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Conventional Partial and Complete Solutions of the Fundamental Equations of Fluid Mechanics in the Problem of Periodic Internal Waves with Accompanying Ligaments Generation

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Abstract: The problem of generating beams of periodic internal waves in a viscous, exponentially stratified fluid by a band oscillating along an inclined plane is considered by the methods of the theory of singular perturbations in the linear and weakly nonlinear approximations. The complete solution to the linear problem, which satisfies the boundary conditions on the emitting surface, is constructed taking into account the previously proposed classification of flow structural components described by complete solutions of the linearized system of fundamental equations without involving additional force or mass sources. Analyses includes all components satisfying the dispersion relation that are periodic waves and thin accompanying ligaments, the transverse scale of which is determined by the kinematic viscosity and the buoyancy frequency. Ligaments are located both near the emitting surface and in the bulk of the liquid in the form of wave beam envelopes. Calculations show that in a nonlinear description of all components, both waves and ligaments interact directly with each other in all combinations: waves-waves, waves-ligaments, and ligaments-ligaments. Direct interactions of the components that generate new harmonics of internal waves occur despite the differences in their scales. Additionally, the problem of generating internal waves by a rapidly bi-harmonically oscillating vertical band is considered. If the difference in the frequencies of the spectral components of the band movement is less than the buoyancy frequency, the nonlinear interacting ligaments generate periodic waves as well. The estimates made show that the amplitudes of such waves are large enough to be observed under laboratory conditions.

Keywords: stratification; viscosity; internal waves; ligaments; generation; non-linear interaction

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1. Introduction

Stable interest in the systematic scientific study of internal gravity waves (IGW), which are hidden relatively slow periodic flows inside the stratified atmosphere and ocean, was only formed in the second half of the last century in connection with the general development of Earth sciences. However, separate observations of this phenomenon were described earlier. Fluctuations of the water/oil interface in a ship lighting lamp were observed by B. Franklin in the second half of the XVIII century [1]. Descriptions of the "dead water" effects, which manifest in the strong impact of relatively weak density stratification on the motion of sailing or rowing ships, were given even by ancient authors [2].

Rayleigh performed the first theoretical study of IGW in a continuously stratified fluid [3]. In the conventional wave approach, he determined the limiting frequency and period of propagating waves, i.e., frequency N and period of buoyancy T_b in modern notation. In the atmosphere and ocean, the buoyancy period is within the range of $T_b \sim 5 \div 10$ min or more, while in laboratory conditions, $T_b \sim 5 \div 10$ s. Nansen, who encountered navigational problems in the Norwegian fjords and the Kara Sea during the famous polar expedition of 1893–1896 years with his ship "Fram", stimulated the development of laboratory investigations of the stratification impact on the sailing ships and boats. On his request, V. Ekman

performed productive laboratory experiments on the generation of internal waves by a model of a ship and definition of the internal wave drag [2].

The development of aeronautics and gliding showed that IGW plays an important role in the dynamics of a stratified atmosphere, where they manifest in the vertical oscillations of meteorological balloons [4] and boundaries of the spectra of variations in physical quantities [5].

The results of the analysis of the group velocities of internal waves, firstly performed in the ideal fluid approximation [6], had a great impact on the theory of internal waves in a stratified fluid and the planning of experiments. The technique of group velocity for a wave with dispersion is widely used in different branches of physics, in particular, to study the propagation of elastic waves in three-dimensional periodic lattices [7]. Calculations of the fundamental mode and first harmonics of two-dimensional internal waves generated by an oscillating surface (a magic carpet) in an ideal fluid were performed by the perturbation theory method presented in [8]. The statement of the problem does not contain the potential to define fine flow components related to velocity dissipation in the theory [8] and low spatial resolution of the optic instrument has no room to visualize fine interfaces formed in the flow.

Moreover, the constructed solutions for the equations of IGW by Lighthill's method [6] contained singularities at critical angles, when the direction of wave propagation θ, and the inclination of the boundaries φ, coincide (hereinafter and below, θ and φ are angles to the horizon). The experiments did not reveal singularities at critical angles and stimulated the search for mechanisms for their elimination in theory taking into account the smoothing effect of dissipative factors that are viscosity, thermal conductivity, and diffusion. The dissipative effects are described by special terms and additional equations in basic systems of fundamental equations, which include the closing state equations, as well [9–11]. The internal wave equations [6] were derived as a simplified form of the systems of fundamental governing equations [9–11] with additional approximations (condition of incompressibility, Boussinesq simplification, reduction of the problem space from 3D to 2D, and even 1D cases).

Initially and in many subsequent publications, the viscosity effect was taken into account by the phenomenological introduction of attenuation for each wavenumber $\mathbf{k}$ in the wave packet [12]. A detailed description of modern methods for calculating waves was given by Lighthill [6], preserving in the calculations only parts of the complete solutions of the equations of motion or equivalent wave equations. Modern analysis of the IGW generation problems with an explanation of the elegant calculation technique was recently presented in [13]. Computation of the mean flow generated close to an undulating horizontal wall that emits internal waves in a viscous, linearly stratified two-dimensional Boussinesq fluid performed in [14].

The solution to the 3D linearized equations was used to derive an analytic expression for the mean vertical vorticity production term, which induces a horizontal mean flow. The solution describes the mean flow generation associated with viscous beam attenuation and also a peculiar inviscid mean flow in the vicinity of the oscillating wall, resulting from line vortices at the lateral edges of the oscillating boundary [15].

The efficiency of the wave generator estimated on data of careful measurement of the wave amplitude based upon group velocity arguments, taking into account the effect of vertical confinement to induce resonance, was discussed in [16]. In the vicinity of resonance, the wave fields undergo a transition to nonlinear behavior that is initiated at the central axis of the domain and proceeds to erode the wave field throughout the domain.

A discussion of the two main mechanisms of instability for periodic internal wave beams in stratified fluids with a constant buoyancy frequency, which is the triadic resonant instability generating two secondary wave beams and the streaming instability corresponding to the spontaneous generation of a mean flow, was analyzed in [17]. Calculation of internal wave generation in the two-layer fluid was analyzed in [18,19], where only wave fields are calculated without accompanying fine flow components.

In the conventional technique of internal wave calculations [12–17], a part of the linearized wave equation solutions and the resulting dispersion relation, respectively, were discarded. An explanation of such a reduction is usually omitted. One of the rare exceptions is the well-known tractate [9], where the wave frequency ω is assumed to be a complex quantity when considering the problem of propagation of surface gravitational waves. Then, based on the analysis of the wave attenuation condition at infinity, some of the solutions were discarded. Explanation of the rules for the selection of real or complex quantities for describing the frequency of waves in [9] was not given. Alternatively, the frequency ω, as a measure of the wave energy, can be defined as a real positive parameter $\omega \geq 0$ of a wave motion. Physically, it is more natural to consider the complex wavenumber $\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$, the imaginary part of which $\mathbf{k}_2$ characterizes the spatial attenuation and restructuring of the wave beam during propagation or reflection from a solid plane [20,21].

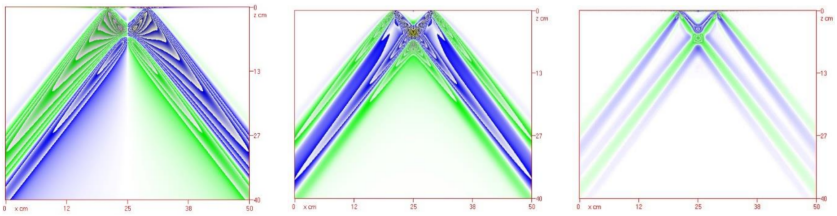
Another independent approach was developed for the description of periodic flows in a viscous stratified fluid based on the construction of complete solutions of the linearized system of governing equations. The technique of IGW and accompanied fine components calculation is based on the analysis of an extremely reduced form of the system of fundamental fluid mechanics equations [9–11], where only the terms, which take into account the effects of stratification and viscosity, were conserved. The impact of compressibility, thermal diffusivity, and diffusion effects [22] was neglected.

In the low-viscosity approximation, the theory of singular perturbations [23] gives room to calculate complete solutions for internal wave equations taking into account the condition of compatibility. Firstly, the problem of periodic internal wave generation by a band oscillating along with a sloping plane in a viscous fluid with a constant buoyancy period was studied. Analysis of the solutions of the 2D linearized system of governing equations showed that regularly perturbed functions describe internal wave beams and singularly perturbed part of the complete solution characterize ligaments, which are extended but thin in transverse direction flow components [24]. The calculated geometry and amplitude distributions in the internal wave beam are in satisfactory agreement with the schlieren visualization data on the flow pattern and measurements of salinity variations by a contact sensor [25].

The developed technique gives room to construct complete solutions in the entire range of problem parameters, including the vertical $\varphi = \frac{\pi}{2}$, horizontal $\varphi = 0$, and critical (along with the wave beam) angle $\varphi = \theta$ inclination of the emitting surface. The problem of the periodic IGW generation by a moving band or circle along with a plane was solved both in 2D and 3D formulations [26,27].

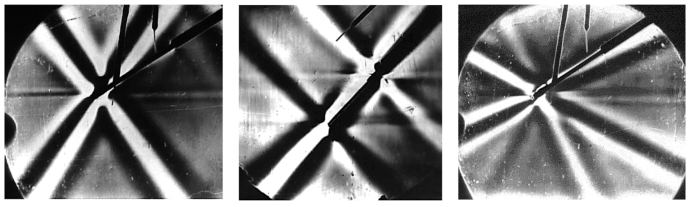
A detailed analysis of the solution for the flow created by a vertically oscillating horizontal disk showed that besides the conic wave beam, a family of ligaments is formed. They are placed on the emitting surface and propagate with wave beams [27]. Ligaments outline the propagating wave beams and are expressed in fields of the velocity derivatives (Figure 1). They include several independent components with the transverse scale $\delta_N^\nu = \sqrt{\nu/N}$ and $\delta_\omega^\nu = \sqrt{\nu/\omega}$ defined by kinematic viscosity of the fluid ν, buoyancy N, and the body oscillation frequency ω.

Conventional schlieren images of periodic internal waves visualized by high resolving instrument with narrow illuminating slit and Foucault knife represent four wave beams and fine horizontal interfaces near their edges in general case when $\theta \neq \varphi$(Figure 2a) and in the critical case $\theta = \varphi$ (Figure 2b).



