(t) = (x - x_1)t^{(x-x_1-1)}, \text{ and } f_2'(t) = (y - y_1)(1-t)^{(y-y_1-1)},$$

functions f_1 and f_2 have the same monotonicity (increasing or decreasing) on $[0,1]$ as $(x-x_1)(y-y_1) \geq 0$.

Then, applying Chebyshev's inequality given in (33), to f_1, f_2, and f_3, we have:

$$\begin{array}{c} \int_0^1 t^{x-1}(1-t)^{y_1-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt \cdot \int_0^1 t^{x_1-1}(1-t)^{y-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt \\ \int_0^1 t^{x-1}(1-t)^{y-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt \cdot \int_0^1 t^{x_1-1}(1-t)^{y_1-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt. \end{array} \quad (36)$$

Then, from definition of extended *beta function* (31), we get our desired result:

$$B^{(u)}_{(u_1,u_2)}(x,y_1) \cdot B^{(u)}_{(u_1,u_2)}(x_1,y) \leq B^{(u)}_{(u_1,u_2)}(x,y) \cdot B^{(u)}_{(u_1,u_2)}(x_1,y_1).$$

□

Corollary 1. *Assuming that* $x, y > 0$, $u_1 \in [0,1]$ *and* $u_2 \in [0,1]$, *then:*

$$\left[B^{(u)}_{(u_1,u_2)}(x,y)\right]^2 \geq B^{(u)}_{(u_1,u_2)}(x,x) \cdot B^{(u)}_{(u_1,u_2)}(y,y). \quad (37)$$

Proof. Put $y_1 = x$ and $x_1 = y$ in theorem (1). Using symmetry property of extended *beta function* $B^{(u)}_{(u_1,u_2)}(x,y) = B^{(u)}_{(u_1,u_2)}(y,x)$, we get our desired result:

$$\left[B^{(u)}_{(u_1,u_2)}(x,y)\right]^2 \geq B^{(u)}_{(u_1,u_2)}(x,x) \cdot B^{(u)}_{(u_1,u_2)}(y,y).$$

□

Theorem 2. *The map* $(x,y) \longmapsto B^{(u)}_{(u_1,u_2)}(x,y)$ *is logarithmically convex on* $\mathbb{R}^+ \times \mathbb{R}^+ \ \forall \ u \geq 0$, *with* $u_1 \in [0,1]$ *and* $u_2 \in [0,1]$. *Moreover:*

$$\left[B^{(u)}_{(u_1,u_2)}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]^2 \leq B^{(u)}_{(u_1,u_2)}(x_1,y_1) \cdot B^{(u)}_{(u_1,u_2)}(x_2,y_2). \quad (38)$$

Proof. Let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and let $a, b \geq 0$ with $a+b=1$.

After some algebraic manipulation on the left-hand side terms of the extended beta function (31), we get

$$B^{(u)}_{(u_1,u_2)}(a(x_1,y_1)+b(x_2,y_2)) = B^{(u)}_{(u_1,u_2)}(ax_1+bx_2, ay_1+by_2)$$

Moreover, using definition of the extended beta function (31), we can write

$$B^{(u)}_{(u_1,u_2)}(ax_1+bx_2, ay_1+by_2) = \int_0^1 t^{ax_1+bx_2-1}(1-t)^{ay_1+by_2-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt;$$

$$\Downarrow$$

$$B^{(r)}_{(u_1,u_2)}(ax_1+bx_2, ay_1+by_2) = \int_0^1 t^{a(x_1-1)}t^{b(x_2-1)}(1-t)^{a(y_1-1)}(1-t)^{b(y_2-1)} \cdot$$
$$\cdot \left[E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)\right]^a \cdot \left[E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)\right]^b dt;$$

$$\Downarrow$$

$$B^{(u)}_{(u_1,u_2)}(ax_1+bx_2, ay_1+by_2) = \int_0^1 \left[t^{x_1-1}(1-t)^{y_1-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)\right]^a \cdot$$
$$\cdot \left[t^{x_2-1}(1-t)^{y_2-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)\right]^b dt.$$

Putting $\theta_1 = \frac{1}{a}$ and $\theta_2 = \frac{1}{b}$, Hölder inequality (34) gives:

$$B_{(u_1,u_2)}^{(u)}(ax_1 + bx_2, ay_1 + by_2) \leq \left[\int_0^1 t^{x_1-1}(1-t)^{y_1-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt\right]^a \cdot$$

$$\cdot\left[\int_0^1 t^{x_2-1}(1-t)^{y_2-1}E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right)dt\right]^b = \left[B_{(u_1,u_2)}^{(u)}(x_1,y_1)\right]^a \cdot \left[B_{(u_1,u_2)}^{(u)}(x_2,y_2)\right]^b.$$

Then, from Definition (1), the function $B_{(u_1,u_2)}^{(u)}(x,y)$ is logarithmically convex on $\mathbb{R}^+ \times \mathbb{R}^+$.

If $a = b = \frac{1}{2}$, we get our desired result:

$$\left[B_{(u_1,u_2)}^{(u)}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]^2 \leq B_{(u_1,u_2)}^{(u)}(x_1,y_1) \cdot B_{(u_1,u_2)}^{(u)}(x_2,y_2).$$

□

3. Extension of Gauss Hypergeometric Function and Confluent Hypergeometric Function

In this section, we introduce a new extension of Gauss hypergeometric function and confluent hypergeometric function by using extended beta function given in (31). In integral representation of Gauss hypergeometric function and confluent hypergeometric function, we introduce Wiman's function as a kernel.

