mathematics

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Optimal Control for a Nonlocal Model of Non-Newtonian Fluid Flows

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Abstract: This paper deals with an optimal control problem for a nonlocal model of the steady-state flow of a differential type fluid of complexity 2 with variable viscosity. We assume that the fluid occupies a bounded three-dimensional (or two-dimensional) domain with the impermeable boundary. The control parameter is the external force. We discuss both strong and weak solutions. Using one result on the solvability of nonlinear operator equations with weak-to-weak and weak-to-strong continuous mappings in Sobolev spaces, we construct a weak solution that minimizes a given cost functional subject to natural conditions on the model data. Moreover, a necessary condition for the existence of strong solutions is derived. Simultaneously, we introduce the concept of the marginal function and study its properties. In particular, it is shown that the marginal function of this control system is lower semicontinuous with respect to the directed Hausdorff distance.

Keywords: optimal control problem; non-Newtonian fluid; second-grade fluid; nonlocal model; influence function; Navier slip condition; existence; strong solution; weak solution; marginal function

Citation: Baranovskii, E.S.; Artemov, M.A. Optimal Control for a Nonlocal Model of Non-Newtonian Fluid Flows. *Mathematics* **2021**, *9*, 275. https://doi.org/10.3390/math9030275

Academic Editors: Ravi P. Agarwal and Bashir Ahmad

Received: 30 December 2020
Accepted: 27 January 2021
Published: 30 January 2021

1. Introduction

It is well known that the behavior of many real fluids cannot be satisfactorily described by the classical Navier–Stokes equations. Such fluids are called non-Newtonian. Since the deviation from the "Newtonian" behavior occurs in widely disparate reasons, numerous mathematical models have been proposed to describe the motion of non-Newtonian fluids. An important example of non-Newtonian models is given by the second-grade fluids model, which forms a subclass of differential type fluids of complexity 2 and is one of basic constitutive models for viscoelastic fluid [1,2]. The motion of second-grade fluids is governed by the following system:

$$\begin{cases} \rho\big(\partial_t \boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y}\big) - \operatorname{div}\boldsymbol{S} + \nabla p = \rho\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{y} = 0, \\ \boldsymbol{S} = \mu\boldsymbol{D}(\boldsymbol{y}) + \alpha\partial_t\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y}) + \alpha\boldsymbol{D}(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{y}) - \alpha\boldsymbol{W}(\boldsymbol{y})\boldsymbol{D}(\boldsymbol{y}), \end{cases} \tag{1}$$

where $\boldsymbol{y} = \boldsymbol{y}(t,\boldsymbol{x})$ is the velocity at a point $\boldsymbol{x} \in \mathbb{R}^n$ $(n = 2,3)$ at time t, $p = p(t,\boldsymbol{x})$ is the pressure, $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x})$ is the external force field, $\boldsymbol{S} = \boldsymbol{S}(t,\boldsymbol{x})$ is the extra stress tensor, the divergence of $\boldsymbol{S}$ is defined as follows

$$\operatorname{div}\boldsymbol{S} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n}\frac{\partial S_{i1}}{\partial x_i},\ldots,\sum_{i=1}^{n}\frac{\partial S_{in}}{\partial x_i}\right)^{\top},$$

$\boldsymbol{D}(\boldsymbol{y})$ and $\boldsymbol{W}(\boldsymbol{y})$ denote the deformation tensor and the spin tensor, respectively,

$$\boldsymbol{D}(\boldsymbol{y}) \stackrel{\text{def}}{=} \frac{1}{2}\Big(\nabla\boldsymbol{y} + (\nabla\boldsymbol{y})^{\top}\Big), \qquad \boldsymbol{W}(\boldsymbol{y}) \stackrel{\text{def}}{=} \frac{1}{2}\Big(\nabla\boldsymbol{y} - (\nabla\boldsymbol{y})^{\top}\Big),$$

the symbol ∇ stands for the gradient with respect to the space variables $x_1, \ldots, x_n$, $\mu > 0$ is the viscosity of the fluid, $\alpha > 0$ is a constant material modulus, $\rho > 0$ is the fluid density. In the sequel, ρ supposed to be equal to 1 for simplicity.

We shall focus on the case when the nonlinear terms $\alpha \boldsymbol{D}(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{y})$ and $\alpha \boldsymbol{W}(\boldsymbol{y})\boldsymbol{D}(\boldsymbol{y})$, which contain products of the first derivatives of the velocity field $\boldsymbol{y}$ with respect to the space variables, are small compared to other terms in the third equality of (1) and can be discarded. In some sense, this simplification is similar the linearization procedure by cutting off all terms of order higher than 1 in a Taylor series. Then, introducing the expression of the extra stress tensor $\boldsymbol{S}$ into the first equation of (1), we arrive at the following system

$$\begin{cases} \partial_t \boldsymbol{y} + (\boldsymbol{y} \cdot \nabla)\boldsymbol{y} - \operatorname{div}\left[\mu \boldsymbol{D}(\boldsymbol{y}) + \alpha \partial_t \boldsymbol{D}(\boldsymbol{y}) + \alpha (\boldsymbol{y} \cdot \nabla)\boldsymbol{D}(\boldsymbol{y})\right] + \nabla p = \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{y} = 0, \end{cases} \tag{2}$$

which is considered as an appropriate model for the motion of viscous fluids with polymer additives, in particular, for aqueous solutions of polyacrylamide and polyethylenoxide [3,4]. Note that even for this simplified model, proving the existence and uniqueness of a solution is difficult since the nonlinear term $\operatorname{div}[\alpha(\boldsymbol{y} \cdot \nabla)\boldsymbol{D}(\boldsymbol{y})]$ contains the third-order derivative whereas the viscous term $\operatorname{div}[\mu \boldsymbol{D}(\boldsymbol{y})]$ is only a Laplace operator. In order to study this problem, one can use an approach that is based on the modified Faedo–Galerkin scheme with a basis of special eigenfunctions [5].

The mathematical analysis of the motion equations plays a major role in the prediction of the fluid flows behavior and the designing of optimal flows that can be successfully applied in technological processes. Starting with the first attempts by Oskolkov [6,7], many scholars have studied various mathematical problems for the second-grade fluid model and its modifications. We mention here only the works [8–14]; a detailed discussion of salient results on the well-posedness of system (1) and some open problems are given in the book [2].

In this paper, we study an optimal control problem for the motion equations of viscous fluids with polymer additives in a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary Γ, assuming the flow is time-independent, that is, $\partial_t \boldsymbol{y} = \boldsymbol{0}$, $\partial_t p = 0$, and $\partial_t \boldsymbol{u} = \boldsymbol{0}$. In contrast to the above-mentioned works, where the viscosity is assumed to be constant, we consider a variable-viscosity fluid model. More precisely, the fluid possesses shear-dependent viscosity $\mu = \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)$, where $\boldsymbol{\mathcal{P}}$ is some operator. Thus, system (2) becomes

$$\begin{cases} (\boldsymbol{y} \cdot \nabla)\boldsymbol{y} - \operatorname{div}\left[\mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha (\boldsymbol{y} \cdot \nabla)\boldsymbol{D}(\boldsymbol{y})\right] + \nabla p = \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{y} = 0. \end{cases} \tag{3}$$

Let us mention three important cases:

- The choice $\boldsymbol{\mathcal{P}} = \boldsymbol{0}$ leads to the stationary model for flows of an incompressible viscous fluid with polymer additives. Clearly, this model includes the classical Newtonian fluid as the limit case for $\alpha = 0$.
- When $\boldsymbol{\mathcal{P}}$ is the identity operator, system (3) reduces to the equations governing the steady-state flow of a viscoelastic fluid with shear-dependent viscosity [15,16]. Note that the concept of variable viscosity plays an important role in simulation of nanoscale liquid flows [17].
- If $\boldsymbol{\mathcal{P}}$ is an averaging operator (for example, the convolution operator with a smooth averaging kernel), then we arrive at a nonlocal model [18] that is concerned with fluids whose the viscosity at a material point $\boldsymbol{x}$ is influenced by the shear rate at all neighboring points of $\boldsymbol{x}$. The analysis of the published literature shows that so far, the well-posedness of boundary-value problems for such models has not been studied.

In studying our problem, it is convenient to remove the pressure function p from (3). For this purpose, we shall use the Leray projector $\boldsymbol{\pi}$ that is defined by the Leray (or Hodge–Helmholtz) decomposition of a vector field into the divergence-free part and the gradient part (see, e.g., [19], Chapter IV). Indeed, applying $\boldsymbol{\pi}$ to both sides of the first equality in (3), we obtain

$$\begin{cases} \boldsymbol{\pi}(\boldsymbol{y}\cdot\nabla)\boldsymbol{y} - \boldsymbol{\pi}\mathrm{div}\big[\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] = \boldsymbol{\pi u}, \\ \nabla\cdot\boldsymbol{y} = 0. \end{cases} \tag{4}$$

This system is equivalent to (3) in the following sense: if a pair $(\boldsymbol{y},\boldsymbol{u})$ satisfies (4), then there exists a function p such that the triplet $(\boldsymbol{y},\boldsymbol{u},p)$ satisfies (3). Therefore, in what follows, we can focus on the analysis of (4).