(**a**) (**b**) (**c**)

Figure 1. Periodic internal waves beams and a family of ligaments (envelopes of beams and thin flows on the emitting surface), generated by vertical oscillations of a horizontal disk D = 8 cm in diameter with a period of T_o = 6.3 s and a velocity amplitude of U_o = 0.25 cm/s in a liquid with a buoyancy period of T_b = 4.2 s: (**a**–**c**)—fields of the vertical component of velocity v_z, the first $\frac{\partial v_z}{\partial r}$ and second derivatives $\frac{\partial^2 v_z}{\partial r^2}$.

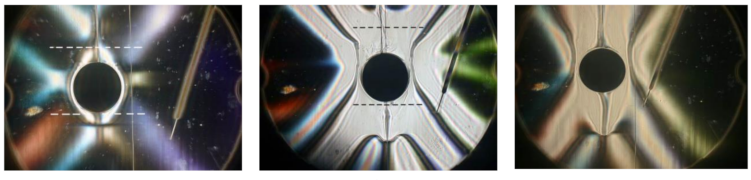


(*a*) (**b**) (**c**)

Figure 2. Schlieren images of the harmonic internal waves beams and ligaments (horizontal interfaces) generated by the sloping band placed under angle $\varphi = 35^\circ$ band oscillating along its plane: (**a**)—period of buoyancy T_b = 5.5 s and band oscillation T = 6.5 s, width a = 1 cm and amplitude of oscillations A = 0.15 cm, beam slope angle $\theta = 57^\circ \neq \varphi$ —general case; (**b**) $\varphi = 45^\circ$, T_b = 5.5 s, T = 7.8 s, a = 6 cm, A = 0.15 cm, $\theta = \varphi = 45^\circ$—critical case: (**c**)—$\varphi = 35^\circ$, T_b = 5.5 s, T = 13 s, a = 1 cm A = 0.15 cm, slope of main beam $\theta_1 = 25^\circ$, second harmonics beam $\theta_2 = 60^\circ$.

When the band oscillation frequency is small and the frequency of harmonics satisfy Raleigh's condition $2\omega < N$ even so gentle a source oscillating along its own surface band generates harmonics of the main mode (Figure 2c).

In a wave field generated by a vertically oscillating sphere, which is a more effective source of internal waves than a band even at small amplitudes of displacements, the schlieren images present periodic internal wave beams together with short vertical and horizontal strips (ligaments) and high-gradient envelopes of the near region (Figure 3a). As the sphere motion amplitude increases, wave beams are bounded by envelopes (dark strips at the boundaries of internal wave beams in Figure 3b). In the case of the large amplitude of the sphere displacements, internal wave beams originate at a thin interface that is bounded to the near zone, including a blocked liquid (Figure 3c), and outlined by a system of ligaments represented by dark strips at the edges of the wave beams.



(**a**) (**b**) (**c**)

Figure 3. Schlieren images of periodic flows induced by oscillating sphere (D = 4.5 cm), (**a**–c)—T_b = 11.2, 7.3, 11.2 s; A = 1, 2.8, 2.8 cm, ω/N = 0.73, 0.8,0.8.

The parameters of ligaments accompanying waves of various types (inertial, internal, acoustic, hybrid) in rotating viscous stratified fluids were calculated in [28].

Accounting for the diffusion effects, which requires the inclusion of an additional equation in the governing system, leads to a correction of the attenuation of internal waves

and an increase in the number of ligaments. Some of them are specific and their thickness reflects the influence of each of the dissipative factors like $\delta_\omega^\nu = \sqrt{\nu/\omega}$ for viscosity or $\delta_\omega^\kappa = \sqrt{\kappa/\omega}$ for diffusivity effects; some are of a mixed nature $\delta_\omega^m = \sqrt[4]{\kappa\nu/\omega^2}$ (κ is the salt diffusion coefficient [29]).

A complete classification of the structural components of periodic flows, taking into account the action of all dissipative factors that are viscosity and diffusivity of temperature and the stratified substance, shows that the number of regular periodic solutions to the fundamental equations system fixed in all formulations, starting with the Euler equation [30]. The number of singular components of the complete solution, which describe thin two-dimensional interfaces and individual 3D fibers, increases with the introduction of the action of each new dissipative factor in the analysis.

As it follows from the compatibility condition, the linearized system of Navier–Stokes equations for both constant and variable density fluids is of the sixth order. Its complete solution includes two regular functions that describe waves and four singularly perturbed functions that define ligaments. Two more singularly perturbed solutions appear with the inclusion of diffusion effects and two extra ones appear when thermal conductivity effects are added. We take into account each new dissipative factor leads to the appearance of a new pair of ligaments, which can be either specific and reflect the independent action of each of the factors, or mixed. In the latter case, their parameters simultaneously depend on several dissipative coefficients. The number of independent functions, which make up the complete solutions of the system of linearized equations, varies from two in the Euler equations to 10 with account for viscosity, thermal diffusivity, and diffusion [30]. Real flows that represent a combination of a number of arbitrary functions, in general with incommensurable spatial and temporal scales, are always unsteady.

For an actually homogeneous incompressible fluid, the density as coefficient can be omitted in all equations, and the set of equations of a fluid motion transforms into a special operator for converting metric space into itself [31]. The velocity and momentum become identical in the approximation $\rho \equiv$ const. In this case, the dispersion equation for harmonic waves in an infinite space has the form for incompressible fluid [30]:

$$\mathbf{k}^2\left(\omega + i\nu\mathbf{k}^2\right)^2 = 0 \tag{1}$$

and for compressible fluid:

$$\left(k^2\left(1 - \frac{i\omega\,\widetilde{\nu}}{c_s^2}\right) - \frac{\omega^2}{c_s^2}\right)\left(\omega + i\nu\,k^2\right)^2 = 0. \tag{2}$$

Here $\widetilde{\nu} = \zeta + 4\nu/3$; ν, ζ are shear (first) and convergence (second) kinematic viscosity, and c_s is the sound velocity. The multiplicity of roots in (1) and (2) indicates the degeneration of the Navier–Stokes equations for a homogeneous compressible and incompressible fluid [30].

The solutions of the Navier–Stokes system of this type characterizing the velocity and pressure fields in actually homogeneous incompressible liquid, do not admit experimental verification with control of accuracy. The matter is that the velocity of a homogeneous liquid is a non-observable parameter. Liquid "Eulerian particles" are not identifiable in a homogeneous medium. Solid markers are not only transported by a vortex or shear flow but also twist around their own axis, which was noted by Descartes [32] and reproduced in recent experiments [33]. Small particles perform a chaotic Brownian motion. The liquid marker, which can be mixed with the carrier liquid, splits into individual fibers in the vortex flow [34]. All these phenomena complicate and even make it impossible to estimate the accuracy of the liquid velocity measurements.

Keeping the density as an independent variable, that is introducing stratification or complete state equation [35], removes the degeneracy of the Navier–Stokes system of governing equations for a homogeneous liquid or a gas. The complete system of fundamental fluid mechanics equations [9–11] with the state equation [35] or in a simpler

form with prescribed density profile becomes correctly formulated and solvable in both 2D and 3D cases in a linear approximation and can be numerically solved in the complete formulation with an estimation of the accuracy.

Since the procedures for analytical construction of complete solutions of the 3D system of governing equations of the inhomogeneous fluid mechanics have not yet been developed, at the first stage it is of interest to study the properties of solutions of wave equations of complete nonlinear systems. In this paper, three problems on generation of 2D periodic waves and accompanying ligaments are considered in a complete nonlinear formulation.

A limited simplified system of nonlinear equations of motion of a viscous fluid is considered, which takes into account the very fact of an inhomogeneous density distribution that creates a stable stratification with a constant buoyancy frequency N. The source of waves in the first two sections is a thin solid band oscillating along the plane. In the first section, the plane is tilted at an arbitrary angle to the horizon, including the critical value, $\varphi = \theta$. In this geometry, two of the four beams of periodic internal waves, which form an oblique "St. Andrew" cross figure, propagate along the plane.

In the second problem, the nonlinear properties of a problem with the edges of the band oscillating with a finite amplitude periodically occupying and releasing part of the space are analyzed.

In the third section, the emission of internal waves by interacting ligaments, which are formed on a vertically oscillating surface, is calculated. In all cases, in contrast with [13–16], all flow components that are large internal waves and thin ligaments are taken into account.

2. A complete Solution to the Problem of Generating 2D Periodic Internal Waves by an Oscillating Tilted Band

The developed approach gives room to solve the problem IGW generation with physically valid initial and boundary conditions both in linear [24–26] and in slightly non-linear approximation without introducing mass or force sources, which are used in conventional theory [6,13]. The method allows determining both a dynamic and fine structure of the flow field.

To construct a complete solution of the wave generation problem analytically, the simplest generator was selected, which is an infinitely thin horizontal band of width a. The band moves along with the infinite plane positioned at an arbitrary angle φ to the horizontal. The geometry of the problem is shown in Figure 4. The band periodically oscillating with frequency ω generates viscous exponentially stratified fluid with vertical density profile $\rho_0 = \rho_{00} \exp(-z/\Lambda)$ and constant buoyancy frequency N periodical perturbations containing internal wave beams and ligaments.

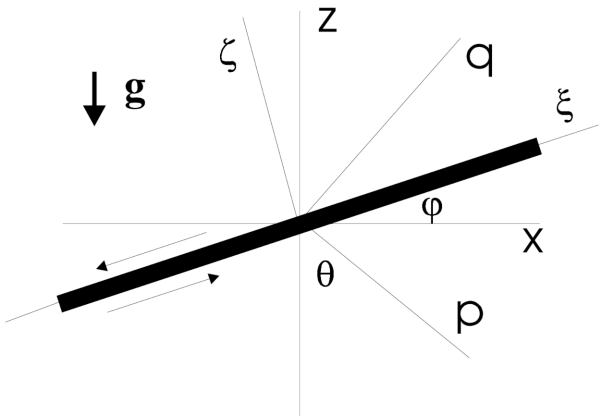


Figure 4. Basic coordinate frames.

The IGW beams propagate under the angle $\theta = \pm\arcsin(\omega/N)$ to the horizon [6]. In an arbitrary case $\varphi \neq \theta$, all beams separate from the emitting plane. The critical case,

$\varphi = \theta$, when two wave beams propagate along with a plane bounding the fluid, is of particular interest for geophysical applications and calls for special consideration.