Definition 2. *A new extended Gauss hypergeometric function* $F_{(r_1,r_2)}^{(r)}(p_0,p_1,p_2;z)$ *is defined as follows:*

$$F_{(r_1,r_2)}^{(r)}(p_0,p_1,p_2;z) = \sum_{n=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(p_1+n, p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_n \frac{z^n}{n!}, \tag{39}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$, $\mid z \mid < 1$ *and* $B_{(r_1,r_2)}^{(r)}(x_1,x_2)$ *extended beta function.*

After substitution of *beta function* value in terms of *gamma function*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

in the above definition, we have another representation of a new extended Gauss hypergeometric function (39):

$$F_{(r_1,r_2)}^{(r)}(p_0,p_1,p_2;z) = \frac{\Gamma(p_2)}{\Gamma(p_1)\Gamma(p_2-p_1)} \sum_{n=0}^{\infty} (p_0)_n B_{(r_1,r_2)}^{(r)}(p_1+n, p_2-p_1)\frac{z^n}{n!}. \tag{40}$$

Definition 3. *A new extended confluent hypergeometric function* $\Phi_{(r_1,r_2)}^{(r)}(p_1,p_2;z)$ *is defined as follows:*

$$\Phi_{(r_1,r_2)}^{(r)}(p_1,p_2;z) = \sum_{n=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(p_1+n, p_2-p_1)}{B(p_1,p_2-p_1)} \frac{z^n}{n!}, \tag{41}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$ *and* $B_{(r_1,r_2)}^{(r)}(x_1,x_2)$ *extended beta function.*

After substitution of *beta function* value in terms of *gamma function*, in the above definition, we have another representation of a new extended confluent hypergeometric function (41):

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = \frac{\Gamma(p_2)}{\Gamma(p_1)\Gamma(p_2-p_1)} \sum_{n=0}^{\infty} B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)\frac{z^n}{n!}, \tag{42}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$ and $r \geq 0$.

Definition 4. (**New**) *A new form of beta function is defined as:*

$$\frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)}{B(p_1,p_2-p_1)} = \widehat{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1). \tag{43}$$

Using the above relation, we have a new form of extended Gauss hypergeometric function $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and extended confluent hypergeometric function $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \sum_{n=0}^{\infty} \widehat{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)(p_0)_n\frac{z^n}{n!}, \tag{44}$$

and

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = \sum_{n=0}^{\infty} \widehat{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)\frac{z^n}{n!}. \tag{45}$$

Remark 4. *We know that Gauss's hypergeometric function does not change if the p_0 and p_1 parameters are swapped while keeping p_2 fixed. This symmetric property with respect to the parameters p_0 and p_1 can also be deduced from the new extended Gauss hypergeometric function* (39)*:*

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = F^{(r)}_{(r_1,r_2)}(p_1,p_0,p_2;z). \tag{46}$$

Then, replacing p_0 with p_1 in (39)*, from above property* (46)*, we have*

$$\begin{aligned} F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) &= \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_n\frac{z^n}{n!} = \\ &= \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_0+n,p_2-p_0)}{B(p_0,p_2-p_0)}(p_1)_n\frac{z^n}{n!}. \end{aligned} \tag{47}$$

The beta function (43) *can be rewritten in a similar form:*

$$\frac{B^{(r)}_{(r_1,r_2)}(p_0+n,p_2-p_0)}{B(p_0,p_2-p_0)} = \tilde{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_0+n,p_2-p_0). \tag{48}$$

Using the above relation (48)*, we can rewrite extended Gauss hypergeometric function $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ in a similar form:*

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \sum_{n=0}^{\infty} \tilde{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_0+n,p_2-p_0)(p_1)_n\frac{z^n}{n!}. \tag{49}$$

Then, from above definitions and relation (44)*,* (46) *and* (49)*, we get*

$$\begin{aligned} F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) &= \sum_{n=0}^{\infty} \tilde{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_0+n,p_2-p_0)(p_1)_n\frac{z^n}{n!} = \\ &= \sum_{n=0}^{\infty} \widehat{\mathcal{B}}^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)\frac{z^n}{n!}. \end{aligned} \tag{50}$$

Remark 5. *(i) If we set $r_2 = 1$ in (39) and (41), then, we get a new extension of Gauss hypergeometric function and confluent hypergeometric function given by (25) and (26), respectively:*

$$F^{(r)}_{(r_1,1)}(p_0, p_1, p_2; z) = F_{r,r_1}(p_0, p_1, p_2; z); \tag{51}$$

and

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \Phi_{r,r_1}(p_1, p_2; z). \tag{52}$$

(ii) If we set $r_1 = r_2 = 1$ in (39) and (41), then we get extended Gauss hypergeometric function and extended confluent hypergeometric function given by (12) and (13), respectively:

$$F^{(r)}_{(1,1)}(p_0, p_1, p_2; z) = F_r(p_0, p_1, p_2; z); \tag{53}$$

and

$$\Phi^{(r)}_{(1,1)}(p_1, p_2; z) = \Phi_r(p_1, p_2; z). \tag{54}$$

(iii) If we set $r_1 = r_2 = 1$ and $r = 0$ in (39) and (41), then we get Gauss hypergeometric function and confluent hypergeometric function given by (3) and (6), respectively:

$$F^{(0)}_{(1,1)}(p_0, p_1, p_2; z) = F(p_0, p_1, p_2; z); \tag{55}$$

and

$$\Phi^{(0)}_{(1,1)}(p_1, p_2; z) = \Phi(p_1, p_2; z). \tag{56}$$

Theorem 3. *Consider $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ functions. Then, the following functional relations hold:*
(1)

$$\begin{aligned} F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) &= \frac{p_1}{p_2} F^{(r)}_{(r_1,r_2)}(p_0, p_1+1, p_2+1; z) \\ &+ \frac{(p_2-p_1)}{p_1} F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2+1; z), \end{aligned} \tag{57}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$ and $\mid z \mid < 1$.

(2)

$$\begin{aligned} \Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) &= \frac{p_1}{p_2} \Phi^{(r)}_{(r_1,r_2)}(p_1+1, p_2+1; z) \\ &+ \frac{(p_2-p_1)}{p_1} \Phi^{(r)}_{(r_1,r_2)}(p_1, p_2+1; z), \end{aligned} \tag{58}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ and $r \geq 0$.