We assume that the boundary of the flow domain Ω is impermeable

$$\boldsymbol{y}\cdot\mathbf{n} = 0 \ \text{ on } \Gamma \tag{5}$$

and use the Navier slip boundary condition [20]

$$\big([\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})]\mathbf{n}\big)_\tau = -\beta\boldsymbol{y} \ \text{ on } \Gamma, \tag{6}$$

where $\beta > 0$ is the slip coefficient, $\mathbf{n}$ is the unit exterior normal to the surface Γ, the symbol τ denotes the tangential component of a vector field defined on Γ, that is, $\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v}\cdot\mathbf{n})\mathbf{n}$.

As the control parameter, we use the external force $\boldsymbol{u}$, assuming that

$$\boldsymbol{u} \in \mathcal{U}, \tag{7}$$

where $\mathcal{U}$ is the set of admissible controls.

The optimization problem is formulated as follows: Find a velocity field $\boldsymbol{y}$ and a control $\boldsymbol{u}$ that minimize a given cost functional $\mathcal{J}$ subject to relations (4)–(7). We formally write

$$\mathcal{J}(\boldsymbol{y},\boldsymbol{u}) \to \min. \tag{8}$$

The strict formulation of problem (4)–(8) in the framework of suitable function spaces will be given in Section 4, where we discuss both strong and weak solutions.

The main aim of the present paper is to prove the solvability of problem (4)–(8) in the class of weak solutions without the assumption that the viscosity function μ is monotone (cf. [21]). The proof is based on the solvability result (see Proposition 1) for a class of nonlinear operator equations with weak-to-weak and weak-to-strong continuous mappings in Sobolev spaces.

Secondly, following [21,22], we introduce the marginal function $\Phi_{\mathcal{J}} = \Phi_{\mathcal{J}}(\mathcal{U})$ that describes the dependence of the optimal value of the cost functional $\mathcal{J}$ on the admissible controls set. It is proved that $\Phi_{\mathcal{J}}$ is lower semicontinuous with respect to the directed Hausdorff distance. For the case of rigid control, it is shown that the marginal function is invariant with respect to applying the projector Leray $\boldsymbol{\pi}$ to the set $\mathcal{U}$. Moreover, we derive a necessary condition for the existence of strong solutions to (4)–(8) in terms of the marginal function concerning a special cost functional.

The plan of the paper is the following. In the next section we introduce notations and function spaces as well as state preparatory results that are required for the study of problem (4)–(8). In Section 3, we describe main assumptions on the model data and discuss some examples. Section 4 contains the functional setting of problem (4)–(8) and the main results (Theorem 1) of this work. Finally, Section 5 is devoted to the proof of Theorem 1.

2. Preliminaries: Notations, Function Spaces, and Preparatory Results

For vectors $\boldsymbol{f},\boldsymbol{g} \in \mathbb{R}^n$ and matrices $\mathbf{F},\mathbf{G} \in \mathbb{R}^{n\times n}$, by $\boldsymbol{f}\cdot\boldsymbol{g}$ and $\mathbf{F}:\mathbf{G}$ we denote the scalar products, respectively:

$$\boldsymbol{f}\cdot\boldsymbol{g} \overset{\text{def}}{=} \sum_{i=1}^n f_i g_i, \qquad \mathbf{F}:\mathbf{G} \overset{\text{def}}{=} \sum_{i,j=1}^n \mathrm{F}_{ij}\mathrm{G}_{ij}.$$

The Euclidean norm $|\cdot|$ is defined as follows:

$$|\boldsymbol{f}| \stackrel{\text{def}}{=} (\boldsymbol{f}\cdot\boldsymbol{f})^{1/2}, \qquad |\mathbf{F}| \stackrel{\text{def}}{=} (\mathbf{F}:\mathbf{F})^{1/2}.$$

The strong (weak) convergence in a Banach space is denoted by $\to$ ($\rightharpoonup$).

As usual, $\rightrightarrows$ indicates the uniform convergence.

The symbol $\hookrightarrow$ denotes a continuous embedding, while $\hookrightarrow\hookrightarrow$ denotes a compact embedding.

Let E_1 and E_2 be Banach spaces. By $\mathcal{L}(E_1, E_2)$ we denote the space of all bounded linear mappings from E_1 to E_2. The space $\mathcal{L}(E_1, E_2)$ is equipped with the norm

$$\|A\|_{\mathcal{L}(E_1,E_2)} \stackrel{\text{def}}{=} \sup_{\|v\|_{E_1}\neq 0} \frac{\|A(v)\|_{E_2}}{\|v\|_{E_1}}.$$

Let U and W be subsets of a Banach space E. By definition, put

$$d_E(U, W) \stackrel{\text{def}}{=} \sup_{u\in U} \inf_{w\in W} \|u - w\|_E.$$

This quantity is termed as the directed Hausdorff distance (or one-sided Hausdorff distance) from the set U to the set W.

Let Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with boundary Γ. We shall use the standard notation for the Lebesgue spaces $\boldsymbol{L}^q(\Omega)$ and $\boldsymbol{L}^q(\Gamma)$, where $q \geq 1$, and the Sobolev space $\boldsymbol{H}^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega)$, $m \in \mathbb{N}$. The definitions and properties of these spaces can be found in [2,23]. For the corresponding classes of vector- and matrix-valued functions, we use the following notations:

$$\boldsymbol{L}^q(\Omega) \stackrel{\text{def}}{=} L^q(\Omega)^n, \qquad \boldsymbol{H}^m(\Omega) \stackrel{\text{def}}{=} H^m(\Omega)^n,$$
$$\mathbb{L}^q(\Omega) \stackrel{\text{def}}{=} L^q(\Omega)^{n\times n}, \qquad \mathbb{H}^m(\Omega) \stackrel{\text{def}}{=} H^m(\Omega)^{n\times n}.$$

Recall that the restriction of a function $v \in H^1(\Omega)$ to the surface Γ is defined by the rule $v|_\Gamma \stackrel{\text{def}}{=} \gamma_0 v$, where γ_0 is the trace operator (see, e.g., [23]). The operator γ_0 is continuous and compact as a map from $H^1(\Omega)$ into $L^2(\Gamma)$.

By $\boldsymbol{P}_{bc}(\boldsymbol{L}^2(\Omega))$ we denote the totality of all bounded sequentially weakly closed sets in $\boldsymbol{L}^2(\Omega)$. Note that if $\mathbf{U}$ is a closed convex bounded set in $\boldsymbol{L}^2(\Omega)$, then $\mathbf{U} \in \boldsymbol{P}_{bc}(\boldsymbol{L}^2(\Omega))$.

Let

$$\begin{aligned}
\mathcal{Y}(\Omega) &\stackrel{\text{def}}{=} \{\boldsymbol{\phi}\colon \overline{\Omega} \to \mathbb{R}^n :\ \boldsymbol{\phi} \in \boldsymbol{C}^\infty(\overline{\Omega}),\ \nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \Omega, \text{ and } \boldsymbol{\phi}\cdot\mathbf{n} = 0 \text{ on } \Gamma\},\\
\boldsymbol{Y}^m(\Omega) &\stackrel{\text{def}}{=} \text{the closure of the set } \mathcal{Y}(\Omega) \text{ in the Sobolev space } \boldsymbol{H}^m(\Omega),\\
[\boldsymbol{Y}^m(\Omega)]' &\stackrel{\text{def}}{=} \text{the dual space of } \boldsymbol{Y}^m(\Omega),
\end{aligned}$$

where $m \in \mathbb{N}$.

It is obvious that $\boldsymbol{Y}^m(\Omega)$ is a Hilbert space with the scalar product $(\cdot,\cdot)_{\boldsymbol{H}^m(\Omega)}$. However, when studying problem (4)–(8), in the space $\boldsymbol{Y}^1(\Omega)$, it is more convenient to use the scalar product and the norm that are defined as follows:

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{Y}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{w})\, d\boldsymbol{x} + \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{w}\, d\Gamma, \qquad \|\boldsymbol{v}\|_{\boldsymbol{Y}^1(\Omega)} \stackrel{\text{def}}{=} (\boldsymbol{v}, \boldsymbol{v})^{1/2}_{\boldsymbol{Y}^1(\Omega)}.$$

From inequalities of Korn's type (see [24], Chapter I, Theorems 2.2 and 2.3) it follows that the scalar product $(\cdot,\cdot)_{\boldsymbol{Y}^1(\Omega)}$ is well defined and the norm $\|\cdot\|_{\boldsymbol{Y}^1(\Omega)}$ is equivalent to the standard $\boldsymbol{H}^1$-norm.

Recall that $\boldsymbol{H}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^4(\Omega)$. Therefore, we have $\boldsymbol{Y}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^4(\Omega)$.

By techniques similar to those employed in the proof of Theorem 3.1 from [25], one can obtain the following statement.