The simplified system, but still satisfying the solvability condition, in which the unperturbed density profile replaces the equation of state for density, the compressibility, heat conductivity, and diffusion effects are neglected, takes the form [9,30]:

$$\begin{array}{c}(\rho_0+\rho)\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}-\nu\Delta\mathbf{u}\right]=-\nabla p-\rho g\mathbf{e}_z,\\ \frac{\partial \rho}{\partial t}+\mathbf{u}\nabla\rho=0,\nabla\mathbf{u}=0,\rho_0(z)=\rho_{00}\exp(-z/\Lambda).\end{array} \tag{3}$$

Here $\rho(x,\ z,t)$ is the density perturbation; $\mathbf{u}$ is the velocity; p is the pressure, minus the hydrostatic pressure; g is the acceleration of gravity; and $\mathbf{e}_z$ is the unit vector in the upward direction of the vertical axis. The calculations are performed in a laboratory $(x,\ z)$ (the gravity acceleration g is opposite to the z-axis) and the sloping plate $(\xi,\ \zeta)$ coordinate frames, which are shown in Figure 1, as well.

The no-slip conditions for velocity on the radiating surface as well as the damping of all disturbances at infinity are selected as boundary conditions:

$$U_\xi(\xi)=u_0\vartheta(a/2-|\xi|),U_\zeta(\xi)=0, \tag{4}$$

where ϑ is the unit Heaviside function. The solution of system (3) is sought in the form of expansions in plane waves with a real frequency ω and a complex wavenumber $\mathbf{k}$. The general time factor $\exp(-i\omega t)$ is omitted everywhere below.

The system (3) is transformed into the equation for the stream function Ψ ($u_\xi=\partial\Psi/\partial\zeta$, $u_\zeta=-\partial\Psi/\partial\xi$)

$$\left[\omega^2\Delta-N^2\left(\cos]\varphi\frac{\partial}{\partial\xi}-\sin\varphi\frac{\partial}{\partial\zeta}\right)^2-i\omega\nu\Delta^2\right]\Psi(\xi,\zeta)=0, \tag{5}$$

with boundary conditions on the band

$$\left.\frac{\partial\Psi}{\partial\zeta}\right|_{\zeta=0}=U_\xi(\xi),\ \left.\frac{\partial\Psi}{\partial\xi}\right|_{\zeta=0}=-U_\xi(\xi). \tag{6}$$

The solutions of the system (3) are searched for by the additional conditions of continuity for function Ψ and to all its derivatives at transverse coordinate ζ including the third at $\zeta=0$, $|\xi|>a/2$, as well as the condition of damping at infinity $\Psi(\xi,\zeta)\to 0$ at $\xi,\zeta\to\pm\infty$ by the method of integral transformations:

$$\Psi(\xi,\zeta)=\int\limits_{-\infty}^{+\infty}\left(be^{i\kappa_w^+\zeta}+ce^{i\kappa_l^+\zeta}\right)e^{ik\xi}dk,\zeta>0, \tag{7}$$

where $b(k)$ and $c(k)$ are the spectral densities, and the wave numbers $\kappa_w^\pm(k)$ and $\kappa_l^\pm(k)$ correspond to the propagating internal waves and ligaments. Propagating waves, as supplementing ligaments, fill the entire space. To satisfy the damping conditions at infinity, the next branches are chosen $\mathrm{Im}\kappa_l^+>0$, $\mathrm{Im}\kappa_w^-<0$, $\mathrm{Im}\kappa_l^-<0$.

The searched wave numbers are the roots of the dispersion equation:

$$\omega^2\left(\kappa^2+k^2\right)-N^2(\kappa\sin\varphi-k\cos\varphi)^2+i\nu\omega\left(\kappa^2+k^2\right)^2=0. \tag{8}$$

The solution to Equation (8). with the foregoing boundary conditions is constructed by the method of successive approximations. Firstly, a zero approximation solution $\Psi_0^{(1)}$ is constructed with boundary conditions (6) and the additional no-slip conditions $u_\xi=u_\zeta=0$ on the motionless part of the plate at $\zeta=0$, $|\xi|>a/2$. The solution $\Psi_0^{(1)}$ does not satisfy the conditions of continuity for all its derivatives with respect to ζ outside the band.

A correction $\Psi_0^{(2)}$ is added, which satisfies Equation (5) and provides the continuity of the sum $\Psi_0^{(1)} + \Psi_0^{(2)}$ and its derivatives up to the third one for $\zeta = 0$, $|\xi| > a/2$.

The sum obtained $\Psi_0^{(1)} + \Psi_0^{(2)}$, which violates the boundary conditions (6). To satisfy these conditions on the moving band, a function $\Psi_1^{(1)}$ is added to the solution. Adding one more function $\Psi_0^{(2)}$ ensures the continuity of the sum $\Psi_0^{(1)} + \Psi_0^{(2)}$ and its derivatives up to the third one outside the plate. Each of the correction functions satisfies Equation (5).

Unlimited repetition of the iterative procedure leads to the following complete solution:

$$\Psi(\xi,\zeta) = \sum_{n=0}^{\infty} \Psi_n(\xi,\zeta), \Psi_n = \Psi_n^{(1)} + \Psi_n^{(2)}. \tag{9}$$

Substitution of (9) into the boundary conditions of the problem leads to the relations connecting the functions $\Psi_0^{(1)}$, $\Psi_n^{(1,2)}$ and their derivatives;

$$\begin{gathered}
\Psi_n^{(2)}\Big|_{\zeta=+0} = \Psi_n^{(2)}\Big|_{\zeta=-0}, \ \Psi_{n\zeta}^{(2)}\Big|_{\zeta=+0} = \Psi_{n\zeta}^{(2)}\Big|_{\zeta=-0}, \\
\Psi_{n\zeta\zeta}^{(2)}\Big|_{\zeta=+0} - \Psi_{n\zeta\zeta}^{(2)}\Big|_{\zeta=-0} = \vartheta\left(|\xi| - \tfrac{a}{2}\right)\left[\Psi_{n\zeta\zeta}^{(1)}\Big|_{\zeta=-0} - \Psi_{n\zeta\zeta}^{(1)}\Big|_{\zeta=+0}\right], \\
\Psi_{n\zeta\zeta\zeta}^{(2)}\Big|_{\zeta=+0} - \Psi_{n\zeta\zeta\zeta}^{(2)}\Big|_{\zeta=-0} = \vartheta\left(|\xi| - \tfrac{a}{2}\right)\left[\Psi_{n\zeta\zeta\zeta}^{(1)}\Big|_{\zeta=-0} - \Psi_{n\zeta\zeta\zeta}^{(1)}\Big|_{\zeta=+0}\right], \\
\Psi_{n+1}^{(1)}\Big|_{\zeta=\pm 0} = 0, \Psi_{n+1,\zeta}^{(1)}\Big|_{\zeta=\pm 0} = -\vartheta\left(\tfrac{a}{2} - |\xi|\right)\Psi_{n\zeta}^{(2)}\Big|_{\zeta=\pm 0}, \\
\Psi_0^{(1)}\Big|_{\zeta=\pm 0} = 0, \Psi_{0\zeta}^{(1)}\Big|_{\zeta=\pm 0} = u_0\vartheta\left(\tfrac{a}{2} - |\xi|\right),
\end{gathered} \tag{10}$$

where each of the indices ζ at $\Psi_n^{(1,2)}$ denotes differentiation with respect to ζ.

Iterations, $\Psi_n^{(1,2)}$, like the complete solution (7), are sought in the form:

$$\begin{aligned}
\Psi_n^{(1)} &= \int_{-\infty}^{+\infty}\left[\vartheta(\zeta)A_n^+\left(e^{i\kappa_w^+\zeta} - e^{i\kappa_l^+\zeta}\right) + \vartheta(-\zeta)A_n^-\left(e^{i\kappa_w^-\zeta} - e^{i\kappa_l^-\zeta}\right)\right]e^{ik\xi}dk, \\
\Psi_n^{(2)} &= \int_{-\infty}^{+\infty}\left[\vartheta(\zeta)\left(B_n^+e^{i\kappa_w^+\zeta} + C_n^+e^{i\kappa_l^+\zeta}\right) + \vartheta(-\zeta)\left(B_n^-e^{i\kappa_w^-\zeta} + C_n^-e^{i\kappa_l^-\zeta}\right)\right]e^{ik\xi}dk.
\end{aligned} \tag{11}$$

After substitutions of the expressions (11) into relations between iterative functions (10), we receive the next system of equations:

$$\begin{gathered}
\left(\kappa_w^+ - \kappa_l^+\right)A_0^+ = \left(\kappa_w^- - \kappa_l^-\right)A_0^- = -\tfrac{iu_0}{\pi k}\sin\tfrac{ka}{2} \equiv A_0(k), \\
A_{n+1}(k) \equiv \left(\kappa_w^+ - \kappa_l^+\right)A_{n+1}^+ = \left(\kappa_w^- - \kappa_l^-\right)A_{n+1}^- = -\tfrac{1}{\pi}\int_{-\infty}^{+\infty}\tfrac{\kappa_w^+(k')B_n^+(k') + \kappa_l^+(k')C_n^+(k')}{k'-k}\sin\tfrac{(k'-k)a}{2}dk', \\
B_n^+ + C_n^+ - B_n^- - C_n^- = 0, \kappa_w^+B_n^+ + \kappa_l^+C_n^+ - \kappa_w^-B_n^- - \kappa_l^-C_n^- = 0, \kappa_w^{+2}B_n^+ + \kappa_l^{+2}C_n^+ - \kappa_w^{-2}B_n^- - \kappa_l^{-2}C_n^- = -D_1A_n + I_1, \\
\kappa_w^{+3}B_n^+ + \kappa_l^{+3}C_n^+ - \kappa_w^{-3}B_n^- - \kappa_l^{-3}C_n^- = -D_2A_n + I_2,
\end{gathered} \tag{12}$$

where

$$\begin{gathered}
I_j = \tfrac{1}{\pi}\int_{-\infty}^{+\infty}\tfrac{D_j(k')A_n(k')}{k'-k}\sin\tfrac{(k'-k)a}{2}dk', \\
D_1 = \kappa_w^+ + \kappa_l^+ - \kappa_w^- - \kappa_l^-, D_2 = \kappa_w^{+2} + \kappa_w^+\kappa_l^+ + \kappa_l^{+2} - \kappa_w^{-2} + \kappa_w^-\kappa_l^- + \kappa_l^{-2}.
\end{gathered} \tag{13}$$