Proof. Using the following known relation from Reference [13]:

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = B^{(u)}_{(u_1,u_2)}(y_1, y_2+1) + B^{(u)}_{(u_1,u_2)}(y_1+1, y_2), \tag{59}$$

in (39), we get

$$\begin{aligned}
&F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \\
= &\sum_{n=0}^{\infty}(p_0)_n\left(\frac{B^{(r)}_{(r_1,r_2)}(p_1+n+1,p_2-p_1)+B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1+1)}{B(p_1,p_2-p_1)}\right)\frac{z^n}{n!} = \\
= &\sum_{n=0}^{\infty}(p_0)_n\left(\frac{B^{(r)}_{(r_1,r_2)}(p_1+n+1,p_2-p_1)}{B(p_1,p_2-p_1)}\right)\frac{z^n}{n!} + \\
+ &\sum_{n=0}^{\infty}(p_0)_n\left(\frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1+1)}{B(p_1,p_2-p_1)}\right)\frac{z^n}{n!} = \\
= &\frac{B(p_1+1,p_2-p_1)}{B(p_1,p_2-p_1)}\sum_{n=0}^{\infty}(p_0)_n\left(\frac{B^{(r)}_{(r_1,r_2)}(p_1+n+1,p_2-p_1)}{B(p_1+1,p_2-p_1)}\right)\frac{z^n}{n!} + \\
+ &\frac{B(p_1,p_2-p_1+1)}{B(p_1,p_2-p_1)}\sum_{n=0}^{\infty}(p_0)_n\left(\frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1+1)}{B(p_1,p_2-p_1+1)}\right)\frac{z^n}{n!}
\end{aligned}$$

Then, using the value of *beta function* in terms of *gamma function* together with (39), allow us to get the desired result.

Similarly, using (59) in (41) and following the same rule that led to the result (57), we obtain the desired statement (58), and the Theorem is fully proved. □

Remark 6. *(i) From (46), we observe that, for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$, another functional relation that also holds is:*

$$\begin{aligned}
F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) &= \frac{p_0}{p_2}F^{(r)}_{(r_1,r_2)}(p_1,p_0+1,p_2+1;z) \\
&+ \frac{(p_2-p_0)}{p_0}F^{(r)}_{(r_1,r_2)}(p_1,p_0,p_2+1;z).
\end{aligned} \tag{60}$$

(ii) If we used (46) on the right-hand side of (60),
we obtain another functional relation for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$:

$$\begin{aligned}
F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) &= \frac{p_0}{p_2}F^{(r)}_{(r_1,r_2)}(p_0+1,p_1,p_2+1;z) \\
&+ \frac{(p_2-p_0)}{p_0}F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2+1;z).
\end{aligned} \tag{61}$$

Theorem 4. *Consider $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ functions. The following Sum relations hold:*
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (p_2-p_1)\sum_{k=0}^{\infty}\frac{(p_1)_k}{(p_2)_{k+1}}F^{(r)}_{(r_1,r_2)}(p_0,p_1+1,p_2+k+1;z), \tag{62}$$

where $\Re(p_2)>0, \Re(p_1)>0, \Re(r_1)>0, \Re(r_2)>0, r\geq 0$ and $\mid z\mid<1$.
(2)

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = (p_2-p_1)\sum_{k=0}^{\infty}\frac{(p_1)_k}{(p_2)_{k+1}}\Phi^{(r)}_{(r_1,r_2)}(p_1+k,p_2+k+1;z), \tag{63}$$

where $\Re(p_2)>0, \Re(p_1)>0, \Re(r_1)>0, \Re(r_2)>0$ and $r\geq 0$.

Proof. Using the following known relation from Reference [13]:

$$B^{(u)}_{(u_1,u_2)}(y_1,y_2) = \sum_{k=0}^{\infty} B^{(u)}_{(u_1,u_2)}(y_1+k, y_2+1), \tag{64}$$

in (39), we have

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \\ &= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{\infty} B^{(r)}_{(r_1,r_2)}(p_1+n+k, p_2-p_1+1)}{B(p_1,p_2-p_1)}(p_0)_n \frac{z^n}{n!} = \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1+n+k, p_2-p_1+1)}{B(p_1,p_2-p_1)}(p_0)_n \frac{z^n}{n!}\right) = \\ &= \sum_{k=0}^{\infty} \frac{B(p_1+k,p_2-p_1+1)}{B(p_1,p_2-p_1)} \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1+k+n, p_2-p_1+1)}{B(p_1+k,p_2-p_1+1)}(p_0)_n \frac{z^n}{n!}. \end{aligned}$$

Moreover, from (39)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \sum_{k=0}^{\infty} \frac{B(p_1+k,p_2-p_1+1)}{B(p_1,p_2-p_1)} F^{(r)}_{(r_1,r_2)}(p_0,p_1+k,p_2+k+1;z).$$

Using value of *beta function* in terms of *gamma function* and value of *gamma function* in terms of Pochhammer symbol in the above equation, we get our desired statement:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (p_2-p_1)\sum_{k=0}^{\infty} \frac{(p_1)_k}{(p_2)_{k+1}} F^{(r)}_{(r_1,r_2)}(p_0,p_1+1,p_2+k+1;z).$$

Similarly, using (64) in (41) and following the same rule getting to result (62), we obtain the desired statement (63), and the Theorem is fully proved. □

Remark 7. *(i) From (46), we observe that, for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$, another summation relation that also holds is:*

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (p_2-p_0)\sum_{k=0}^{\infty} \frac{(p_0)_k}{(p_2)_{k+1}} F^{(r)}_{(r_1,r_2)}(p_1,p_0+1,p_2+k+1;z). \tag{65}$$

(ii) If we used (46) on the right-hand side of (65), we obtain another summation relation for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (p_2-p_0)\sum_{k=0}^{\infty} \frac{(p_0)_k}{(p_2)_{k+1}} F^{(r)}_{(r_1,r_2)}(p_0+1,p_1,p_2+k+1;z). \tag{66}$$

Theorem 5. *Consider $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ functions. The following Sum relations hold:*
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = \sum_{k=0}^{\infty} \frac{(p_1-p_2+1)_k B(p_1+k,1)}{k!B(p_1,p_2-p_1)} F^{(r)}_{(r_1,r_2)}(p_0,p_1+k,p_1+k+1;z), \tag{67}$$

where $\Re(p_2) > 0$, $\Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$ *and* $| z | < 1$.