Proposition 1. *Let $\boldsymbol{A}$ be an operator from $\boldsymbol{Y}^1(\Omega)$ into $[\boldsymbol{Y}^1(\Omega)]'$, let $\boldsymbol{B}$ be an operator from $\boldsymbol{Y}^1(\Omega)$ into $[\boldsymbol{Y}^3(\Omega)]'$, and let $\mathcal{G}_{\boldsymbol{A}}$ be the functional defined by*

$$\mathcal{G}_{\boldsymbol{A}}\colon \boldsymbol{Y}^1(\Omega) \to \mathbb{R}, \quad \mathcal{G}_{\boldsymbol{A}}(\boldsymbol{v}) \stackrel{\text{def}}{=} \langle \boldsymbol{A}(\boldsymbol{v}), \boldsymbol{v}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times \boldsymbol{Y}^1(\Omega)}.$$

Assume that

- *the operator $\boldsymbol{A}$ is a weak-to-weak continuous operator, that is, the convergence $\boldsymbol{v}_m \rightharpoonup \boldsymbol{v}_0$ in $\boldsymbol{Y}^1(\Omega)$ implies that $\boldsymbol{A}(\boldsymbol{v}_m) \rightharpoonup \boldsymbol{A}(\boldsymbol{v}_0)$ in $[\boldsymbol{Y}^1(\Omega)]'$ as $m \to \infty$;*
- *the operator $\boldsymbol{B}$ is a weak-to-strong continuous operator, that is, the convergence $\boldsymbol{v}_m \rightharpoonup \boldsymbol{v}_0$ in $\boldsymbol{Y}^1(\Omega)$ implies that $\boldsymbol{B}(\boldsymbol{v}_m) \to \boldsymbol{B}(\boldsymbol{v}_0)$ in $[\boldsymbol{Y}^3(\Omega)]'$ as $m \to \infty$;*
- *the functional $\mathcal{G}_{\boldsymbol{A}}$ is lower semicontinuous with respect to the weak convergence in $\boldsymbol{Y}^1(\Omega)$, that is, for any sequence $\{\boldsymbol{v}_m\}_{m=1}^{\infty}$ such that $\boldsymbol{v}_m \rightharpoonup \boldsymbol{v}_0$ in $\boldsymbol{Y}^1(\Omega)$, we have*

$$\mathcal{G}_{\boldsymbol{A}}(\boldsymbol{v}_0) \le \liminf_{m\to\infty} \mathcal{G}_{\boldsymbol{A}}(\boldsymbol{v}_m);$$

- *the inequality $\mathcal{G}_{\boldsymbol{A}}(\boldsymbol{v}) \ge C_0\|\boldsymbol{v}\|^2_{\boldsymbol{Y}^1(\Omega)}$, $C_0 = \text{const}$, holds for all $\boldsymbol{v} \in \boldsymbol{Y}^1(\Omega)$;*
- *the equality $\langle \boldsymbol{B}(\boldsymbol{w}), \boldsymbol{w}\rangle_{[\boldsymbol{Y}^3(\Omega)]'\times \boldsymbol{Y}^3(\Omega)} = 0$ holds for all $\boldsymbol{w} \in \boldsymbol{Y}^3(\Omega)$.*

Then, for any $\boldsymbol{h} \in [\boldsymbol{Y}^1(\Omega)]'$, the following equation

$$\langle \boldsymbol{A}(\boldsymbol{y}), \boldsymbol{w}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times \boldsymbol{Y}^1(\Omega)} + \langle \boldsymbol{B}(\boldsymbol{y}), \boldsymbol{w}\rangle_{[\boldsymbol{Y}^3(\Omega)]'\times \boldsymbol{Y}^3(\Omega)} = \langle \boldsymbol{h}, \boldsymbol{w}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times \boldsymbol{Y}^1(\Omega)}, \quad \forall \boldsymbol{w} \in \boldsymbol{Y}^3(\Omega),$$

has at least one solution $\boldsymbol{y}_{\boldsymbol{h}} \in \boldsymbol{Y}^1(\Omega)$ such that

$$\langle \boldsymbol{A}(\boldsymbol{y}_{\boldsymbol{h}}), \boldsymbol{y}_{\boldsymbol{h}}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times \boldsymbol{Y}^1(\Omega)} \le \langle \boldsymbol{h}, \boldsymbol{y}_{\boldsymbol{h}}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times \boldsymbol{Y}^1(\Omega)}.$$

3. Main Assumptions on the Model Data and Some Examples

Let us assume that the following conditions are fulfilled:

(C1) the surface Γ is of class $C^{0,1}$;
(C2) the function $\mu\colon [0,+\infty) \to [0,+\infty)$ is continuous and there exist constants μ_0 and μ_1 such that $0 < \mu_0 \le \mu(s) \le \mu_1$ for any $s \in [0,+\infty)$;
(C3) the operator $\boldsymbol{\mathcal{P}}$ maps $\mathbb{L}^2(\Omega)$ into $\mathbb{L}^2(\Omega)$ and, for any sequence $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$ such that $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}_0$ in $\boldsymbol{Y}^1(\Omega)$ as $k \to \infty$, we have $\boldsymbol{\mathcal{P}D}(\boldsymbol{v}_k) \to \boldsymbol{\mathcal{P}D}(\boldsymbol{v}_0)$ in $\mathbb{L}^2(\Omega)$ as $k \to \infty$;
(C4) the admissible controls set $\mathcal{U}$ belongs to the space $\boldsymbol{P}_{bc}(\boldsymbol{L}^2(\Omega))$;
(C5) the cost functional $\mathcal{J}\colon \boldsymbol{Y}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ is lower semicontinuous with respect to the weak convergence in $\boldsymbol{Y}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$, that is, for any sequence $\{(\boldsymbol{y}_k, \boldsymbol{u}_k)\}_{k=1}^{\infty}$ such that $\boldsymbol{y}_k \rightharpoonup \boldsymbol{y}_0$ in $\boldsymbol{Y}^1(\Omega)$ and $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_0$ in $\boldsymbol{L}^2(\Omega)$, we have

$$\mathcal{J}(\boldsymbol{y}_0, \boldsymbol{u}_0) \le \liminf_{k\to\infty} \mathcal{J}(\boldsymbol{y}_k, \boldsymbol{u}_k).$$

As the main example of an operator satisfying condition (C3), we consider the averaging operator (see, e.g., [26], Chapter 1) defined as follows

$$\boldsymbol{\mathcal{P}}_{\omega}\boldsymbol{Q}(\boldsymbol{x}) \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^n} \omega(\boldsymbol{x}-\boldsymbol{x}')\widetilde{\boldsymbol{Q}}(\boldsymbol{x}')d\boldsymbol{x}', \qquad \forall \boldsymbol{Q} \in \mathbb{L}^2(\Omega),$$

where

$$\widetilde{\boldsymbol{Q}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{Q}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega}, \\ \boldsymbol{0} & \text{if } \boldsymbol{x} \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

and $\omega\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support such that

$$\int_{\mathbb{R}^n} \omega(\boldsymbol{x}')d\boldsymbol{x}' = 1$$

and $\omega(\boldsymbol{x}') = \omega(\boldsymbol{x}'')$ whenever $|\boldsymbol{x}'| = |\boldsymbol{x}''|$.

Remark 1. *From the physical point of view, ω is the influence function [18].*

Lemma 1. *Let $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$ be a sequence such that $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}_0$ in $\boldsymbol{Y}^1(\Omega)$. Then*

$$\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_k) \rightrightarrows \boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_0) \text{ on } \overline{\Omega} \text{ as } k \to \infty.$$

Proof. First, we observe that $\boldsymbol{v}_k \to \boldsymbol{v}_0$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{v}_k|_\Gamma \to \boldsymbol{v}_0|_\Gamma$ in $\boldsymbol{L}^2(\Gamma)$, because $\boldsymbol{Y}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$ and the trace operator $\gamma_0\colon \boldsymbol{Y}^1(\Omega) \to \boldsymbol{L}^2(\Gamma)$ is compact. Therefore, we have

$$\lim_{k\to\infty} \|\boldsymbol{v}_k - \boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Omega)} = 0, \qquad \lim_{k\to\infty} \|\boldsymbol{v}_k - \boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Gamma)} = 0. \tag{9}$$

Further, using integration by parts, we obtain

$$\begin{aligned}
[\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{\psi})]_{ij}(\boldsymbol{x}) =& \frac{1}{2}\int_{\Omega} \omega(\boldsymbol{x}-\boldsymbol{x}')\left(\frac{\partial \psi_i(\boldsymbol{x}')}{\partial x_j'} + \frac{\partial \psi_j(\boldsymbol{x}')}{\partial x_i'}\right) d\boldsymbol{x}' \\
=& \frac{1}{2}\int_{\Gamma} \omega(\boldsymbol{x}-\boldsymbol{x}')[\psi_i(\boldsymbol{x}')n_j + \psi_j(\boldsymbol{x}')n_i]\, d\Gamma \\
& - \frac{1}{2}\int_{\Omega} \left(\frac{\partial[\omega(\boldsymbol{x}-\boldsymbol{x}')]}{\partial x_j'}\psi_i(\boldsymbol{x}') + \frac{\partial[\omega(\boldsymbol{x}-\boldsymbol{x}')]}{\partial x_i'}\psi_j(\boldsymbol{x}')\right) d\boldsymbol{x}',
\end{aligned} \tag{10}$$

for any $\boldsymbol{x} \in \overline{\Omega}$ and $\boldsymbol{\psi} \in \boldsymbol{Y}^1(\Omega)$. Applying the Cauchy–Schwarz inequality, we derive from (10) the following estimate

$$\begin{aligned}
\|\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{\psi})\|_{C(\overline{\Omega})} \leq& \max\{|\omega| :\ \boldsymbol{x} \in \mathbb{R}^n\} \operatorname{meas}(\Gamma)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Gamma)} \\
& + \max\left\{\left|\frac{\partial \omega}{\partial x_s}\right| :\ \boldsymbol{x} \in \mathbb{R}^n,\, s = 1,\dots,n\right\} \operatorname{meas}(\Omega)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Setting $\boldsymbol{\psi} = \boldsymbol{v}_k - \boldsymbol{v}_0$ into the last inequality, we get

$$\begin{aligned}
\|\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_k - \boldsymbol{v}_0)\|_{C(\overline{\Omega})} \leq& \max\{|\omega| :\ \boldsymbol{x} \in \mathbb{R}^n\} \operatorname{meas}(\Gamma)^{\frac{1}{2}} \|\boldsymbol{v}_k - \boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Gamma)} \\
& + \max\left\{\left|\frac{\partial \omega}{\partial x_s}\right| :\ \boldsymbol{x} \in \mathbb{R}^n,\, s = 1,\dots,n\right\} \operatorname{meas}(\Omega)^{\frac{1}{2}} \|\boldsymbol{v}_k - \boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