By solving the system (12), substituting the result into (11) and (9), and comparing it with (7), we find the spectral densities for the wave $b(k)$ and the ligament component $c(k)$ of flows:

$$b(k) = \frac{1}{\pi}\int_{-\infty}^{+\infty}R_w(k,k')g(k')dk', c(k) = \frac{1}{\pi}\int_{-\infty}^{+\infty}R_l(k,k')g(k')dk', \tag{14}$$

where:

$$R_w(k,k') = \frac{D_2(k') + \kappa_w^+ D_1(k')}{(\kappa_w^+ - \kappa_w^-)(\kappa_w^+ - \kappa_l^+)(\kappa_w^+ - \kappa_l^-)}, R_l(k,k') = \frac{D_2(k') + \kappa_l^+ D_1(k')}{(\kappa_l^+ - \kappa_l^-)(\kappa_l^+ - \kappa_w^+)(\kappa_l^+ - \kappa_w^-)}. \quad (15)$$

In (14), all wave numbers $\kappa_w^\pm$ and $\kappa_l^\pm$ are functions of k and do not depend on k'. The universal function $g(k)$ in (14) is a solution of an integral equation:

$$g(k) - \frac{1}{\pi}\int_{-\infty}^{+\infty} R(k,k')g(k')dk' = -\frac{iu_0}{\pi k}\sin\frac{ka}{2}, \quad (16)$$

with kernel:

$$R(k,k') = \frac{1}{k'-k}\sin\frac{(k'-k)a}{2} + \frac{1}{\pi}\int_{-\infty}^{+\infty}\sin\frac{(k''-k)a}{2}\sin\frac{(k''-k')a}{2}\frac{D_3(k'')D_2(k') - D_4(k'')D_1(k')}{(k''-k)(k''-k')D_0(k'')}dk'', \quad (17)$$

where:

$$\begin{gathered} D_0 = (\kappa_w^+ - \kappa_w^-)(\kappa_w^+ - \kappa_l^-)(\kappa_l^+ - \kappa_w^-)(\kappa_l^+ - \kappa_l^-), D_3 = \kappa_w^+\kappa_l^+ - \kappa_w^-\kappa_l^-, \\ D_4 = \kappa_w^+\kappa_l^+(\kappa_w^+ + \kappa_l^+) - \kappa_w^-\kappa_l^-(\kappa_w^- + \kappa_l^-), \end{gathered} \quad (18)$$

and D_1 $И D_2$ were defined in (13). The performed calculations of the stream function at $\zeta < 0$ show that the wave field has central symmetry: $\Psi(-\xi,-\zeta) = \Psi(\xi,\zeta)$.

Since further calculations use approximate solutions of the dispersion Equation (8), the cases of the general position of the band ($\varphi \neq \pm\theta$) and critical angles ($\varphi = \pm\theta$) require separate consideration.

In the general non-degenerated case ($\varphi \neq \pm\theta$), approximate solutions of the dispersion Equation (8) have the form:

$$\begin{gathered} \kappa_w^\pm(k) = \mp|k|\mathrm{ctg}(\theta \mp \lambda\varphi) \pm \frac{i\nu|k|^3}{2N\cos\theta\sin^4(\theta\mp\lambda\varphi)}, \lambda = \operatorname{sign} k, \\ \kappa_l^\pm(k) = \pm k_l - \frac{k\sin\varphi\cos\varphi}{\sin^2\varphi - \sin^2\theta}, k_l = \left[i + sign\left(\sin^2\varphi - \sin^2\theta\right)\right]\sqrt{\frac{N\left|\sin^2\varphi - \sin^2\theta\right|}{2\nu\sin\theta}}. \end{gathered} \quad (19)$$

Substituting (19) into (15) and (17) gives approximate expressions for the kernels R, R_b and R_c:

$$R(k,k') = \frac{2}{\pi k_l \sin 2\theta\left(\sin^2\varphi - \sin^2\theta\right)}\int_{-\infty}^{+\infty}\sin\frac{(k''-k)a}{2}\sin\frac{(k''-k')a}{2}\frac{\operatorname{sign}k''(k''-k') + (|k''| - |k'|)\sin^2 2\varphi}{4(k''-k)(k''-k')}dk'', \quad (20)$$

$$R_b(k,k') = -\frac{1}{k_l}\left[\frac{1}{k'-k} - \frac{\sin 2\varphi}{|k|\sin 2\theta}\right]\sin\frac{(k'-k)a}{2} \quad (21)$$

$$R_c(k,k') = \frac{1}{k_l(k'-k)}\sin\frac{(k'-k)a}{2} \quad (22)$$

In the low-viscosity approximation for $\nu \to 0$, kernel (20) of the integral Equation (16) also tends to zero, and in the first approximation solution, (16) is of the form:

$$g(k) = -\frac{iu_0}{\pi k}\sin\frac{ka}{2}. \quad (23)$$

Substitution (23) with kernels (20) and (21) in the expressions (14) specifies spectral density of internal waves and ligaments

$$b(k) = \frac{iu_0}{\pi k_b}\left[\frac{1}{k} + \frac{1}{2|k|}\frac{\sin 2\varphi}{\sin 2\theta}\right]\sin\frac{ka}{2}, c(k) = -\frac{iu_0}{\pi k_l k}\sin\frac{ka}{2}. \quad (24)$$

In the critical case, $\varphi = \theta$, roots of dispersion Equation (8) have the form:

$$\begin{gathered}\kappa_w^+ = \vartheta(k)\left[\frac{i-\sqrt{3}}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)k\mathrm{ctg}2\theta,\\ \kappa_w^+ = \vartheta(k)\left[\frac{i-\sqrt{3}}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)k\mathrm{ctg}2\theta,\\ \kappa_l^+ = \vartheta(k)\left[\frac{\sqrt{3}+i}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)\left[i\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right],\\ \kappa_l^- = -\vartheta(k)\left[i\gamma + \frac{k\mathrm{ctg}2\theta}{3}\right] - \vartheta(-k)\left[\frac{\sqrt{3}+i}{2}\gamma + \frac{k\mathrm{ctg}2\theta}{3}\right], \gamma = \sqrt[3]{\frac{2|k|N\cos\theta}{\nu}}.\end{gathered} \tag{25}$$

Substitution of (25) in (16) gives:

$$g(k) = -\frac{iu_0 a}{2\pi}F\left(\frac{ka}{2}\right), \tag{26}$$

where the reference function $F(k)$ is the solution of the integral equation:

$$F(x) - \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\frac{\sin(x'-x)}{x'-x} - \frac{1}{3\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{sign}(x'x'')}{(x''-x')(x''-x)}\sqrt[3]{\frac{x'}{x''}}\left(2+\sqrt[3]{\frac{x'}{x''}}\right)\sin(x''-x')\sin(x''-x)dx''\right]F(x')dx' = \\ = \frac{\sin x}{x}. \tag{27}$$

The kernels R_b and R_c are equal to:

$$\begin{gathered}R_b(k,k') = -\frac{i}{3|k|(k'-k)}\sqrt[3]{\frac{\nu k'|k'|}{2N\cos\theta}}\left\{3\vartheta(-k) + \vartheta(k)\left[1-(1+i\sqrt{3})\sqrt[3]{\frac{k}{k'}}\right]\right\}\sin\frac{(k'-k)a}{2},\\ R_c(k,k') = -\frac{i}{3|k|(k'-k)}\sqrt[3]{\frac{\nu k'|k'|}{2N\cos\theta}}\left\{\vartheta(-k)\left[1-2\sqrt[3]{\frac{|k|}{k'}}\right]\right\}\sin\frac{(k'-k)a}{2}.\end{gathered} \tag{28}$$

The Equation (27) for $F(x)$ does not contain parameters of medium or the wave source. They appear in the formula only after transformations (14) with kernels (28), which were applied to the function $g(k)$.

In the critical case, $\varphi = \theta$, the coordinate system (ξ, ζ) becomes the intrinsic reference frame for two beams propagating along the plane of the band oscillation. Taking into account these properties, the expression for the stream function in the region $\xi > a/2$ can be found immediately without solving Equation (27):

$$\Psi = -\frac{i\beta u_0}{\pi^2}\sqrt[3]{\frac{\nu^2 a^2}{2N^2\cos^2\theta}}\int_0^\infty \exp\left[-ik\zeta - \frac{\nu k^3\xi}{2N\cos\theta}\right]dk, \tag{29}$$

with the universal coefficient of the problem:

$$\beta = \int_{-\infty}^{+\infty}\frac{\sin x}{\sqrt[3]{x|x|}}F(x)dx. \tag{30}$$

As follows from (29), the periodic IGW beam propagating along the plate is unimodular in the critical case. The vertical-displacement amplitude on its axis is determined by the formula:

$$h(\xi,0) = \frac{\beta b\sin\theta}{3\pi^2}\Gamma\left(\frac{2}{3}\right)\sqrt[3]{\frac{2a^2}{\xi^2}}, \tag{31}$$

where $b = u_0/\omega$ is the amplitude of the band oscillations. Thus, employing the solutions (25) to the dispersion Equation (8) resolves the difficulty with critical angles in the wave generation:

$$\begin{gathered}\kappa_w^+ = \vartheta(k)\left[\frac{i-\sqrt{3}}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)k\mathrm{ctg}2\theta,\\ \kappa_w^+ = \vartheta(k)\left[\frac{i-\sqrt{3}}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)k\mathrm{ctg}2\theta,\\ \kappa_l^+ = \vartheta(k)\left[\frac{\sqrt{3}+i}{2}\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right] + \vartheta(-k)\left[i\gamma - \frac{k\mathrm{ctg}2\theta}{3}\right],\\ \kappa_l^- = -\vartheta(k)\left[i\gamma + \frac{k\mathrm{ctg}2\theta}{3}\right] - \vartheta(-k)\left[\frac{\sqrt{3}+i}{2}\gamma + \frac{k\mathrm{ctg}2\theta}{3}\right], \gamma = \sqrt[3]{\frac{2|k|N\cos\theta}{\nu}}.\end{gathered} \tag{32}$$

Substitution of (32) in (16) gives:

$$g(k) = -\frac{iu_0 a}{2\pi}F\left(\frac{ka}{2}\right), \tag{33}$$

where the reference function $F(k)$ is the solution of the integral equation:

$$F(x) - \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\frac{\sin(x'-x)}{x'-x} - \frac{1}{3\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{sign}(x'x'')}{(x''-x')(x''-x)}\sqrt[3]{\frac{x'}{x''}}\left(2+\sqrt[3]{\frac{x'}{x''}}\right)\sin(x''-x')\sin(x''-x)dx''\right]F(x')dx' = \frac{\sin x}{x}. \tag{34}$$

The kernels R_b and R_c are equal to:

$$\begin{gathered}R_b(k,k') = -\frac{i}{3|k|(k'-k)}\sqrt[3]{\frac{\nu k'|k'|}{2N\cos\theta}}\left\{3\vartheta(-k) + \vartheta(k)\left[1-(1+i\sqrt{3})\sqrt[3]{\frac{k}{k'}}\right]\right\}\sin\frac{(k'-k)a}{2},\\ R_c(k,k') = -\frac{i}{3|k|(k'-k)}\sqrt[3]{\frac{\nu k'|k'|}{2N\cos\theta}}\left\{\vartheta(-k)\left[1-2\sqrt[3]{\frac{|k|}{k'}}\right]\right\}\sin\frac{(k'-k)a}{2}.\end{gathered} \tag{35}$$

The Equation (34) for $F(x)$ does not contain parameters of medium or the wave source. They appear in the formula only after transformations (14) (with kernels (28), which were applied to the function $g(k)$.