(2)

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{(p_1 - p_2 + 1)_k B(p_1 + k, 1)}{k! B(p_1, p_2 - p_1)} \Phi^{(r)}_{(r_1,r_2)}(p_1 + k, p_1 + k + 1; z), \tag{68}$$

where $\Re(p_2) > 0$, $\Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$ *and* $r \geq 0$.

Proof. Using the following known relation from Reference [13]:

$$B^{(u)}_{(u_1,u_2)}(y_1, 1 - y_2) = \sum_{k=0}^{\infty} \frac{(y_2)_k}{k!} B^{(u)}_{(u_1,u_2)}(y_1 + k, 1), \tag{69}$$

in (39), we get

$$\begin{aligned}
&F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\
&= \sum_{n=0}^{\infty} \frac{(p_0)_n}{B(p_1, p_2 - p_1)} \left(\sum_{k=0}^{\infty} \frac{(1 + p_1 - p_2)_k}{k!} B^{(r)}_{(r_1,r_2)}(p_1 + n + k, 1) \right) \frac{z^n}{n!} = \\
&= \sum_{k=0}^{\infty} \frac{B(p_1 + k, 1)(1 + p_1 - p_2)_k}{B(p_1, p_2 - p_1) k!} \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1 + k + n, 1)}{B(p_1 + k, 1)} (p_0)_n \frac{z^n}{n!} = \\
&= \sum_{k=0}^{\infty} \frac{(p_1 - p_2 + 1)_k B(p_1 + k, 1)}{k! B(p_1, p_2 - p_1)} F^{(r)}_{(r_1,r_2)}(p_0, p_1 + k, p_1 + k + 1; z).
\end{aligned}$$

Similarly, using (69) in (41) and following the same rule getting to result (67), we obtain the desired statement (68), and the Theorem is fully proved. □

Remark 8. *(i) From (46), we observe that, for function* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$*, another summation relation that also holds is:*

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{(p_0 - p_2 + 1)_k B(p_0 + k, 1)}{k! B(p_0, p_2 - p_0)} F^{(r)}_{(r_1,r_2)}(p_1, p_0 + k, p_0 + k + 1; z). \tag{70}$$

(ii) If we used (46) on the right-hand side of (70),
we have another summation relation for $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$*:*

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{k=0}^{\infty} \frac{(p_0 - p_2 + 1)_k B(p_0 + k, 1)}{k! B(p_0, p_2 - p_0)} F^{(r)}_{(r_1,r_2)}(p_0 + k, p_1, p_0 + k + 1; z). \tag{71}$$

Theorem 6. *Consider* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ *and* $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ *functions. The following results hold:*
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_1 + l, p_2 - p_1 - l + k)}{B(p_1, p_2 - p_1)} F^{(r)}_{(r_1,r_2)}(p_0, p_1 + l, p_2 + k; z), \tag{72}$$

where $\Re(p_2) > 0, \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0, k \in N$ *and* $| z | < 1$.

(2)

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_1 + l, p_2 - p_1 - l + k)}{B(p_1, p_2 - p_1)} \Phi^{(r)}_{(r_1,r_2)}(p_1 + l, p_2 + k; z), \tag{73}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$ *and* $k \in N$.

Proof. Using the following known relation from Reference [13]:

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = \sum_{l=0}^{k} \binom{k}{l} B^{(u)}_{(u_1,u_2)}(y_1 + l, y_2 + k - l), \tag{74}$$

in (39), we have

$$\begin{aligned} & F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ = & \sum_{n=0}^{\infty} \frac{(p_0)_n}{B(p_1, p_2 - p_1)} \left(\sum_{l=0}^{k} \binom{k}{l} B^{(r)}_{(r_1,r_2)}(p_1 + n + l, p_2 - p_1 + k - l) \right) \frac{z^n}{n!} = \\ = & \sum_{l=0}^{k} \binom{k}{l} \frac{1}{B(p_1, p_2 - p_1)} \cdot \sum_{n=0}^{\infty} \left(B^{(r)}_{(r_1,r_2)}(p_1 + n + l, p_2 - p_1 + k - l)(p_0)_n \frac{z^n}{n!} \right) = \\ = & \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_1 + l, p_2 - p_1 - l + k)}{B(p_1, p_2 - p_1)} \sum_{n=0}^{\infty} (p_0)_n \frac{B^{(r)}_{(r_1,r_2)}(p_1 + n + l, p_2 - p_1 + k - l)}{B(p_1 + l, p_2 - p_1 - l + k)} \frac{z^n}{n!}. \end{aligned}$$

From (39), we get our desired result:

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_1 + l, p_2 - p_1 - l + k)}{B(p_1, p_2 - p_1)} F^{(r)}_{(r_1,r_2)}(p_0, p_1 + l, p_2 + k; z).$$

Similarly, using (74) in (41) and following the same rule getting to result (72), we obtain the desired statement (73), and the Theorem is fully proved. □

Remark 9. *(i) From (46), we observe that, for* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$, *the following result holds:*

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_0 + l, p_2 - p_0 + k)}{B(p_0, p_2 - p_0)} F^{(r)}_{(r_1,r_2)}(p_1, p_0 + l, p_2 + k; z). \tag{75}$$

(ii) If we used (46) on the right-hand side of (75), we have another result for $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$:

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(p_0 + l, p_2 - p_0 + k)}{B(p_0, p_2 - p_0)} F^{(r)}_{(r_1,r_2)}(p_0 + l, p_1, p_2 + k; z). \tag{76}$$

Theorem 7. *Consider* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ *and* $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ *functions. The following integral representations hold:*
(1)

$$\begin{aligned} & F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ = & \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 t^{p_1 - 1}(1 - t)^{p_2 - p_1 - 1}(1 - zt)^{-p_0} E_{r_1,r_2}\left(-r(t(1 - t))^{-1}\right) dt, \end{aligned} \tag{77}$$

where $\Re(p_2) > \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$ *and* $\mid z \mid < 1$.
(2)

$$\begin{aligned} & \Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \\ = & \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 t^{p_1 - 1}(1 - t)^{p_2 - p_1 - 1} e^{zt} E_{r_1,r_2}\left(-r(t(1 - t))^{-1}\right) dt, \end{aligned} \tag{78}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$ *and* $r \geq 0$.

Proof. Using the following known relation from Reference [13]:

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} E_{u_1,u_2}\left(-u(t(1-t))^{-1}\right) dt \tag{79}$$

in (39), we have:

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ &= \frac{1}{B(p_1, p_2 - p_1)} \sum_{n=0}^{\infty} (p_0)_n \cdot \left(\int_0^1 t^{p_1+n-1}(1-t)^{p_2-p_1-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt \right) \frac{z^n}{n!} \\ &= \frac{1}{B(p_1, p_2 - p_1)} \sum_{n=0}^{\infty} (p_0)_n \cdot \left(\int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) t^n dt \right) \frac{z^n}{n!} \end{aligned}$$

Changing the order of integration and summation, we get:

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ &= \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt \sum_{n=0}^{\infty} (p_0)_n \frac{(zt)^n}{n!} \end{aligned}$$

Since

$$(1 - zt)^{-p_0} = \sum_{n=0}^{\infty} \frac{(p_0)_n}{n!} (zt)^n, \mid t \mid < 1, \tag{80}$$

the last expression becomes the searched result:

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ &= \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1}(1-zt)^{-p_0} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt. \end{aligned}$$

Similarly, using (79) in (41) and following the same rule getting to result (77), we obtain the desired statement (78), and the Theorem is fully proved. □

Remark 10. *From* (46)*, we observe that, for* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$*, the following integral representation holds:*

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \\ &= \frac{1}{B(p_0, p_2 - p_0)} \int_0^1 t^{p_0-1}(1-t)^{p_2-p_0-1}(1-zt)^{-p_1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt. \end{aligned} \tag{81}$$

Corollary 2. *Consider* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ *and* $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ *functions. The following integral representations hold:*

(1)

$$\begin{aligned} F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) &= \frac{2}{B(p_1, p_2 - p_1)} \int_0^{\frac{\pi}{2}} (\cos(x))^{2p_1-1} (\sin(x))^{2p_2-2p_1-1} \cdot \\ &\cdot E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2 (\sin(x))^2} \right) \left((1 - z(\cos(x))^2)^{-p_0} \right) dx. \end{aligned} \tag{82}$$

(2)

$$\begin{aligned}\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) &= \frac{2}{B(p_1,p_2-p_1)}\int_0^{\frac{\pi}{2}}(\cos(x))^{2p_1-1}(\sin(x))^{2p_2-2p_1-1}\,.\\ &\cdot\ E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2(\sin(x))^2}\right)e^{z(\cos(x))^2}dx.\end{aligned} \tag{83}$$

Proof. The corollary is easily proved by choosing $t=\cos^2(x)$ in (77) and (78). □

Corollary 3. *Consider $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2,z)$ functions. The following integral representations hold:*
(1)

$$\begin{aligned}F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) &= \frac{2}{B(p_1,p_2-p_1)}\int_0^{\frac{\pi}{2}}(\sin(x))^{2p_1-1}(\cos(x))^{2p_2-2p_1-1}\,.\\ &\cdot\ E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2(\sin(x))^2}\right)\left(1-z((\sin(x))^2)^{-p_0}\right)dx.\end{aligned} \tag{84}$$

(2)

$$\begin{aligned}\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) &= \frac{2}{B(p_1,p_2-p_1)}\int_0^{\frac{\pi}{2}}(\sin(x))^{2p_1-1}(\cos(x))^{2p_2-2p_1-1}\,.\\ &\cdot\ E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2(\sin(x))^2}\right)e^{z(\sin(x))^2}dx.\end{aligned} \tag{85}$$

Proof. The corollary is easily proved by choosing $t=\sin^2(x)$ in (77) and (78). □

Theorem 8. *The Mellin Transformations for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ are given by:*
(1)

$$\boldsymbol{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \frac{\Gamma_0^{(r_1,r_2)}(s)B(p_1+s,p_2-p_1+s)}{B(p_1,p_2-p_1)}F(p_0,p_1+s,p_2+2s;z), \tag{86}$$

where $\Re(p_2)>\Re(p_1)>0, \Re(r_1)>0, \Re(r_2)>0, s\geq 0, r\geq 0$ *and* $|\,z\,|<1$.
(2)

$$\boldsymbol{M}\{\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z);s\} = \frac{\Gamma_0^{(r_1,r_2)}(s)B(p_1+s,p_2-p_1+s)}{B(p_1,p_2-p_1)}\Phi(p_1+s,p_2+2s;z), \tag{87}$$

where $\Re(p_2)>\Re(p_1)>0, \Re(r_1)>0, \Re(r_2)>0, s\geq 0$ *and* $r\geq 0$.