This inequality, together with (9), yields that

$$\lim_{k\to\infty} \|\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_k - \boldsymbol{v}_0)\|_{\boldsymbol{C}(\overline{\Omega})} = 0,$$

and hence $\boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_k) \rightrightarrows \boldsymbol{\mathcal{P}}_\omega \boldsymbol{D}(\boldsymbol{v}_0)$ on $\overline{\Omega}$ as $k \to \infty$. Lemma 1 is proved. □

Corollary 1. *The operator $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_\omega$ satisfies condition* (C3).

Here, we also give two examples of cost functionals satisfying condition (C5):

- $\mathcal{J} = \mathcal{J}_1(\boldsymbol{y}, \boldsymbol{u}) \stackrel{\text{def}}{=} (1-\xi)\|\boldsymbol{y} - \boldsymbol{y}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \xi\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}$, where $\boldsymbol{y}_0$ is a given vector function that describes the desirable velocity field in the domain Ω, while ξ is a numerical parameter, $0 \leq \xi \leq 1$;

- $\mathcal{J} = \mathcal{J}_2(\boldsymbol{y},\boldsymbol{u}) \stackrel{\text{def}}{=} -\|\boldsymbol{y}-\boldsymbol{y}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}$, where $\boldsymbol{y}_1$ is an unfavourable velocity field, that is, a velocity field whose occurrence in the domain Ω is undesirable.

In the above-mentioned examples, we consider compromise cost functionals, assuming that the cost of control $\boldsymbol{u}$ must be minimal. However, in some cases, the cost of control is not significative, while the state function $\boldsymbol{y}$ is of primary importance. From the mathematical point of view, this means that the functional $\mathcal{J}$ does not contain the term with $\boldsymbol{u}$ or the value of the parameter ξ tends to zero. A control with such a cost functional is referred to as *rigid* (see [27], Chapter 1, Section 5).

Definition 1. *We shall say that* (4)–(8) *is a rigid control problem if the cost functional* $\mathcal{J}$ *does not depend on the control function* $\boldsymbol{u}$ *explicitly, that is,* $\mathcal{J} = \mathcal{J}(\boldsymbol{y})$.

4. Functional Setting of the Problem and Main Results

This section introduces the mathematical framework for studying problem (4)–(8) in Sobolev spaces. To begin with, we shall consider the concept of strong solutions, which have all derivatives occuring in the equations, in the sense of Sobolev.

Definition 2. *We shall say that* $(\boldsymbol{y},\boldsymbol{u})$ *is a strong solution of control system* (4)–(7) *if* $(\boldsymbol{y},\boldsymbol{u}) \in \boldsymbol{H}^3(\Omega)\times\mathcal{U}$ *and the pair* $(\boldsymbol{y},\boldsymbol{u})$ *satisfies* (4)–(6).

By $\mathfrak{M}_s(\mathcal{U})$ denote the set of strong solutions to (4)–(7).

Lemma 2. *If* $(\boldsymbol{y},\boldsymbol{u})$ *is a strong solution of control system* (4)–(7), *then*

(a) *the following equality holds*

$$-\sum_{i=1}^{n}\int_\Omega y_i\boldsymbol{y}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,d\boldsymbol{x} + \int_\Omega \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}):\boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - \alpha\sum_{i=1}^{n}\int_\Omega y_i\boldsymbol{D}(\boldsymbol{y}):\frac{\partial\boldsymbol{D}(\boldsymbol{\varphi})}{\partial x_i}\,d\boldsymbol{x} + \beta\int_\Gamma \boldsymbol{y}\cdot\boldsymbol{\varphi}\,d\Gamma = \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}, \tag{11}$$

for any $\boldsymbol{\varphi}\in\boldsymbol{Y}^2(\Omega)$;

(b) *the pair* $(\boldsymbol{y},\boldsymbol{u})$ *satisfies the energy equality*

$$\int_\Omega \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)|\boldsymbol{D}(\boldsymbol{y})|^2\,d\boldsymbol{x} + \beta\int_\Gamma |\boldsymbol{y}|^2\,d\Gamma = \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{y}\,d\boldsymbol{x}. \tag{12}$$

Proof. Let $\boldsymbol{\varphi}$ be a vector function from the space $\boldsymbol{Y}^2(\Omega)$. On taking the scalar product of both the left-hand and right-hand sides of the first equation in (4) with $\boldsymbol{\varphi}$ and integrating over the flow domain Ω, we obtain

$$\underbrace{\int_\Omega [\boldsymbol{\pi}(\boldsymbol{y}\cdot\nabla)\boldsymbol{y}]\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}}_{=I_1} - \underbrace{\int_\Omega \boldsymbol{\pi}\mathrm{div}\left[\mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\right]\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}}_{=I_2} = \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{13}$$

Since $(\nabla q,\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)} = 0$ for any $q\in H^1(\Omega)$, we see that

$$I_1 = \int_\Omega [(\boldsymbol{y}\cdot\nabla)\boldsymbol{y}]\cdot\boldsymbol{\varphi}\,d\boldsymbol{x},$$

$$I_2 = \int_\Omega \operatorname{div}\big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big]\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}.$$

Using integration by parts, the incompressibility condition $\nabla\cdot\boldsymbol{y} = 0$ in Ω, and boundary conditions (5) and (6), we get

$$I_1 = \int_\Gamma \underbrace{(\boldsymbol{y}\cdot\mathbf{n})}_{=0}(\boldsymbol{y}\cdot\boldsymbol{\varphi})\,d\Gamma + \int_\Omega \underbrace{(\nabla\cdot\boldsymbol{y})}_{=0}(\boldsymbol{y}\cdot\boldsymbol{\varphi})\,d\boldsymbol{x} - \sum_{i=1}^n \int_\Omega y_i\boldsymbol{y}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,d\boldsymbol{x}$$
$$= -\sum_{i=1}^n \int_\Omega y_i\boldsymbol{y}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,d\boldsymbol{x}, \tag{14}$$

$$I_2 = \int_\Gamma \underbrace{\big([\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})]\mathbf{n}\big)_\tau}_{=-\beta\boldsymbol{y}}\cdot\boldsymbol{\varphi}\,d\Gamma$$
$$- \int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \nabla\boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{15}$$

Taking into account that the matrix $\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})$ is symmetric, it is easily shown that

$$\int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \nabla\boldsymbol{\varphi}\,d\boldsymbol{x}$$
$$= \frac{1}{2}\int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \nabla\boldsymbol{\varphi}\,d\boldsymbol{x}$$
$$+ \frac{1}{2}\int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big]^\top : (\nabla\boldsymbol{\varphi})^\top\,d\boldsymbol{x}$$
$$= \frac{1}{2}\int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \big(\nabla\boldsymbol{\varphi} + (\nabla\boldsymbol{\varphi})^\top\big)\,d\boldsymbol{x}$$
$$= \int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}. \tag{16}$$

Combining (15) and (16), we obtain

$$I_2 = -\beta\int_\Gamma \boldsymbol{y}\cdot\boldsymbol{\varphi}\,d\Gamma - \int_\Omega \big[\mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) + \alpha(\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y})\big] : \boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}. \tag{17}$$

Substituting (14) and (17) into (13), we arrive at the equality

$$-\sum_{i=1}^n \int_\Omega y_i\boldsymbol{y}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,d\boldsymbol{x} + \beta\int_\Gamma \boldsymbol{y}\cdot\boldsymbol{\varphi}\,d\Gamma + \int_\Omega \mu(|\mathcal{P}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}) : \boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}$$
$$+ \alpha\underbrace{\int_\Omega (\boldsymbol{y}\cdot\nabla)\boldsymbol{D}(\boldsymbol{y}) : \boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}}_{=I_3} = \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{18}$$

Finally, applying integration by parts to the term I_3, we derive from (18) relation (11).