In the critical case, $\varphi = \theta$, the coordinate system (ξ, ζ) becomes the intrinsic reference frame for two beams propagating along the plane of the band oscillation. Taking into account these properties, the expression for the stream function in the region $\xi > a/2$ can be found immediately without solving Equation (27):

$$\Psi = -\frac{i\beta u_0}{\pi^2}\sqrt[3]{\frac{\nu^2 a^2}{2N^2\cos^2\theta}}\int_0^\infty \exp\left[-ik\zeta - \frac{\nu k^3\xi}{2N\cos\theta}\right]dk, \tag{36}$$

with the universal coefficient of the problem:

$$\beta = \int_{-\infty}^{+\infty}\frac{\sin x}{\sqrt[3]{x|x|}}F(x)dx. \tag{37}$$

As follows from (36), the periodic IGW beam propagating along the plate is unimodular in the critical case. The vertical-displacement amplitude on its axis is determined by the formula:

$$h(\xi,0) = \frac{\beta b\sin\theta}{3\pi^2}\Gamma\left(\frac{2}{3}\right)\sqrt[3]{\frac{2a^2}{\xi^2}}, \tag{38}$$

where $b = u_0/\omega$ is the amplitude of the band oscillations. Thus, employing the solutions (32) to the dispersion Equation (8) resolves the difficulty with critical angles in the wave

generation problem as well as in the problem on reflection of internal waves from a rigid surface in a viscous, continuously stratified fluid [21].

In a 3D case, fine flow components, which in contrast to the conventional boundary layer can occupy any domains in the entire space, has a more complex structure. They include components with parameters similar to periodic Stokes flow, whose length scale $\delta_\omega^\nu = \sqrt{\frac{\nu}{\omega}}$ does not depend on the presence of stratification, and transient length scale of ligaments $\delta_N^\nu = \sqrt{\frac{\nu}{N}}$ is specific for a given stratification and geometry [29,30]. The extensions of the ligaments form a fine structure of the medium at the horizon of the band edges, which manifests itself in the form of thin horizontal interfaces in Figure 2. Non-linearly interacting ligaments form thin interfaces of finite length in the domain of the intersecting internal wave from independent sources [36,37] and in the vicinity of a free oscillating sphere on a neutral buoyancy horizon [38]. Nonlinear interaction between ligaments and IGW can serve as an additional mechanism of the internal wave generation.

3. A Non-Linear Model of Periodic Flow Formation by a Band Oscillating Along a Tilted Plate

Linear models of IGW [6,13–16] generated by the periodically oscillating body do not describe a number of important properties of wave fields experimentally observed. The phenomenon of generation of the second and higher harmonics of internal waves, as well as the formation of fine high-gradient interfaces, which were interpreted as ligaments [30] or internal waves of zero frequency (dissipative-gravitational waves [39]) are among them.

When a body oscillates in a fluid, two types of nonlinearity are presented. One type follows directly from the nonlinear terms of the equations of motion. A second type, which can be named by the consequence of the boundary conditions nonlinearity, is associated with the fact that there is a domain near the body edge, in which either part of the body or the fluid is alternately present. In this case, even with monochromatic oscillating of the body, harmonics of the fundamental oscillation have appeared in the wave field.

Just as in the case of linear problems, a convenient source for constructing exact solutions is part of an infinite plane of performing monochromatic oscillations. Here the source of a two-dimensional wave in an exponentially stratified viscous fluid is a horizontal band of width a placed on an infinite plane, inclined at an arbitrary angle φ to the horizon (Figure 4). The band oscillates along the plane with frequency ω_0 and amplitude b. The only non-zero tangential component of the surface velocity in the coordinate system (ξ, ζ) associated with the plane in Figure 4 has the form:

$$u(\xi, t) = \omega_0 b \cos \omega_0 t \vartheta\left(\frac{a}{2} - |\xi - b \sin \omega_0 t|\right), \tag{39}$$

where ϑ is the unit Heaviside function. In the linear problem of generating internal waves by the band on plate, the oscillation amplitude is assumed to be infinitely small, and the second term $b \sin \omega_0 t$ in the argument of the Heaviside function is absent. It represents in (39) the nonlinearity following from the boundary conditions.

The same simplified system (3) with the unperturbed density profile $\rho_0 = \rho_{00} e^{-z/\Lambda}$, which replaces the equation of state for density, was selected for analysis. The no-slip conditions for velocity on the radiating surface and the damping of all disturbances at infinity are selected as boundary conditions:

$$w_\xi(\xi, \zeta = 0, t) = u(\xi, t), w_\zeta(\xi, \zeta = 0, t) = 0 \tag{40}$$

The solution of system (3) in the first order of the perturbation theory is constructed in the form of the following sums for density $\rho = \rho_1 + \rho_2$, fluid velocity $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$,

and pressure $P = P_1 + P_2$. The quantities with the index 1 satisfy the homogeneous linear system:

$$\rho_0\left(\frac{\partial \mathbf{w}_1}{\partial t} - \nu\Delta\mathbf{w}_1\right) = -\nabla P_1 - \rho_1 g\mathbf{e}_z, \frac{\partial \rho_1}{\partial t} + w_{1z}\frac{d\rho_0}{dz} = 0, \ \nabla\mathbf{w}_1 = 0, \tag{41}$$

with boundary conditions on the band (40).

The quantities with index 2 satisfy the inhomogeneous linear system:

$$\rho_0\left(\frac{\partial \mathbf{w}_2}{\partial t} - \nu\Delta\mathbf{w}_2\right) = -\nabla P_2 - \rho_2 g\mathbf{e}_z + \mathbf{f}, \ \frac{\partial \rho_2}{\partial t} + w_{2z}\frac{d\rho_0}{dz} = m\frac{\partial \rho_2}{\partial t} + w_{2z}\frac{d\rho_0}{dz} = m, \tag{42}$$

with zero boundary conditions on the plane.

Here the sources $\mathbf{f}$ and m, which are formed from the nonlinear terms of system (3) by substituting the solutions of system (41), have the form:

$$\mathbf{f} = -\rho_0(\mathbf{w}_1\nabla)\mathbf{w}_1, \ m = -\mathbf{w}\nabla\rho_1. \tag{43}$$

In the expression for $\mathbf{f}$ in (43), the term $-\rho_1\partial\mathbf{w}_1/\partial t$ is omitted, which is possible if the inequality $\lambda << \Lambda$ holds, where λ is the natural wave length of the generated disturbances.

For a stream function Ψ_1 such that $u_{1\xi} = \partial\Psi_1/\partial\zeta$, $u_{1\zeta} = -\partial\Psi_1/\partial\xi$, from Equation (42), we obtain in the Boussinesq approximation:

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right) + N^2\left(\cos\varphi\frac{\partial}{\partial\xi} - \sin\varphi\frac{\partial}{\partial\zeta}\right)^2 - \nu\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)^2\right]\Psi_1 = 0, \tag{44}$$

where N is the buoyancy frequency. Following [24] the solution to Equation (44) with boundary conditions (40) can be represented as

$$\Psi_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(\omega,k)\left[e^{ik_w\zeta} - e^{ik_l\zeta}\right]e^{i(k\xi-\omega t)}dkd\omega, \tag{45}$$

where $k_w(\omega,k)$ and $k_l(\omega,k)$ are the roots of the dispersion equation corresponding to (44 for which the next approximate expressions are valid:

$$\begin{aligned} k_w &= k\mathrm{ctg}(\varphi - \theta\mathrm{sign}k) + \tfrac{i|k|^3}{2N\cos\theta\sin^4(\varphi-\theta\mathrm{sign}k)}, \ |\omega|< N, \\ k_w &= \tfrac{i|\omega k|\sqrt{\omega^2-N^2}-kN^2\sin\varphi\cos\varphi}{\omega^2-N^2\sin^2\varphi}, \ |\omega|> N, \\ k_l &= (i+\mathrm{sign}\widetilde{\omega})\sqrt{\tfrac{\widetilde{\omega}}{2\nu}}, \ \widetilde{\omega} = \tfrac{\omega^2-N^2\sin^2\varphi}{\omega}. \end{aligned} \tag{46}$$

Here, the root k_w describes the wave (or pseudo-wave) that is evanescent at $|\omega|> N$ field, k_l is the fine-structured ligament, and $\theta = \arcsin(\omega/N)$ is the angle of inclination of the beam to the horizontal or wave vector $\mathbf{k}$ to vertical.

The spectral density $A(\omega,k)$ is given by the expression:

$$A(\omega,k) = \frac{iU(\omega,k)}{k_l - k_w}, \tag{47}$$

where:

$$U(\omega,k) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(\xi,t)e^{-i(k\xi-\omega t)}d\xi dt. \tag{48}$$

Substituting (40) into (48) gives:

$$U = \frac{\omega_0}{\pi k^2} \sin\frac{ka}{2} \sum_{n=-\infty}^{+\infty} nJ_n(kb)\delta(\omega - n\omega_0), \tag{49}$$

where J_n is the Bessel functions, and δ is the delta function. The spectrum (49) does not lead to the generation of dissipative-gravity waves [39].

For propagating internal waves with frequencies $n\omega_0 < N$, vertical displacement of particles in each beam are described by the expression:

$$h_1^{(n)}(p_n, q_n) = \frac{2i\sin\theta_n}{\pi k_{bn}\sin(\varphi+\theta_n)} e^{-in\omega_0 t} \int_{-\infty}^{+\infty} \frac{1}{k} \sin\frac{ka'}{2} J_n(kb') \exp\left[ikp_n - \frac{\nu k^3 q_n}{2N\cos\theta_n}\right] dk. \tag{50}$$

Here, $\theta_n = \arcsin(n\omega_0/N)$ is such an angle, which the n-th harmonic beam makes with the horizon, (p_n, q_n) is the coordinate system accompanying the beam with the axis q_n directed along the beam, $a' = a\sin(\varphi+\theta_n)$, $b' = b\sin(\varphi+\theta_n)$, and $k_{ln} = k_l(\omega = n\omega_0)$.