Proof. From the definition of Mellin Transformation, we have:

$$\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \int_0^\infty r^{s-1}F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)dr.$$

Using (77) on the right-hand side, we have

$$\begin{aligned}\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} &= \int_0^\infty\{\frac{1}{B(p_1,p_2-p_1)}\int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1}(1-zt)^{-p_0}\,.\\ &\cdot\ E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right)dt\}r^{s-1}dr\end{aligned}$$

Interchanging the order of integrations, the last expression reads as:

$$\begin{aligned}\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} &= \frac{1}{B(p_1,p_2-p_1)}\int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1}(1-zt)^{-p_0}\cdot \\ &\cdot \left[\int_0^\infty r^{s-1}E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right)dr\right]dt.\end{aligned}$$

Let $v=\frac{r}{t(1-t)}$. Then:

$$\begin{aligned}\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} &= \frac{1}{B(p_1,p_2-p_1)}\int_0^1 t^{p_1+s-1}(1-t)^{p_2-p_1+s-1}(1-zt)^{-p_0}\cdot \\ &\cdot \left[\int_0^\infty v^{s-1}E_{r_1,r_2}(-v)dv\right]dt\end{aligned}$$

From the known result in Reference [21]:

$$\Gamma^{(r_1,r_2)}_r(x_1)=\int_0^\infty t^{x_1-1}E_{r_1,r_2}\left(-t-rt^{-1}\right)dt, \tag{88}$$

and, using it with $r=0$ in the above equation, we get

$$\begin{aligned}&\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \\ &= \frac{1}{B(p_1,p_2-p_1)}\int_0^1 t^{p_1+s-1}(1-t)^{p_2-p_1+s-1}(1-zt)^{-p_0}[\Gamma^{(r_1,r_2)}_0(s)]dt = \\ &= \frac{B(p_1+s,p_2-p_1+s)}{B(p_1,p_2-p_1)}\int_0^1 \frac{t^{p_1+s-1}(1-t)^{p_2-p_1+s-1}(1-zt)^{-p_0}}{B(p_1+s,p_2-p_1+s)}[\Gamma^{(r_1,r_2)}_0(s)]dt = \\ &= \frac{B(p_1+s,p_2-p_1+s)\Gamma^{(r_1,r_2)}_0(s)}{B(p_1,p_2-p_1)}\int_0^1 \frac{t^{p_1+s-1}(1-t)^{p_2-p_1+s-1}(1-zt)^{-p_0}}{B(p_1+s,p_2-p_1+s)}dt.\end{aligned}$$

Finally, from integral representation of Gauss hypergeometric function (5), we get our desired result:

$$\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \frac{\Gamma^{(r_1,r_2)}_0(s)B(p_1+s,p_2-p_1+s)}{B(p_1,p_2-p_1)}F(p_0,p_1+s,p_2+2s;z).$$

Similarly, using Mellin transformation, Equation (78), and following the same rule getting to result (86), we obtain the desired statement (87), and the Theorem is fully proved. □

Remark 11. *(i) From (46), we observe that, for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$, the following Mellin representation also holds:*

$$M\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \frac{\Gamma^{(r_1,r_2)}_0(s)B(p_0+s,p_2-p_0+s)}{B(p_0,p_2-p_0)}F(p_1,p_0+s,p_2+2s;z). \tag{89}$$

(ii) Using symmetric property of Gauss hypergeometric function $F(p_o,p_1,p_2;z)=F(p_1,p_0,p_2;z)$ on the right-hand side of (89), we have another Mellin representation for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$:

$$M\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z);s\} = \frac{\Gamma^{(r_1,r_2)}_0(s)B(p_0+s,p_2-p_0+s)}{B(p_0,p_2-p_0)}F(p_0+s,p_1,p_2+2s;z). \tag{90}$$

Theorem 9. *The Pfaff's transformations for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ is given by:*
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{-p_0} F^{(r)}_{(r_1,r_2)}\left(p_0, p_2-p_1, p_2; \frac{z}{z-1}\right), \tag{91}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$ *and* $\mid \frac{z}{z-1} \mid < 1$.
(2)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{-p_1} F^{(r)}_{(r_1,r_2)}\left(p_2-p_0, p_1, p_2; \frac{z}{z-1}\right), \tag{92}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$ *and* $\mid \frac{z}{z-1} \mid < 1$.

Proof. From (77), we deduce:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) =$$
$$= \frac{1}{B(p_1,p_2-p_1)} \cdot \int_0^1 v^{p_1-1}(1-v)^{p_2-p_1-1}(1-zv)^{-p_0} E_{r_1,r_2}\left(-r(v(1-v))^{-1}\right) dv.$$

Let $v = 1-t$. After some algebraic manipulation, the last expression reads as:

$$[1-z(1-t)]^{-p_0} = (1-z)^{-p_0}\left(1+\frac{z}{1-z}t\right)^{-p_0}.$$

Thus, we get our desired result (91):

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{-p_0} F^{(r)}_{(r_1,r_2)}\left(p_0, p_2-p_1, p_2; \frac{z}{z-1}\right).$$

From (46) and (91), we deduce that:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{-p_1} F^{(r)}_{(r_1,r_2)}\left(p_1, p_2-p_0, p_2; \frac{z}{z-1}\right).$$

Moreover, using same property (46) on the right-hand side of the above equation, we get our statement (92):

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{-p_1} F^{(r)}_{(r_1,r_2)}\left(p_2-p_0, p_1, p_2; \frac{z}{z-1}\right).$$

□

Theorem 10. *The Euler's transformation for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ is given by:*

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{p_2-p_0-p_1} F^{(r)}_{(r_1,r_2)}(p_2-p_0, p_2-p_1, p_2; z), \tag{93}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$ *and* $\mid z \mid < 1$.

Proof. From (91) and (92), we obtain:

$$(1-z)^{-p_0} F^{(r)}_{(r_1,r_2)}\left(p_0, p_2-p_1, p_2; \frac{z}{z-1}\right) = (1-z)^{-p_1} F^{(r)}_{(r_1,r_2)}\left(p_2-p_0, p_1, p_2; \frac{z}{z-1}\right).$$

Now, replacing p_0 with p_2-p_0 and z with $\frac{z}{z-1}$, we complete the theorem's proof:

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{p_2-p_0-p_1} F^{(r)}_{(r_1,r_2)}(p_2-p_0, p_2-p_1, p_2; z).$$

□

Remark 12. *Using (46) on the right-hand side of (93), we deduce another Euler's transformation for* $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (1-z)^{p_2-p_0-p_1} F^{(r)}_{(r_1,r_2)}(p_2-p_1,p_2-p_0,p_2;z). \tag{94}$$

Theorem 11. *For* $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$*, we obtain the extension of Kummer transformation:*

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = e^z \Phi^{(r)}_{(r_1,r_2)}(p_2-p_1,p_2;-z), \tag{95}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ *and* $r \geq 0$.