Next, by setting $\boldsymbol{\varphi} = \boldsymbol{y}$ into (11), we obtain

$$-\underbrace{\sum_{i=1}^{n}\int_{\Omega} y_i \boldsymbol{y}\cdot\frac{\partial \boldsymbol{y}}{\partial x_i}\,d\boldsymbol{x}}_{=I_4} + \int_{\Omega}\mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)\boldsymbol{D}(\boldsymbol{y}):\boldsymbol{D}(\boldsymbol{y})\,d\boldsymbol{x}$$
$$-\alpha\underbrace{\sum_{i=1}^{n}\int_{\Omega} y_i\boldsymbol{D}(\boldsymbol{y}):\frac{\partial \boldsymbol{D}(\boldsymbol{y})}{\partial x_i}\,d\boldsymbol{x}}_{=I_5} + \beta\int_{\Gamma}\boldsymbol{y}\cdot\boldsymbol{y}\,d\Gamma = \int_{\Omega}\boldsymbol{\pi u}\cdot\boldsymbol{y}\,d\boldsymbol{x}. \quad (19)$$

Using integration by parts, we find

$$\begin{aligned} I_4 &= \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega} y_i\frac{\partial|\boldsymbol{y}|^2}{\partial x_i}\,d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\Gamma}\underbrace{(\boldsymbol{y}\cdot\mathbf{n})}_{=0}|\boldsymbol{y}|^2\,d\Gamma - \frac{1}{2}\int_{\Omega}\underbrace{(\nabla\cdot\boldsymbol{y})}_{=0}|\boldsymbol{y}|^2\,d\boldsymbol{x} \\ &= 0, \end{aligned}$$

$$\begin{aligned} I_5 &= \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega} y_i\frac{\partial|\boldsymbol{D}(\boldsymbol{y})|^2}{\partial x_i}\,d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\Gamma}\underbrace{(\boldsymbol{y}\cdot\mathbf{n})}_{=0}|\boldsymbol{D}(\boldsymbol{y})|^2\,d\Gamma - \frac{1}{2}\int_{\Omega}\underbrace{(\nabla\cdot\boldsymbol{y})}_{=0}|\boldsymbol{D}(\boldsymbol{y})|^2\,d\boldsymbol{x} \\ &= 0. \end{aligned}$$

Thus, (19) reduces to (12). The proof is complete. □

The question of the existence of strong solutions is delicate, especially in the case when the forcing term $\boldsymbol{u}$ is non-smooth and/or has a large norm. Since the set of admissible controls $\mathcal{U}$ contains many such elements, it is reasonable to go from strong solutions to weak solutions, which can be found for a large class of vector functions $\boldsymbol{u}$. The lemma just proved suggests how to define a weak solution in a suitable way.

Definition 3. *We shall say that $(\boldsymbol{y},\boldsymbol{u})$ is a weak solution of control system (4)–(7) if $(\boldsymbol{y},\boldsymbol{u}) \in \boldsymbol{Y}^1(\Omega)\times\mathcal{U}$, equality (11) holds for any $\boldsymbol{\varphi}\in\boldsymbol{Y}^3(\Omega)$, and*

$$\int_{\Omega}\mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y})|)|\boldsymbol{D}(\boldsymbol{y})|^2\,d\boldsymbol{x} + \beta\int_{\Gamma}|\boldsymbol{y}|^2\,d\Gamma \le \int_{\Omega}\boldsymbol{\pi u}\cdot\boldsymbol{y}\,d\boldsymbol{x}. \quad (20)$$

By $\mathfrak{M}_w(\mathcal{U})$ denote the set of weak solutions to (4)–(7).
Clearly, we have the inclusion $\mathfrak{M}_s(\mathcal{U}) \subset \mathfrak{M}_w(\mathcal{U})$.

Definition 4. *A pair $(\boldsymbol{y}_*,\boldsymbol{u}_*)$ is called an optimal weak solution to problem (4)–(8) if $(\boldsymbol{y}_*,\boldsymbol{u}_*)$ belongs to the set $\mathfrak{M}_w(\mathcal{U})$ and*

$$\mathcal{J}(\boldsymbol{y}_*,\boldsymbol{u}_*) = \inf_{(\boldsymbol{y},\boldsymbol{u})\in\mathfrak{M}_w(\mathcal{U})}\mathcal{J}(\boldsymbol{y},\boldsymbol{u}). \quad (21)$$

By $\mathfrak{M}_w^{\mathrm{opt}}(\mathcal{U})$ denote the set of optimal weak solutions to (4)–(8). As it will be shown, $\mathfrak{M}_w^{\mathrm{opt}}(\mathcal{U}) \neq \varnothing$ under conditions (C1)–(C5).

In order to examine the situation when the set of admissible controls is changeable, we introduce the concept of the marginal function.

By definition, put

$$\Phi_{\mathcal{J}}\colon \boldsymbol{P}_{bc}(\boldsymbol{L}^2(\Omega)) \to \mathbb{R}\cup\{-\infty\}, \quad \Phi_{\mathcal{J}}(\mathcal{U}) \stackrel{\text{def}}{=} \inf_{(\boldsymbol{y},\boldsymbol{u})\in\mathfrak{M}_w(\mathcal{U})} \mathcal{J}(\boldsymbol{y},\boldsymbol{u}).$$

Clearly, if $(\boldsymbol{y}_*,\boldsymbol{u}_*)$ is a pair that belongs to the set $\mathfrak{M}_w^{\text{opt}}(\mathcal{U})$, then

$$\Phi_{\mathcal{J}}(\mathcal{U}) = \mathcal{J}(\boldsymbol{y}_*,\boldsymbol{u}_*).$$

Definition 5. *The function $\Phi_{\mathcal{J}}$ is called the marginal function of system* (4)–(8)*.*

The main results of the present work are summarized as follows:

Theorem 1. *Suppose that conditions* (C1)–(C5) *hold. Then*

(i) *problem* (4)–(8) *has at least one optimal weak solution;*

(ii) *if $\mathfrak{M}_s(\mathcal{U}) \neq \varnothing$, then $\Phi_{\mathcal{J}_0}(\mathcal{U}) = 0$, where*

$$\mathcal{J}_0(\boldsymbol{y},\boldsymbol{u}) \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{y}\,\boldsymbol{dx} - \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y})|)|\boldsymbol{D}(\boldsymbol{y})|^2\,\boldsymbol{dx} - \beta\int_\Gamma |\boldsymbol{y}|^2\,d\Gamma;$$

(iii) *if* (4)–(8) *is a rigid control problem, then $\Phi_{\mathcal{J}}(\mathcal{U}) = \Phi_{\mathcal{J}}(\boldsymbol{\pi}\mathcal{U})$;*

(iv) *the marginal function $\Phi_{\mathcal{J}}$ is lower semicontinuous in the following sense:*
if $\mathcal{U}_k \in \boldsymbol{P}_{bc}(\boldsymbol{L}^2(\Omega))$, for any $k\in\mathbb{N}\cup\{0\}$, and

$$\lim_{k\to\infty} d_{\boldsymbol{L}^2(\Omega)}(\mathcal{U}_k,\mathcal{U}_0) = 0, \tag{22}$$

then

$$\Phi_{\mathcal{J}}(\mathcal{U}_0) \le \liminf_{k\to\infty}\Phi_{\mathcal{J}}(\mathcal{U}_k).$$

The proof of this theorem is given in Section 5.

Remark 2. *For polymeric fluid flows with constant viscosity* (*in our notation, the case when* $\boldsymbol{\mathcal{P}} = \boldsymbol{0}$), *sufficient conditions for the existence of optimal boundary controls are derived in [28,29]. The paper [30] deals with optimal control for two-dimensional stochastic second-grade fluids.*

5. Proof of Theorem 1

First we shall establish the existence result (i). The proof of this statement is derived in four steps.

Step 1. Our first step is to show that $\mathfrak{M}_w(\mathcal{U}) \neq \varnothing$. Fix a vector-valued function $\boldsymbol{u}$ belonging to the set $\mathcal{U}$. Let us introduce operators $\boldsymbol{A}$ and $\boldsymbol{B}$ by the formulas:

$$\boldsymbol{A}\colon \boldsymbol{Y}^1(\Omega) \to [\boldsymbol{Y}^1(\Omega)]',$$

$$\langle \boldsymbol{A}(\boldsymbol{v}),\boldsymbol{\omega}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times\boldsymbol{Y}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{v})|)\boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\boldsymbol{\omega})\,\boldsymbol{dx} + \beta\int_\Gamma \boldsymbol{y}\cdot\boldsymbol{\omega}\,d\Gamma,$$

$$\boldsymbol{B}\colon \boldsymbol{Y}^1(\Omega) \to [\boldsymbol{Y}^3(\Omega)]',$$

$$\langle \boldsymbol{B}(\boldsymbol{v}),\boldsymbol{w}\rangle_{[\boldsymbol{Y}^3(\Omega)]'\times\boldsymbol{Y}^3(\Omega)} \stackrel{\text{def}}{=} -\sum_{i=1}^n\int_\Omega v_i\boldsymbol{v}\cdot\frac{\partial\boldsymbol{w}}{\partial x_i}\,\boldsymbol{dx} - \alpha\sum_{i=1}^n\int_\Omega v_i\boldsymbol{D}(\boldsymbol{v}):\frac{\partial\boldsymbol{D}(\boldsymbol{w})}{\partial x_i}\,\boldsymbol{dx},$$

where $\boldsymbol{v}\in\boldsymbol{Y}^1(\Omega)$, $\boldsymbol{\omega}\in\boldsymbol{Y}^1(\Omega)$, $\boldsymbol{w}\in\boldsymbol{Y}^3(\Omega)$, and define the functional $\widehat{\boldsymbol{u}}\in[\boldsymbol{Y}^1(\Omega)]'$ as follows

$$\langle\widehat{\boldsymbol{u}},\boldsymbol{\omega}\rangle_{[\boldsymbol{Y}^1(\Omega)]'\times\boldsymbol{Y}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\pi u}\cdot\boldsymbol{\omega}\,\boldsymbol{dx}.$$

Using these operators, we can rewrite (11) in the form

$$\langle \boldsymbol{A}(\boldsymbol{y}), \boldsymbol{\varphi} \rangle_{[\boldsymbol{Y}^1(\Omega)]' \times \boldsymbol{Y}^1(\Omega)} + \langle \boldsymbol{B}(\boldsymbol{y}), \boldsymbol{\varphi} \rangle_{[\boldsymbol{Y}^3(\Omega)]' \times \boldsymbol{Y}^3(\Omega)} = \langle \widehat{\boldsymbol{u}}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{Y}^1(\Omega)]' \times \boldsymbol{Y}^1(\Omega)}, \tag{23}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{Y}^3(\Omega)$.