From (50), it follows that if the band oscillates with an infinitesimally small amplitude b, then there is only a beam with a frequency ω_0, the expression for which, when using the asymptotic $J_1(kb') = kb'/2$, coincides with the linear solution [24].

If the amplitude of band oscillations is large enough, then the use of the asymptotic behavior of the Bessel function J_n for large values of the argument allows us to assert that each of the fields (50) is created by four point sources located on the plane at points with coordinates $\xi = -(b+0.5a)$, $-|b-0.5a|$, $|b-0.5a|$, $b+0.5a$. An analysis of the modality of the emitted beams, similar to that carried out in [25], shows that the following situations are possible. If $2b' + a' < L_\nu$, where $L_\nu = \sqrt[3]{g\nu}/N$ is the viscous wave scale, a unimodal beam is excited. In the case $2b' + a' > L_\nu$, the following options are possible: If $b' + a' - |b' - a'| < L_\nu$, the excited beam will be bimodal; if $2|b' - a'| < L_\nu$ it will be three-modal; if $2|b' - a'| > L_\nu$ and $b' + a' - |b' - a'| > L_\nu$ it will be four-modal.

The solution of system (42) expressed through the stream function Ψ_2, $(u_{2\xi} = \partial\Psi_2/\partial\zeta u_{2\zeta} = -\partial\Psi_2/\partial\xi)$, can be written in the form:

$$\Psi_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(\zeta;\Omega,\kappa) e^{i\kappa\xi} e^{-i\Omega t} d\kappa d\Omega, \tag{51}$$

where the function ψ satisfies the equation:

$$\left[\Omega^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right) - N^2\left(\cos\varphi\frac{\partial}{\partial\xi} - \sin\varphi\frac{\partial}{\partial\zeta}\right)^2 - i\nu\Omega\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)^2\right]\psi = -F(\zeta;\Omega,\kappa). \tag{52}$$

Here the function F is obtained by substituting solution (45) into (43) and has the form:

$$F = i\Omega \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(\omega,k)A(\Omega-\omega,\kappa-k)\left[D\left(k_w,\widetilde{k}_w\right) - D\left(k_l,\widetilde{k}_w\right) - D\left(k_w,\widetilde{k}_l\right) + D\left(k_l,\widetilde{k}_l\right)\right] dk d\omega, \tag{53}$$

where $k_{w,l} = k_{w,l}(\omega,k)$, $\widetilde{k}_{w,l} = k_{w,l}(\Omega-\omega,\kappa-k)$, function D is defined by expressing $D(\sigma,\widetilde{\sigma}) = e^{i(\sigma+\widetilde{\sigma})\zeta} D_0(\sigma,\widetilde{\sigma})$ and

$$D_0(\sigma,\widetilde{\sigma}) = [k\widetilde{\sigma} - (\kappa-k)\sigma]\left\{\sigma^2 + k^2 + \frac{N^2}{\omega\Omega}(\sigma\sin\varphi - k\cos\varphi)[(\sigma+\widetilde{\sigma})\sin\varphi - \kappa\cos\varphi]\right\}. \tag{54}$$

The solution to Equation (52) can be represented as a convolution of its right-hand side with the Green's function:

$$\psi(\zeta;\Omega,\kappa) = \int_0^\infty G(\zeta,\zeta';\Omega,\kappa)F(\zeta';\Omega,\kappa)d\zeta'. \tag{55}$$

Green's function, which is the solution to Equation (52), with the right-hand side $-\delta(\zeta-\zeta')$ and boundary conditions $G(0,\zeta';\Omega,\kappa)=\left.\frac{\partial G(\zeta,\zeta';\Omega,\kappa)}{\partial\zeta}\right|_{\zeta=0}=0$ has the form:

$$G=\left[\alpha_w^+e^{i\kappa_w^+(\zeta-\zeta')}+\alpha_l^+e^{i\kappa_l^+(\zeta-\zeta')}\right]\vartheta(\zeta-\zeta')+\left[\alpha_w^-e^{i\kappa_w^-(\zeta-\zeta')}+\alpha_b^-e^{i\kappa_l^-(\zeta-\zeta')}\right]\vartheta(\zeta'-\zeta)+\beta_we^{i\kappa_w^+\zeta}+\beta_le^{i\kappa_l^+\zeta}. \quad (56)$$

where:

$$\begin{gathered}\alpha_w^\pm=\pm\frac{1}{\nu\Omega\left(\kappa_w^\pm-\kappa_w^\mp\right)\left(\kappa_w^\pm-\kappa_l^+\right)\left(\kappa_w^\pm-\kappa_l^-\right)},\ \alpha_l^\pm=\pm\frac{1}{\nu\Omega\left(\kappa_l^\pm-\kappa_l^\mp\right)\left(\kappa_l^\pm-\kappa_w^+\right)\left(\kappa_l^\pm-\kappa_w^-\right)},\\ \beta_w=\frac{e^{-i\kappa_w^-\zeta'}}{\nu\Omega\left(\kappa_w^--\kappa_w^+\right)\left(\kappa_w^--\kappa_l^-\right)\left(\kappa_w^+-\kappa_l^+\right)}+\frac{e^{-i\kappa_l^-\zeta'}}{\nu\Omega\left(\kappa_l^--\kappa_w^+\right)\left(\kappa_l^--\kappa_w^-\right)\left(\kappa_w^+-\kappa_l^+\right)},\\ \beta_l=\frac{e^{-i\kappa_w^-\zeta'}}{\nu\Omega\left(\kappa_w^--\kappa_l^+\right)\left(\kappa_w^--\kappa_l^-\right)\left(\kappa_l^+-\kappa_w^+\right)}+\frac{e^{-i\kappa_b^-\zeta'}}{\nu\Omega\left(\kappa_l^--\kappa_w^-\right)\left(\kappa_l^--\kappa_l^+\right)\left(\kappa_l^+-\kappa_w^+\right)}.\end{gathered} \quad (57)$$

Here, $\kappa_{w,l}^\pm$ are the four roots of the dispersion equation:

$$\Omega^2\left(\widetilde{\kappa}^2+\kappa^2\right)-N^2(\widetilde{\kappa}\sin\varphi-\kappa\cos\varphi)^2+i\nu\Omega\left(\widetilde{\kappa}^2+\kappa^2\right)^2=0, \quad (58)$$

resolved relative to $\widetilde{\kappa}$. The perturbations decay at infinity when the inequalities $\mathrm{Im}\kappa_{w,l}^+>0$, $\mathrm{Im}\kappa_{w,l}^-<0$ are valid.

Substituting (53) and (56) with (57) into (55) and performing integration over, we obtain, taking into account (51), the expression for the wave part of the field excited due to nonlinear interactions:

$$\begin{gathered}\Psi_{2w}=\frac{1}{\nu}\int\int\int\int\frac{U(\omega,k)U(\Omega-\omega)}{\left(\kappa_w^+-\kappa_l^+\right)\left(k_l-k_w\right)\left(\tilde{k}_l-\tilde{k}_w\right)}\Big[H(k_w,\tilde{k}_w)-H(k_w,\tilde{k}_l)-H(k_l,\tilde{k}_w)+\\ +H(k_l,\tilde{k}_l)\Big]e^{i\kappa_w^+\zeta}e^{i\kappa\xi}e^{-i\Omega t}dkd\kappa d\omega d\Omega,\\ H(\sigma,\tilde{\sigma})=\frac{D_0(\sigma,\tilde{\sigma})}{\left(\sigma+\tilde{\sigma}-\kappa_w^-\right)\left(\sigma+\tilde{\sigma}-\kappa_w^+\right)\left(\sigma+\tilde{\sigma}-\kappa_l^-\right)}.\end{gathered} \quad (59)$$

The first term in square brackets in (59) describes the nonlinear interaction of wave (or pseudowave) fields with each other, the second and third—the interaction of wave fields with ligaments, and the fourth—the interaction of ligaments with each other.

All terms have a different order of smallness in viscosity, and the main terms are those that describe the interaction of ligaments with each other and with wave fields. This indicates the exceptional importance of small-scale perturbations in nonlinear generation processes. Substituting expressions (47) for $U(\omega,k)$ and in (59), integrating over ω and Ω, we obtain the following representation for Ψ_{2w}:

$$\Psi_{2w}=\frac{1}{2}\sum_{n=1}^{\infty}\left[\psi_n(\xi,\zeta)e^{-in\omega_0t}+\psi_n^*(\xi,\zeta)e^{in\omega_0t}\right], \quad (60)$$

where for the ψ_n, approximate expression is valid:

$$\begin{gathered}\psi_n=\frac{2\omega_0^2}{\pi^2\nu\kappa_{ln}^+}\int_{-\infty}^{+\infty}S_n(\kappa)e^{i\kappa_{wn}^+\zeta}e^{i\kappa\xi}d\kappa,\\ S_n(\kappa)'=\sum_{m=-\infty}^{+\infty}{}^{-}\frac{m(n-m)\gamma_{nm}}{\tilde{k}_l}\int_{-\infty}^{+\infty}\frac{1}{k^2(\kappa-k)}\sin\frac{ka}{2}\sin\frac{(\kappa-k)a}{2}J_m[(\kappa-k)b]J_{n-m}(kb)dk,\end{gathered} \quad (61)$$

$$\begin{gathered}\gamma_{nm}=\frac{1}{(k_l+\tilde{k}_l)(k_l+\tilde{k}_l-\kappa_l^-)}\left[k_l-\widetilde{k}_l+\frac{N^2\sin^2\varphi}{n\omega_0^2}\left(\frac{\tilde{k}_l}{m}-\frac{k_l}{n-m},\right)\right]\\ a-\frac{1}{k_l-\kappa_l^-}\left[1+\frac{N^2}{\omega_0^2}\frac{\sin^2\varphi}{n(n-m)}\right].\end{gathered} \quad (62)$$

Here, the sign (–) at the sum means the exclusion of terms with $m=0$ and $m=n$.