Proof. From (78), we get:

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = \frac{1}{B(p_1,p_2-p_1)} \int_0^1 u^{p_1-1}(1-u)^{p_2-p_1-1} e^{zu} E_{r_1,r_2}\left(-r(u(1-u))^{-1}\right) du.$$

Let $u = 1-t$. After some algebraic manipulation, the last expression becomes:

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z) = e^z \frac{1}{B(p_1,p_2-p_1)} \int_0^1 t^{p_2-p_1-1}(1-t)^{p_1-1} e^{-zt} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) du.$$

Then, from (78), we obtain our statement (95). □

Theorem 12. *Consider* $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ *function. The following relation holds:*

$$\int_0^\infty z^{p_0-1} \Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z) dz = \Gamma(p_0) F^{(r)}_{(r_1,r_2)}(p_0,p_2-p_1,p_2;1), \tag{96}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(p_0) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ *and* $r \geq 0$.

Proof. Replacing z by $-z$ in (95), we can write

$$\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z) = e^{-z} \Phi^{(r)}_{(r_1,r_2)}(p_2-p_1,p_2;z).$$

Multiplying by z^{p_0-1} and integrating the above equation with respect to z from $z=0$ to $z=\infty$, we get:

$$\int_0^\infty z^{p_0-1} \Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z) dz = \int_0^\infty z^{p_0-1} \{e^{-z} \Phi^{(r)}_{(r_1,r_2)}(p_2-p_1,p_2;z)\} dz.$$

Using (41) on the right-hand side of last expression, we obtain:

$$\int_0^\infty z^{p_0-1} \Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z) dz = \int_0^\infty z^{p_0-1} e^{-z} \sum_{n=0}^\infty \frac{B^{(r)}_{(r_1,r_2)}(p_2-p_1+n,p_1)}{B(p_2-p_1,p_1)} \frac{z^n}{n!} dz.$$

Then, changing order of integration and summation, we can read:

$$\int_0^\infty z^{p_0-1} \Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z) dz = \sum_{n=0}^\infty \frac{B^{(r)}_{(r_1,r_2)}(p_2-p_1+n,p_1)}{B(p_2-p_1,p_1) n!} \int_0^\infty z^{n+p_0-1} e^{-z} dz.$$

From definition of *gamma function* and from its relation with Pochhammer symbol

$$\Gamma(n+p_0) = (p_0)_n \Gamma(p_0),$$

we obtain:

$$\int_0^\infty z^{p_0-1}\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z)dz = \sum_{n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(p_2-p_1+n,p_1)}{B(p_2-p_1,p_1)n!}(p_0)_n\Gamma(p_0).$$

Finally, from (39), we get our desired result:

$$\int_0^\infty z^{p_0-1}\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z)dz = \Gamma(p_0)F^{(r)}_{(r_1,r_2)}(p_0,p_2-p_1,p_2;1).$$

□

Remark 13. *Using* (46) *on the right-hand side of Equation* (96)*, we get another relation for* $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$*:*

$$\int_0^\infty z^{p_0-1}\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;-z)dz = \Gamma(p_0)F^{(r)}_{(r_1,r_2)}(p_2-p_1,p_0,p_2;1). \quad (97)$$

Theorem 13. *Consider* $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ *and* $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ *functions. The following differentiation formulas hold:*
(1)

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{(p_0)_m(p_1)_m}{(p_2)_m}F^{(r)}_{(r_1,r_2)}(p_0+m,p_1+m,p_2+m;z), \quad (98)$$

where $\Re(p_2) > \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0, r \geq 0, m \geq 0$ *and* $|z| < 1$.
(2)

$$\frac{d^m}{dz^m}\{\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)\} = \frac{(p_1)_m}{(p_2)_m}\Phi^{(r)}_{(r_1,r_2)}(p_1+m,p_2+m;z), \quad (99)$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0, m \geq 0$ *and* $r \geq 0$.

Proof. Let $\frac{d^m}{dz^m}$ be the m-th derivative with respect to the variable z. Differentiating (39), we obtain

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{d^m}{dz^m}\sum_{n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_n\frac{z^n}{n!},$$

that is:

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \sum_{n=m}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(p_1+n,p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_n\frac{z^{n-m}n!}{n!(n-m)!}.$$

Replacing $n-m$ with m, we can read:

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \sum_{m=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(p_1+m+m,p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_{2m}\frac{z^m}{m!}.$$

Using the property of Pochhammer symbol $(p_0)_{m+k} = (p_0)_m(p_0+k)_m$, and after manipulation, we obtain:

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{(p_1)_m}{(p_2)_m}\left\{\sum_{m=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(p_1+m+m,p_2-p_1)}{B(p_1+m,p_2-p_1)}(p_0+m)_m\frac{z^m}{m!}\right\}(p_0)_m.$$

By exploiting (39), we get our statement:

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{(p_0)_m(p_1)_m}{(p_2)_m}F^{(r)}_{(r_1,r_2)}(p_0+m,p_1+m,p_2+m;z).$$

Similarly, following the same rule getting to result (98) applied in (41), we obtain the desired statement (99), and the Theorem is fully proved. □

Corollary 4. *If we consider $m=1$ in (98) and (99), we get following derivative formulas:*

$$\frac{d}{dz}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{p_0p_1}{p_2}F^{(r)}_{(r_1,r_2)}(p_0+1,p_1+1,p_2+1;z); \tag{100}$$

$$\frac{d}{dz}\{\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)\} = \frac{p_1}{p_2}\Phi^{(r)}_{(r_1,r_2)}(p_1+1,p_2+1;z). \tag{101}$$