Taking into account conditions (C2) and (C3), the inclusion $\boldsymbol{Y}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^4(\Omega)$, and the following relations

$$\langle \boldsymbol{A}(\boldsymbol{v}), \boldsymbol{v} \rangle_{[\boldsymbol{Y}^1(\Omega)]' \times \boldsymbol{Y}^1(\Omega)} \geq \min\{\mu_0, \beta\} \|\boldsymbol{v}\|^2_{\boldsymbol{Y}^1(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{Y}^1(\Omega),$$

$$\langle \boldsymbol{B}(\boldsymbol{w}), \boldsymbol{w} \rangle_{[\boldsymbol{Y}^3(\Omega)]' \times \boldsymbol{Y}^3(\Omega)} = 0, \quad \forall \boldsymbol{w} \in \boldsymbol{Y}^3(\Omega),$$

it is not hard to check that all conditions of Proposition 1 hold. Then, by applying Proposition 1 to problem (23), we deduce that (23) has a solution $\boldsymbol{y}_{\boldsymbol{u}} \in \boldsymbol{Y}^1(\Omega)$ such that

$$\langle \boldsymbol{A}(\boldsymbol{y}_{\boldsymbol{u}}), \boldsymbol{y}_{\boldsymbol{u}} \rangle_{[\boldsymbol{Y}^1(\Omega)]' \times \boldsymbol{Y}^1(\Omega)} \leq \langle \widehat{\boldsymbol{u}}, \boldsymbol{y}_{\boldsymbol{u}} \rangle_{[\boldsymbol{Y}^1(\Omega)]' \times \boldsymbol{Y}^1(\Omega)}.$$

Clearly, the pair $(\boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{u})$ is a weak solution to problem (4)–(7), and hence $\mathfrak{M}_w(\mathcal{U}) \neq \varnothing$.

Step 2. Let us show that the set $\mathfrak{M}_w(\mathcal{U})$ is sequentially weakly closed in the space $\boldsymbol{Y}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$. Consider a sequence $\{(\boldsymbol{y}_m, \boldsymbol{u}_m)\}_{m=1}^{\infty}$ such that

$$(\boldsymbol{y}_m, \boldsymbol{u}_m) \in \mathfrak{M}_w(\mathcal{U}), \quad \forall m \in \mathbb{N},$$

$$\boldsymbol{y}_m \rightharpoonup \boldsymbol{y}_0 \text{ in } \boldsymbol{Y}^1(\Omega) \text{ as } m \to \infty, \tag{24}$$

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}_0 \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } m \to \infty. \tag{25}$$

We must prove that $(\boldsymbol{y}_0, \boldsymbol{u}_0) \in \mathfrak{M}_w(\mathcal{U})$.

Since $(\boldsymbol{y}_m, \boldsymbol{u}_m)$ is a weak solution to problem (4)–(7), we have

$$\begin{aligned} &-\sum_{i=1}^{n} \int_{\Omega} y_{mi} \boldsymbol{y}_m \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_i} \, \boldsymbol{dx} + \int_{\Omega} \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|) \boldsymbol{D}(\boldsymbol{y}_m) : \boldsymbol{D}(\boldsymbol{\psi}) \, \boldsymbol{dx} \\ &\quad - \alpha \sum_{i=1}^{n} \int_{\Omega} y_{mi} \boldsymbol{D}(\boldsymbol{y}_m) : \frac{\partial \boldsymbol{D}(\boldsymbol{\psi})}{\partial x_i} \, \boldsymbol{dx} + \beta \int_{\Gamma} \boldsymbol{y}_m \cdot \boldsymbol{\psi} \, d\Gamma = \int_{\Omega} \boldsymbol{\pi u}_m \cdot \boldsymbol{\psi} \, \boldsymbol{dx}, \quad \forall m \in \mathbb{N}, \end{aligned} \tag{26}$$

for an arbitrary vector-valued function $\boldsymbol{\psi} \in \mathcal{Y}(\Omega)$. Moreover, the energy inequality

$$\int_{\Omega} \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|) |\boldsymbol{D}(\boldsymbol{y}_m)|^2 \, \boldsymbol{dx} + \beta \int_{\Gamma} |\boldsymbol{y}_m|^2 \, d\Gamma \leq \int_{\Omega} \boldsymbol{\pi u}_m \cdot \boldsymbol{y}_m \, \boldsymbol{dx}, \quad \forall m \in \mathbb{N}, \tag{27}$$

holds.

In view of (24) and condition (C3), we have

$$\boldsymbol{D}(\boldsymbol{y}_m) \rightharpoonup \boldsymbol{D}(\boldsymbol{y}_0) \text{ in } \mathbb{L}^2(\Omega) \text{ as } m \to \infty, \tag{28}$$

$$\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m) \to \boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0) \text{ in } \mathbb{L}^2(\Omega) \text{ as } m \to \infty. \tag{29}$$

Next, using the Krasnoselskii theorem on the continuity of a superposition operator in Lebesgue spaces (see, e.g., [21]) and condition (C2), we derive from (29) that

$$\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|) \to \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|) \text{ in } L^2(\Omega) \text{ as } m \to \infty, \tag{30}$$

$$\sqrt{\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|)} \to \sqrt{\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|)} \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{31}$$

Moreover, since $\boldsymbol{Y}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^4(\Omega)$ and the trace operator $\gamma_0 \colon \boldsymbol{Y}^1(\Omega) \to \boldsymbol{L}^2(\Gamma)$ is compact, we see that

$$\boldsymbol{y}_m \to \boldsymbol{y}_0 \text{ in } \boldsymbol{L}^4(\Omega) \text{ as } m \to \infty, \tag{32}$$

$$\boldsymbol{y}_m|_\Gamma \to \boldsymbol{y}_0|_\Gamma \text{ in } \boldsymbol{L}^2(\Gamma) \text{ as } m \to \infty. \tag{33}$$

Taking into account (25), (28), (30), (32) and (33), we can pass to the limit $m \to \infty$ in (26); this gives

$$-\sum_{i=1}^{n} \int_\Omega y_{0i} \boldsymbol{y}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_i}\, \boldsymbol{dx} + \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|)\boldsymbol{D}(\boldsymbol{y}_0) : \boldsymbol{D}(\boldsymbol{\psi})\, \boldsymbol{dx} \\ - \alpha \sum_{i=1}^{n} \int_\Omega y_{0i} \boldsymbol{D}(\boldsymbol{y}_0) : \frac{\partial \boldsymbol{D}(\boldsymbol{\psi})}{\partial x_i}\, \boldsymbol{dx} + \beta \int_\Gamma \boldsymbol{y}_0 \cdot \boldsymbol{\psi}\, d\Gamma = \int_\Omega \boldsymbol{\pi u}_0 \cdot \boldsymbol{\psi}\, \boldsymbol{dx}.$$

Since the set $\mathcal{Y}(\Omega)$ is dense in the space $\boldsymbol{Y}^3(\Omega)$, the last equality remains valid if we replace $\boldsymbol{\psi}$ with arbitrary vector function $\boldsymbol{\varphi} \in \boldsymbol{Y}^3(\Omega)$:

$$-\sum_{i=1}^{n} \int_\Omega y_{0i} \boldsymbol{y}_0 \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\, \boldsymbol{dx} + \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|)\boldsymbol{D}(\boldsymbol{y}_0) : \boldsymbol{D}(\boldsymbol{\varphi})\, \boldsymbol{dx} \\ - \alpha \sum_{i=1}^{n} \int_\Omega y_{0i} \boldsymbol{D}(\boldsymbol{y}_0) : \frac{\partial \boldsymbol{D}(\boldsymbol{\varphi})}{\partial x_i}\, \boldsymbol{dx} + \beta \int_\Gamma \boldsymbol{y}_0 \cdot \boldsymbol{\varphi}\, d\Gamma = \int_\Omega \boldsymbol{\pi u}_0 \cdot \boldsymbol{\varphi}\, \boldsymbol{dx}. \tag{34}$$