The quantities k_l, $\widetilde{k}_l$ and κ_l^- in (62) are determined by the formulas

$$(k_l)^2 = i\frac{(n-m)^2\omega_0^2 - N^2\sin^2\varphi}{\nu(n-m)\omega_0},\ \left(\widetilde{k}_l\right)^2 = i\frac{m^2\omega_0^2 - N^2\sin^2\varphi}{\nu m\omega_0},\ \left(\kappa_l^-\right)^2 = i\frac{n^2\omega_0^2 - N^2\sin^2\varphi}{\nu n\omega_0}. \tag{63}$$

The calculations show that, despite the differences in characteristic scales, all components of the flows described by both regular (waves) and singularly perturbed solutions (ligaments) interact with each other and generate new waves and ligaments with multiple frequencies and specific transient size. All solution components propagate along radius vectors inclined at an angle to the horizon in the beam with the width and modality determined by the ratio of the source size to the viscous wave scale L_ν.

4. Non-Linear Generation of Flows by a Band Oscillating along a Vertical Plate

The complete solution to the linearized problem of generating disturbances by a body performing a periodic motion in a viscous continuously stratified fluid describes a family of thin ligaments supplementing either propagating internal waves ($\omega < N$) or evanescent periodic flows around the source ($\omega > N$) depending on the relation between the oscillation ω and buoyancy N frequencies. Since the governing Equations (3) are nonlinear, the interacting ligaments can be one of the direct sources of waves even when propagating internal waves cannot be generated directly due to Rayleigh frequency limit [3]. The parameters of waves generated by ligaments in a flow on a horizontal disc performing torsional oscillations are in satisfactory agreement with measurements [40]. Here the generation of internal waves by the nonlinearly interacting ligaments with each other and with propagating internal waves or residual motions at $\omega > N$ is analyzed.

As a source of flows, we consider an infinite motionless vertical plane and infinitely thin band on it performing complex two-dimensional motion. The displacement of the band is the superposition of two vertical oscillations with frequencies ω_1 and ω_2. In this case, only the vertical component of the band velocity is nonzero:

$$U(z,t) = U_1(z)e^{-i\omega_1 t} + U_2(z)e^{-i\omega_2 t}. \tag{64}$$

The system of governing Equation (3) is supplemented by the no-slip boundary conditions on the plane:

$$u_x(x=0,z,t) = 0,\ u_z(x=0,\ S(z),t) = U(S(z),t), \tag{65}$$

where function $S(z)$ describes the geometry of moving part of the plane and damping of all perturbations at infinity.

In the approximation of weak nonlinearity, the solution of the problem in the first-order perturbation theory is represented as the sum of solutions of a linearized system (3) with boundary conditions (65) and the solutions of the inhomogeneous linearized system:

$$\begin{gathered}\rho_0\frac{\partial\widetilde{u}_x}{\partial t} = -\frac{\partial\widetilde{P}}{\partial x} + \rho_0\nu\Delta\widetilde{u}_x + \rho_0 f^x,\ \rho_0\frac{\partial\widetilde{u}_z}{\partial t} = -\frac{\partial\widetilde{P}}{\partial z} + \rho_0\nu\Delta\widetilde{u}_z - \widetilde{\rho}g + \rho_0 f^z,\\ \frac{\partial\widetilde{\rho}}{\partial t} + \widetilde{u}_z\frac{d\rho_0}{dz} = m,\ \frac{\partial\widetilde{u}_x}{\partial x} + \frac{\partial\widetilde{u}_z}{\partial z} = 0,\end{gathered} \tag{66}$$

with zero boundary conditions in the plane. The sources f^x, f^z and m in the right part of equations (66) are results of substitutions of stream function Ψ ($u_x = \Psi_z$, $u_z = -\Psi_x$), which is the solution of the linearized system, into the quadratic terms of Equation (3) and have the form:

$$f^x = \Psi_x\Psi_{zz} - \Psi_z\Psi_{xz},\ f^z = \Psi_z\Psi_{xx} - \Psi_x\Psi_{xz},\ m = \Psi_x\rho_z - \Psi_z\rho_x, \tag{67}$$

where the subscripts x and z denote the partial derivatives with respect to the corresponding variables. In Equation (66), the terms $\rho\partial\mathbf{u}/\partial t$ and $\rho\nu\Delta\mathbf{u}$ are omitted under the assumption that the typical internal wave length λ and the typical spatial scale of viscous ligaments $\delta_N^\nu = \sqrt{2\nu/N}$ are small with respect to the stratification scale Λ: $\lambda << \Lambda$, $\left(\delta_N^\nu\right)^2 << \lambda\Lambda$.

For the correction to the stream function, $\widetilde{\Psi}$ ($\widetilde{u}_x = \widetilde{\Psi}_z$, $\widetilde{u}_z = -\widetilde{\Psi}_x$), we obtain the next equations, which follows from Equation (66):

$$\left[\frac{\partial^2}{\partial t^2}\Delta + N^2\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial t}\Delta^2\right]\widetilde{\Psi} = \frac{\partial}{\partial t}\left(\frac{\partial f^x}{\partial z} - \frac{\partial f^z}{\partial x}\right) + \frac{g}{\rho_0}\frac{\partial m}{\partial x} \equiv F. \tag{68}$$

Substituting (67) into (68), the solution of linearized system (66) with boundary conditions (65) is represented as the plane-wave expansion [21]:

$$\begin{gathered}\Psi = \tfrac{1}{2}\left(\Psi_1 e^{-i\omega_1 t} + \Psi_2 e^{-i\omega_2 t}\right) + c.c.,\ \rho = \tfrac{i\rho'_0}{2\omega_1}\Psi_{1x}e^{-i\omega_1 t} + \tfrac{i\rho'_0}{2\omega_2}\Psi_{2x}e^{-i\omega_2 t} + c.c.,\\ \Psi_j = \int\limits_{-\infty}^{+\infty} A_j(k)\left(e^{ik_{jw}x} - e^{ik_{jl}x}\right)e^{ikz}dk,\ A_j(k) = \tfrac{i}{2\pi}\tfrac{1}{k_{jw}-k_{jl}}\int\limits_{-\infty}^{+\infty} U_j(z)e^{-ikz}dz,\end{gathered} \tag{69}$$

which describes both the internal waves and the ligaments (the terms with the subscripts w and l, the term $c.c$ means the complex conjugate). Wavenumbers $k_{jw}(k)$ and $k_{jl}(k)$ satisfy the dispersion equation:

$$\omega_j^2\left(k_j^2 + k^2\right) - N^2k_j^2 + i\omega_j\nu\left(k_j^2 + k^2\right)^2 = 0. \tag{70}$$

The substitution of solutions (69) into expressions (67) and (68) results in the appearance of the terms with different combination frequencies 0, $2\omega_1$, $2\omega_2$, and $\omega_1 \pm \omega_2$ at the right-hand side of Equation (68). To calculate the generation of waves with the frequency $\Omega = \omega_1 - \omega_2$, we seek a solution of Equation (68) in the form:

$$\widetilde{\Psi} = \frac{1}{2}\left[\psi(x,z)e^{-i\Omega t} + \psi^*(x,z)e^{i\Omega t}\right].$$

Here, the asterisk denotes complex conjugation, and ψ satisfies the equation:

$$\left[\Omega^2\Delta - N^2\frac{\partial^2}{\partial x^2} - i\Omega\nu\Delta^2\right]\psi = F_\Omega(x,z), \tag{71}$$

and

$$\begin{gathered}F_\Omega = \tfrac{i\Omega}{2}\{(1+\alpha_1)[\Psi^*_{2x}\Psi_{1xxz} - \Psi^*_{2z}\Psi_{1xxx}](1-\alpha_2)[\Psi_{1x}\Psi^*_{2xxz} - \Psi_{1z}\Psi^*_{2xxx}] +\\ +(\alpha_1+\alpha_2)[\Psi_{1xz}\Psi^*_{2xx} - \Psi_{1xx}\Psi^*_{2xz}] + \Psi_{1x}\Psi^*_{2zzz} - \Psi_{1z}\Psi^*_{2xzz} + \Psi^*_{2x}\Psi_{1zzz} - \Psi^*_{2z}\Psi_{1xzz}\},\end{gathered} \tag{72}$$

where $\alpha_j = \frac{N^2}{(\Omega\omega_j)}$.

A solution for Equation (71) is constructed in the form of the convolution of its right-hand side with the Green's function of this equation:

$$\psi(x,z) = \int\limits_0^\infty\int\limits_{-\infty}^{+\infty} G(x,\xi;z-\zeta)F_\Omega(\xi,\zeta)d\zeta d\xi, \tag{73}$$

which for boundary conditions, $\partial G/\partial x = \partial G/\partial z = 0$, at $x = 0$, has the form:

$$G = \frac{1}{4\pi\Omega\nu}\int\limits_{-\infty}^{+\infty}\widetilde{G}(\kappa;x,\xi)e^{i\kappa(z-\zeta)}d\kappa, \tag{74}$$

and

$$\begin{gathered}\widetilde{G} = \tfrac{1}{\kappa_l^2-\kappa_w^2}\left\{\tfrac{1}{\kappa_w}\left[e^{i\kappa_w|x-\xi|} - \tfrac{\kappa_l+\kappa_w}{\kappa_l-\kappa_w}e^{i\kappa_w(x+\xi)} + \tfrac{2\kappa_w}{\kappa_l-\kappa_w}e^{i\kappa_w x}e^{i\kappa_l\xi}\right] -\right.\\ \left. -\tfrac{1}{\kappa_l}\left[e^{i\kappa_l|x-\xi|} + \tfrac{\kappa_l+\kappa_w}{\kappa_l-\kappa_w}\ e^{i\kappa_l(x+\xi)} - \tfrac{2\kappa_l}{\kappa_l-\kappa_w}e^{i\kappa_l x}e^{i\kappa_w\xi}\right]\right\},\end{gathered} \tag{75}$$

where the solutions $\kappa_w(\kappa)$ and $\kappa_l(\kappa)$ of the dispersion equation corresponding to the Equation (71) have the form

$$\kappa_w = |\kappa|\mathrm{tg}\theta + \frac{i\nu|\kappa|^3}{2N\cos^5\theta},\ \kappa_l = (i-1)\mathrm{ctg}\theta\sqrt{\frac{\Omega}{2\nu}}, \tag{76}$$

and $\theta = \arcsin(\Omega/N)$ is the angle between the direction of the beam propagation and the horizontal plane.