Remark 14. *Using (46) on the right-hand side of the (98), we can write another derivative formula for $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$*

$$\frac{d^m}{dz^m}\{F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)\} = \frac{(p_0)_m(p_1)_m}{(p_2)_m}F^{(r)}_{(r_1,r_2)}(p_1+m,p_0+m,p_2+m;z). \tag{102}$$

4. Conclusions

We conclude our analysis by mentioning that the results obtained in this paper are new and potentially useful. First, we have investigated some inequalities of the extended beta function. Then, we have introduced a new extension of Gauss hypergeometric and confluent hypergeometric function and investigated some properties of these extended functions. We are also trying to find certain applications of the results obtained here in some significant research areas, such as statistics, physics, engineering, applied mathematics, and computer algebra. In the future, we will work on matrix forms of these extended functions and also try to develop algorithms for these extended functions by using maple computer software.

Author Contributions: Conceptualization, S.J. and R.G.; methodology, P.A.; software, S.J. and R.G.; validation, S.J., R.G., P.A., A.L. and C.C.; investigation, C.C.; formal analysis, P.A. and C.C.; resources, P.A.; data writing—original draft preparation, S.J.; writing-review and editing, S.J., R.G., P.A. and C.C.; visualization, P.A.; funding acquisition, P.A. and C.C. All authors have read and agreed to the published version of the manuscript.

Funding: Shilpi Jain is very thankful to SERB (project number: MTR/2017/000194) for providing the necessary facility and support during this work.

Acknowledgments: Shilpi Jain is very thankful to SERB (project number: MTR/2017/000194) for providing the necessary facility and support during this work and Rahul Goyal and Praveen Agarwal are thankful to NBHM(project number: 02011/12/2020NBHM(R.P)/R&DII/7867).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Agarwal, P.; Choi, J.; Jain, S. Extended hypergeometric functions of two and three variables. *Commun. Korean Math. Soc.* **2015**, *30*, 403–414. [CrossRef]
2. Luo, M.J.; Milovanovic, G.V.; Agarwal, P. Some results on the extended beta and extended hypergeometric functions. *Commun. Appl. Math. Comput.* **2014**, *248*, 631–651. [CrossRef]
3. Rainville, E.D. *Special Functions*; Macmillan: New York, NY, USA, 1960.
4. Olver, W.J.F.; Lozier, W.D.; Boisvert, F.R.; Clark, W.C. *NIST Handbook of Mathematical Functions*; Cambridge University Press: New York, NY, USA, 2010.
5. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittage Leffler Functions Related Topics and Application*, 2nd ed.; Springer: New York, NY, USA, 2020.
6. Mittag-Leffler, G.M. Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci.* **1903**, *137*, 554–558.
7. Wiman, A. Über die Nullstellun der Funktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 217–234. [CrossRef]
8. Chaudhry, M.A.; Qadir, A.; Rafique, M.; Zubair, S.M. Extension of Euler's Beta function. *J. Comput. Appl. Math.* **1997**, *78*, 19–32. [CrossRef]
9. Chaudhry, M.A.; Qadir, A.; Srivastava, H.M.; Paris, R.B. Extended hypergeometric and confluent hypergeometric functions. *Appl. Math. Comput.* **2004**, *159*, 589–602. [CrossRef]
10. Özergin, E.; Özarslan, M.A.; Altın, A. Extension Gamma, Beta and Hypergeometric Functions. *J. Comput. Appl. Math.* **2011**, *235*, 4601–4610. [CrossRef]
11. Choi, J.; Rathie, A.K.; Parmar, R.K. Extension of extended beta, hypergeometric and confluent hypergeometric functions. *Honam Math. J.* **2014**, *36*, 357–385. [CrossRef]
12. Shadab, M.S.; Jabee, S.J.; Choi, J.C. An extended Beta function and its applications. *Far East J. Math. Sci. (FJMS)* **2018**, *103*, 235–251. [CrossRef]
13. Goyal, R.; Agarwal, P.; Momami, S.; Rassias, M.T. An Extension of Beta Function by Using wiman's function. *Axioms* **2021**, *10*, 187. [CrossRef]
14. Butt, S.I.; Pečarić, J.; Rehman, A.U. Exponential convexity of Petrović and related functional. *J. Inequal. Appl.* **2011**, *2011*. [CrossRef]
15. Qi, F.; Li, W.H. logarithmically completely monotonic function involving the ratio of gamma functions. *J. Appl. Anal. Comput.* **2015**, *5*, 626–634.
16. Dragomir, S.S.; Agarwal, R.P.; Barnett, N.S. Inequalities for Beta and Gamma functions via some classical and new integral inequalities. *J. Inequal. Appl.* **1999**, *5*, 103–165. [CrossRef]
17. Kumar, P.; Singh, S.P.; Dragomir, S.S. Some inequalities involving beta and gamma functions, Nonlinear analysis and its applications. *Nonlinear Anal. Forum.* **2001**, *6*, 143–150.
18. Qi, F.; Li, L.H.; Xu, S.L. Some inequalities constructed by Tchebysheff's integral inequality. *Math. Inequal. Appl.* **1999**, *2*, 517–528. [CrossRef]
19. Rudin, W. *Real and Complex Analysis*, 3rd ed.; McGraw-Hill Book Co.: New York, NY, USA, 1987.
20. Tian, J.F.; Ha, M.H. Properties of generalized sharp Hölder's inequalities. *J. Math. Inequal.* **2017**, *11*, 511–525. [CrossRef]
21. Atash, A.A.; Barahmah, S.S.; Kulib, M.A. On a new extensions of extended Gamma and Beta functions. *Int. J. Stat. Appl. Math.* **2018**, *3*, 4–18.