Moreover, by (28) and (31), we obtain

$$\sqrt{\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|)}\boldsymbol{D}(\boldsymbol{y}_m) \rightharpoonup \sqrt{\mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|)}\boldsymbol{D}(\boldsymbol{y}_0) \text{ in } \mathbb{L}^2(\Omega),$$

which, together with (27), (32) and (33), implies

$$\int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_0)|)|\boldsymbol{D}(\boldsymbol{y}_0)|^2\, \boldsymbol{dx} \le \liminf_{m\to\infty} \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}_m)|)|\boldsymbol{D}(\boldsymbol{y}_m)|^2\, \boldsymbol{dx} \\ \le \liminf_{m\to\infty} \int_\Omega \boldsymbol{\pi u}_m \cdot \boldsymbol{y}_m\, \boldsymbol{dx} - \beta \int_\Gamma |\boldsymbol{y}_m|^2\, d\Gamma \\ = \int_\Omega \boldsymbol{\pi u}_0 \cdot \boldsymbol{y}_0\, \boldsymbol{dx} - \beta \int_\Gamma |\boldsymbol{y}_0|^2\, d\Gamma. \tag{35}$$

Since $\{\boldsymbol{u}_m\}_{m=1}^\infty \subset \mathcal{U}$ and the set $\mathcal{U}$ is sequentially weakly closed in $\boldsymbol{L}^2(\Omega)$, from (25) it follows that

$$\boldsymbol{u}_0 \in \mathcal{U}. \tag{36}$$

Thus, taking into account relations (34)–(36), we see that the pair $(\boldsymbol{y}_0, \boldsymbol{u}_0)$ is a weak solution of (4)–(7), and hence $(\boldsymbol{y}_0, \boldsymbol{u}_0) \in \mathfrak{M}_w(\mathcal{U})$.

Step 3. We shall show that the set $\mathfrak{M}_w(\mathcal{U})$ is bounded in the space $\boldsymbol{Y}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$.

Let us consider the projection $\mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}$ defined by

$$\mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)} : \boldsymbol{Y}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \to \boldsymbol{Y}^1(\Omega), \qquad \mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}(\boldsymbol{y}, \boldsymbol{u}) \stackrel{\text{def}}{=} \boldsymbol{y}.$$

Since the set $\mathcal{U}$ is bounded in $\boldsymbol{L}^2(\Omega)$, we must only prove that the set $\mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\mathcal{U})$ is bounded in $\boldsymbol{Y}^1(\Omega)$.

From estimate (20) it follows that

$$\int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y})|)|\boldsymbol{D}(\boldsymbol{y})|^2\, \boldsymbol{dx} + \beta \int_\Gamma |\boldsymbol{y}|^2\, d\Gamma \le \sup_{\boldsymbol{u}\in\mathcal{U}} \left| \int_\Omega \boldsymbol{\pi u} \cdot \boldsymbol{y}\, \boldsymbol{dx} \right| \\ \le \|\boldsymbol{\pi}\|_{\mathcal{L}(\boldsymbol{L}^2(\Omega), \boldsymbol{L}^2(\Omega))} \sup_{\boldsymbol{u}\in\mathcal{U}} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)}, \tag{37}$$

for any $\boldsymbol{y} \in \mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\mathcal{U})$. Moreover, by condition (C2), we get

$$\min\{\mu_0, \beta\}\|\boldsymbol{y}\|^2_{\boldsymbol{Y}^1(\Omega)} \le \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y})|)|\boldsymbol{D}(\boldsymbol{y})|^2\, \boldsymbol{dx} + \beta\int_\Gamma |\boldsymbol{y}|^2\, d\Gamma. \tag{38}$$

By comparing (37) and (38), we find

$$\min\{\mu_0, \beta\}\|\boldsymbol{y}\|^2_{\boldsymbol{Y}^1(\Omega)} \le \|\boldsymbol{\pi}\|_{\mathcal{L}(\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega))} \sup_{\boldsymbol{u}\in\mathcal{U}} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)}. \tag{39}$$

Let $\boldsymbol{I}$ be the embedding operator of $\boldsymbol{Y}^1(\Omega)$ into $\boldsymbol{L}^2(\Omega)$. Then, we have

$$\|\boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{I}\|_{\mathcal{L}(\boldsymbol{Y}^1(\Omega),\boldsymbol{L}^2(\Omega))}\|\boldsymbol{y}\|_{\boldsymbol{Y}^1(\Omega)}. \tag{40}$$

From relations (39) and (40) we derive the follwing estimate

$$\|\boldsymbol{y}\|_{\boldsymbol{Y}^1(\Omega)} \le \max\{\mu_0^{-1}, \beta^{-1}\}\|\boldsymbol{\pi}\|_{\mathcal{L}(\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega))}\|\boldsymbol{I}\|_{\mathcal{L}(\boldsymbol{Y}^1(\Omega),\boldsymbol{L}^2(\Omega))} \sup_{\boldsymbol{u}\in\mathcal{U}} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}. \tag{41}$$

Since $\boldsymbol{y}$ is an arbitrary element of the set $\mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\mathcal{U})$, estimate (41) means that this set is bounded in the space $\boldsymbol{Y}^1(\Omega)$.

Step 4. Taking into account the results of Steps 1–3, we can apply the generalized Weierstrass theorem (see [31], Section 38.3) and conclude that there exists a pair $(\boldsymbol{y}_*, \boldsymbol{u}_*) \in \mathfrak{M}_w(\mathcal{U})$ such that (21) holds.

The assertion (ii) is an immediate consequence of Lemma 2(b).

Now we shall prove (iii). In view of the following equality

$$\boldsymbol{\pi}(\boldsymbol{\pi w}) = \boldsymbol{\pi w}, \quad \forall \boldsymbol{w} \in \boldsymbol{L}^2(\Omega),$$

it is easily shown that

$$\mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\mathcal{U}) = \mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\boldsymbol{\pi}\mathcal{U}).$$

Therefore, we have

$$\begin{aligned}
\Phi_{\mathcal{J}}(\mathcal{U}) &= \inf\{\mathcal{J}(\boldsymbol{y}) : \ (\boldsymbol{y}, \boldsymbol{u}) \in \mathfrak{M}_w(\mathcal{U})\} \\
&= \inf\{\mathcal{J}(\boldsymbol{y}) : \ \boldsymbol{y} \in \mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\mathcal{U})\} \\
&= \inf\{\mathcal{J}(\boldsymbol{y}) : \ \boldsymbol{y} \in \mathbf{Pr}_{\boldsymbol{Y}^1(\Omega)}\mathfrak{M}_w(\boldsymbol{\pi}\mathcal{U})\} \\
&= \inf\{\mathcal{J}(\boldsymbol{y}) : \ (\boldsymbol{y}, \boldsymbol{u}) \in \mathfrak{M}_w(\boldsymbol{\pi}\mathcal{U})\} \\
&= \Phi_{\mathcal{J}}(\boldsymbol{\pi}\mathcal{U}).
\end{aligned}$$

We turn to proving statement (iv). Assume the converse. Then, there exists a subsequence $\{k_\ell\}_{\ell=1}^\infty$ such that

$$\lim_{\ell\to\infty} \Phi_{\mathcal{J}}(\mathcal{U}_{k_\ell}) < \Phi_{\mathcal{J}}(\mathcal{U}_0). \tag{42}$$

Consider a sequence $\{(\boldsymbol{y}^*_{k_\ell}, \boldsymbol{u}^*_{k_\ell})\}_{\ell=1}^\infty$ such that $(\boldsymbol{y}^*_{k_\ell}, \boldsymbol{u}^*_{k_\ell}) \in \mathfrak{M}_w^{\mathrm{opt}}(\mathcal{U}_{k_\ell})$, for each $\ell \in \mathbb{N}$. Directly from Definition 3 it follows that

$$\begin{aligned}
&-\sum_{i=1}^n \int_\Omega y^*_{k_\ell i}\boldsymbol{y}^*_{k_\ell} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\, \boldsymbol{dx} + \int_\Omega \mu(|\boldsymbol{\mathcal{P}D}(\boldsymbol{y}^*_{k_\ell})|)\boldsymbol{D}(\boldsymbol{y}^*_{k_\ell}) : \boldsymbol{D}(\boldsymbol{\varphi})\, \boldsymbol{dx} \\
&\qquad - \alpha\sum_{i=1}^n \int_\Omega y^*_{k_\ell i}\boldsymbol{D}(\boldsymbol{y}^*_{k_\ell}) : \frac{\partial \boldsymbol{D}(\boldsymbol{\varphi})}{\partial x_i}\, \boldsymbol{dx} + \beta\int_\Gamma \boldsymbol{y}^*_{k_\ell} \cdot \boldsymbol{\varphi}\, d\Gamma = \int_\Omega \boldsymbol{u}^*_{k_\ell} \cdot \boldsymbol{\varphi}\, \boldsymbol{dx}, \\
&\qquad\qquad \forall \boldsymbol{\varphi} \in \boldsymbol{Y}^3(\Omega),\ \ell \in \mathbb{N},
\end{aligned} \tag{43}$$

$$\int_\Omega \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y}^*_{k_\ell})|)|\boldsymbol{D}(\boldsymbol{y}^*_{k_\ell})|^2\,\boldsymbol{dx} + \beta\int_\Gamma |\boldsymbol{y}^*_{k_\ell}|^2\,d\Gamma \le \int_\Omega \boldsymbol{u}^*_{k_\ell}\cdot\boldsymbol{y}^*_{k_\ell}\,\boldsymbol{dx}, \quad \forall \ell\in\mathbb{N}. \tag{44}$$

Moreover, in accordance with definition of the marginal function $\Phi_{\mathcal{J}}$, we have

$$\Phi_{\mathcal{J}}(\mathcal{U}_{k_\ell}) = \mathcal{J}(\boldsymbol{y}^*_{k_\ell}, \boldsymbol{u}^*_{k_\ell}), \quad \forall \ell\in\mathbb{N}. \tag{45}$$

Let us show that the norms $\|\boldsymbol{y}^*_{k_\ell}\|_{\boldsymbol{Y}^1(\Omega)}$ and $\|\boldsymbol{u}^*_{k_\ell}\|_{\boldsymbol{L}^2(\Omega)}$ are uniformly bounded with respect to $\ell\in\mathbb{N}$. Using the triangle inequality, it is easily shown that

$$\sup_{\boldsymbol{u}\in\mathcal{U}_{k_\ell}} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \sup_{\boldsymbol{u}\in\mathcal{U}_0} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \varrho_{k_\ell}, \tag{46}$$

where

$$\varrho_{k_\ell} \overset{\text{def}}{=} d_{\boldsymbol{L}^2(\Omega)}(\mathcal{U}_{k_\ell}, \mathcal{U}_0).$$

From (22) it follows that the sequence $\{\varrho_{k_\ell}\}_{\ell=1}^\infty$ is convergent, and hence this sequence is bounded. Denoting by ϱ_* the supremum of the set $\{\varrho_{k_\ell} :\ \ell\in\mathbb{N}\}$, we derive from (46) the following estimate

$$\sup_{\ell\in\mathbb{N}} \|\boldsymbol{u}^*_{k_\ell}\|_{\boldsymbol{L}^2(\Omega)} \le \sup_{\boldsymbol{u}\in\mathcal{U}_0} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \varrho_*. \tag{47}$$

Next, setting $\boldsymbol{y} = \boldsymbol{y}^*_{k_\ell}$ and $\mathcal{U} = \mathcal{U}_{k_\ell}$ into (41), we obtain the inequality

$$\|\boldsymbol{y}^*_{k_\ell}\|_{\boldsymbol{Y}^1(\Omega)} \le \max\{\mu_0^{-1}, \beta^{-1}\}\|\boldsymbol{\pi}\|_{\mathcal{L}(\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega))}\|\boldsymbol{I}\|_{\mathcal{L}(\boldsymbol{Y}^1(\Omega),\boldsymbol{L}^2(\Omega))} \sup_{\boldsymbol{u}\in\mathcal{U}_{k_\ell}} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)},$$

which, together with (46), yields the estimate

$$\begin{aligned}\sup_{\ell\in\mathbb{N}} \|\boldsymbol{y}^*_{k_\ell}\|_{\boldsymbol{Y}^1(\Omega)} \le{} & \max\{\mu_0^{-1}, \beta^{-1}\}\|\boldsymbol{\pi}\|_{\mathcal{L}(\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega))}\|\boldsymbol{I}\|_{\mathcal{L}(\boldsymbol{Y}^1(\Omega),\boldsymbol{L}^2(\Omega))} \\ & \times \Big(\sup_{\boldsymbol{u}\in\mathcal{U}_0} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \varrho_*\Big).\end{aligned} \tag{48}$$

Since the set $\mathcal{U}_0$ is bounded in $\boldsymbol{L}^2(\Omega)$, from (47) and (48) it follows that the set $\{(\boldsymbol{y}^*_{k_\ell}, \boldsymbol{u}^*_{k_\ell})\}_{\ell=1}^\infty$ is bounded in $\boldsymbol{Y}^1(\Omega)\times\boldsymbol{L}^2(\Omega)$. Therefore, without loss of generality, it can be assumed that

$$\boldsymbol{y}^*_{k_\ell} \rightharpoonup \boldsymbol{y}^* \ \text{ in } \boldsymbol{Y}^1(\Omega) \text{ as } \ell\to\infty, \tag{49}$$

$$\boldsymbol{u}^*_{k_\ell} \rightharpoonup \boldsymbol{u}^* \ \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \ell\to\infty, \tag{50}$$

for some pair $(\boldsymbol{y}^*, \boldsymbol{u}^*) \in \boldsymbol{Y}^1(\Omega)\times\boldsymbol{L}^2(\Omega)$. Then, arguing as in Step 2 of the proof of assertion (i), we can pass to the limit $\ell\to\infty$ in relations (43) and (44); this gives

$$\begin{aligned}-\sum_{i=1}^n \int_\Omega y_i^*\boldsymbol{y}^*\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\boldsymbol{dx} + \int_\Omega \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y}^*)|)\boldsymbol{D}(\boldsymbol{y}^*):\boldsymbol{D}(\boldsymbol{\varphi})\,\boldsymbol{dx} \\ -\alpha\sum_{i=1}^n \int_\Omega y_i^*\boldsymbol{D}(\boldsymbol{y}^*):\frac{\partial\boldsymbol{D}(\boldsymbol{\varphi})}{\partial x_i}\,\boldsymbol{dx} + \beta\int_\Gamma \boldsymbol{y}^*\cdot\boldsymbol{\varphi}\,d\Gamma = \int_\Omega \boldsymbol{u}^*\cdot\boldsymbol{\varphi}\,\boldsymbol{dx},\end{aligned}$$

$$\int_\Omega \mu(|\boldsymbol{\mathcal{P}}\boldsymbol{D}(\boldsymbol{y}^*)|)|\boldsymbol{D}(\boldsymbol{y}^*)|^2\,\boldsymbol{dx} + \beta\int_\Gamma |\boldsymbol{y}^*|^2\,d\Gamma \le \int_\Omega \boldsymbol{u}^*\cdot\boldsymbol{y}^*\,\boldsymbol{dx}.$$

Next, note that for any $\ell\in\mathbb{N}$, there exists a vector function $\widehat{\boldsymbol{u}}_\ell\in\mathcal{U}_0$ such that

$$\|\boldsymbol{u}^*_{k_\ell} - \widehat{\boldsymbol{u}}_\ell\|_{\boldsymbol{L}^2(\Omega)} \le \varrho_{k_\ell}.$$

In view of (22), we have $\lim_{\ell\to\infty}\varrho_{k_\ell}=0$, and hence

$$\lim_{\ell\to\infty}\|\boldsymbol{u}^*_{k_\ell}-\widehat{\boldsymbol{u}}_\ell\|_{L^2(\Omega)}=0.$$

This equality, together with (50), yields that

$$\widehat{\boldsymbol{u}}_\ell \rightharpoonup \boldsymbol{u}^* \ \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \ell\to\infty. \tag{51}$$

Since $\{\widehat{\boldsymbol{u}}_\ell\}_{\ell=1}^\infty\in\mathcal{U}_0$ and the set $\mathcal{U}_0$ is sequentially weakly closed in the space $\boldsymbol{L}^2(\Omega)$, we derive from (51) the inclusion $\boldsymbol{u}^*\in\mathcal{U}_0$.

Thus, we have established that the pair $(\boldsymbol{y}^*,\boldsymbol{u}^*)$ belongs to the set $\mathfrak{M}_w(\mathcal{U}_0)$. This is one of the key points in proving statement (iv). Indeed, using (45), (49), (50), and condition (C5), we obtain

$$\begin{aligned}\Phi_{\mathcal{J}}(\mathcal{U}_0)&=\inf\{\mathcal{J}(\boldsymbol{y},\boldsymbol{u}):\ (\boldsymbol{y},\boldsymbol{u})\in\mathfrak{M}_w(\mathcal{U}_0)\}\\&\le\mathcal{J}(\boldsymbol{y}^*,\boldsymbol{u}^*)\\&\le\liminf_{\ell\to\infty}\mathcal{J}(\boldsymbol{y}^*_{k_\ell},\boldsymbol{u}^*_{k_\ell})\\&=\lim_{\ell\to\infty}\Phi_{\mathcal{J}}(\mathcal{U}_{k_\ell}),\end{aligned}$$

that contradicts inequality (42). This contradiction concludes the proof.

6. Conclusions

In this work, we initiated the mathematical study of nonlocal models of fluid dynamics by using methods of nonlinear functional analysis. For the motion equations of a differential type fluid of complexity 2 with variable viscosity, we proved the existence of weak solutions that minimize a given cost functional subject to natural assumptions on the model data and the set of admissible controls. For this control system, we also proposed the concept of the marginal function and established that this function is lower semicontinuous with respect to the directed Hausdorff distance. This means that the optimal control is stable in the following sense: it is impossible to achieve a large improvement of the optimal value for the cost functional by small changes in the set of admissible controls.

Author Contributions: Conceptualization, E.S.B.; methodology, E.S.B.; writing—original draft preparation, E.S.B. and M.A.A.; writing—review and editing, M.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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