Substituting Equations (69), (72), (74), and (75) into Equation (73) and integrating with respect to ζ and k'' we obtain:

$$\psi = \frac{i}{4\nu}\int\limits_{-\infty}^{+\infty} e^{i\kappa z}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{\infty} A_1(k)A_2^*(k-\kappa)\widetilde{G}(\kappa;x,\xi)\widetilde{F}(k,\kappa;\xi)d\xi dkd\kappa, \tag{77}$$

where:

$$\begin{gathered}\widetilde{F} = H(k_w,\widetilde{k}_w) - H(k_l,\widetilde{k}_w) - H(k_w,\widetilde{k}_l) + H(k_l,\widetilde{k}_l),\\ H(\sigma,\widetilde{\sigma}) = e^{i(\sigma-\widetilde{\sigma})\xi}[(k-\kappa)\sigma - k\widetilde{\sigma}]\{(\sigma-\widetilde{\sigma})[(1+\alpha_1)\sigma + (1-\alpha_2)\widetilde{\sigma}] + \kappa(2k-\kappa\}.\end{gathered} \tag{78}$$

The notations introduced here are:

$$k_w = k_{1w}(k),\ \widetilde{k}_w = k_{2w}^*(k-\kappa),\ k_l = k_{1l}(k),\ \widetilde{k}_l = k_{2l}^*(k-\kappa). \tag{79}$$

Separate terms in expression for $\widetilde{F}$ in (4.15) describe the nonlinear interaction between the internal waves, which is the interaction of the internal waves with ligaments and between the different ligaments.

Separate terms in the expression for $\widetilde{F}$ in (78) describe three types of different nonlinear interactions: between the internal waves, the interaction of the internal waves with ligaments, and between the different ligaments.

When integrating in Equation (77) with respect to the coordinate ξ in the low viscosity approximation, we retain only the terms of the minimum order in $\nu \to 0$. Then the terms presenting the direct interaction between internal waves disappear in Equation (77), since they have a higher order in ν. The nonlinear interaction of the ligaments with each other and with the internal waves leads to the generation of a wave field, whose stream function is:

$$\begin{aligned}\psi = \tfrac{1}{2\nu\kappa_l}\int\limits_{-\infty}^{+\infty} e^{i\kappa_w x}e^{i\kappa z}\int\limits_{-\infty}^{+\infty} A_1(k)A_2^*(k-\kappa)\times\Big\{&\tfrac{(1+\alpha_1)(k-\kappa)k_l}{\kappa_l+k_l} + \tfrac{(1-\alpha_2)k\widetilde{k}_l}{\kappa_l-\widetilde{k}_l} -\\ &- \tfrac{[(k-\kappa)k_l - k\widetilde{k}_l][(1+\alpha_1)k_l+(1-\alpha_2)\widetilde{k}_l]}{(\kappa_l+k_l-\widetilde{k}_l)(k_l-\widetilde{k}_l)}\Big\}dkd\kappa.\end{aligned} \tag{80}$$

Solution (80) exists for arbitrary frequencies ω_1 and ω_2, in particular, exceeding the buoyancy frequency N. Since the density of energy increases with the wave frequency, only the case $\omega_1,\ \omega_2 >> N$ is of practical importance. In these conditions, the direct generation of propagating internal waves is forbidden by the properties of the dispersion equation. In this case, it follows from Equation (70) that the roots of the dispersion equation corresponding to the ligaments take the form:

$$k_l = (1+i)\sqrt{\frac{\omega_1^2-N^2}{2\omega_1\nu}},\ \widetilde{k}_l = (1-i)\sqrt{\frac{\omega_2^2-N^2}{2\omega_2\nu}}.$$

Substituting values for $A_j(k)$ in (69) into expression (79) and performing integration, we obtain:

$$\begin{gathered}\psi = -\frac{i}{4\pi\nu\kappa_l(\kappa_l+k_l-\widetilde{k}_l)(k_l-\widetilde{k}_l)}\int\limits_{-\infty}^{+\infty} e^{i\kappa_w x}e^{i\kappa z}[\beta_1 I_1(\kappa)+\beta_2 I_2(\kappa)]d\kappa,\\ I_1(\kappa) = \int\limits_{-\infty}^{+\infty} U'_1(z)U_2^*(z)e^{-i\kappa z}dz,\ I_2(\kappa) = \int\limits_{-\infty}^{+\infty} U_1(z)U'^*_2(z)e^{-i\kappa z}dz,\\ \beta_1 = 2+\alpha_1-\alpha_2+(1-\alpha_2)\tfrac{k_l-\widetilde{k}_l}{\kappa_l-\widetilde{k}_l},\ \beta_2 = 2+\alpha_1-\alpha_2+(1+\alpha_1)\tfrac{k_l-\widetilde{k}_l}{\kappa_l+k_l}.\end{gathered} \tag{81}$$

If the geometry of the part of the plane moving with different frequencies is characterized by a common function $S(z)$, i.e., $U_j(z) = -i\omega_j b_j S(z)$, where b_j is the amplitude of corresponding oscillations and $\max S(z) = 1$, we have:

$$\psi = \frac{\omega_1\omega_2 b_1 b_2\beta}{8\pi\nu\kappa_l}\int\limits_{-\infty}^{+\infty}\kappa e^{i\kappa_w x}e^{i\kappa z}\int\limits_{-\infty}^{+\infty} S^2(\zeta)e^{-i\kappa\zeta}d\zeta d\kappa,\ \beta = \frac{\beta_1+\beta_2}{(\kappa_l+k_l-\widetilde{k}_l)(k_l-\widetilde{k}_l)}. \tag{82}$$

Since the frequencies ω_1 and ω_2 are high ($\omega_1 \approx \omega_2 = \omega >> \Omega$), it follows from (81) that:

$$\psi = -\frac{3\omega b_1 b_2}{8\pi\kappa_l}\int\limits_{-\infty}^{+\infty}\kappa e^{i\kappa_w x}e^{i\kappa z}\int\limits_{-\infty}^{+\infty} S^2(\zeta)e^{-i\kappa\zeta}d\zeta d\kappa. \tag{83}$$

As an illustration, we consider the generation of waves by a band with width a when $S(z) = \vartheta\left(\frac{a}{2}-|z|\right)$, where ϑ is the unit step function. Integrating with respect to ζ in expression (83), we obtain the stream function in the form:

$$\psi = -\frac{3\omega b_1 b_2}{4\pi\kappa_l}\int\limits_{-\infty}^{+\infty}\sin\frac{\kappa a}{2}e^{i\kappa_w x}e^{i\kappa z}d\kappa. \tag{84}$$

Considering propagation of a single beam in the first quadrant and introducing the attached coordinate system (p, q)) with the q-axis directed along the beam, and using Equations (76) and (83), we obtain the vertical displacements h of particles:

$$h(p,q) = \frac{3\omega b_1 b_2(1+i)\sin^2\theta}{8\pi\Omega}\sqrt{\frac{2\nu}{\Omega}}\int\limits_0^{\infty}\kappa\sin\frac{\kappa a\cos\theta}{2}\exp\left(i\kappa p-\frac{\nu\kappa^3 q}{2N\cos\theta}\right)d\kappa, \tag{85}$$

which can be expressed by means of reference functions $F(p, q)$ (27).

The amplitude of the displacements in the axis of the single-mode beam generated by the motion of a narrow band ($a < L_\nu$) with respect to viscous wave scale $L_\nu = \frac{\sqrt[3]{8\nu}}{N}$ and for large distances from a source, $q >> \frac{2Na^3\cos\theta}{\nu}$, takes the form:

$$h_m(q) = \frac{\omega a b_1 b_2\cos^2\theta}{4\pi q}\sqrt{\frac{\sin\theta}{\nu N}}, \tag{86}$$

and is proportional to the product of the oscillation amplitudes, a; average frequency, ω; and band width, b_1 and b_2. Under the laboratory conditions, when $q = 20$ cm, $N = 1$ s^{-1} s, $a = b_1 = b_2 = 1$ cm, $\theta = 45^\circ$, and $\omega = 10$ s^{-1}, formula (85) gives the estimate $h_m(q) \approx 2$ mm that is quite available for experimental observation. The presented technique gives a room to evaluate the parameters of wave beams for other combination frequencies: double, summary, and zero.

The effects of nonlinear generation are also manifested in the cases of more complex motions of a generating surface. In particular, let a part of the plane perform frequency-modulated oscillations with a constant amplitude b so that the surface velocity is:

$$U(z,t) = -i\phi'(t)be^{-i\phi(t)}S(z),\ \phi'(t) \equiv \omega(t) = \omega_0(1+\mu\sin\Omega t), \tag{87}$$

where μ is the frequency-modulation depth and the function $S(z)$ specifies, as above, the geometry of the moving band on the plane. Following the above method, we find the stream function of the generated wave field:

$$\psi = -\frac{3\omega_0 b^2\mu}{16\pi\kappa_b}\int\limits_{-\infty}^{+\infty}\kappa J(\kappa)e^{i\kappa_w x}e^{i\kappa z}d\kappa,\ J(\kappa) = \int\limits_{-\infty}^{+\infty}S^2(z)e^{-i\kappa z}dz. \tag{88}$$

When a wave beam is generated by an oscillating band with width a, $S(z) = \vartheta\left(\frac{a}{2}-|z|\right)$ the expression (88) has a form:

$$\widetilde{\Psi}_\Omega = \frac{3\mu\omega_0 b^2}{8\pi\kappa_b}\int\limits_{-\infty}^{+\infty}\sin\frac{\kappa a}{2}e^{i\kappa_w x}e^{i\kappa z}d\kappa, \tag{89}$$

which determines both the beams of internal waves propagating away from the source to the right. Considering motion only in the first quadrant of the coordinate system (p,q) attached to the beam, we find the vertical displacements h of particles in the beam in the form:

$$h(p,q) = \frac{3\mu\omega_0 b^2(1+i)\sin^2\theta}{16\pi\Omega}\sqrt{\frac{2\nu}{\Omega}}\int\limits_0^\infty\kappa\sin\frac{\kappa a\cos\theta}{2}\exp\left(i\kappa p-\frac{\nu\kappa^3 q}{2N\cos\theta}\right)d\kappa. \tag{90}$$

At large distances from the source $q >> \frac{2Na^3\cos\theta}{\nu}$ when the beam is single-mode, the integral in formula (89) is calculated and the expression for the displacements in the beam axis becomes:

$$h_m(q) = \frac{3\mu\omega_0 a b^2(1+i)\sin\theta\cos^2\theta}{8\pi q\sqrt{2\nu\Omega}}. \tag{91}$$

The amplitude of the generated wave is proportional to the depth of the frequency modulation μ.

Under natural conditions, when the flows are substantially nonstationary, the mechanisms under consideration can significantly contribute to the generation and formation of the internal-wave spectrum. Similar effects can be observed in the dynamics of other types of waves (acoustic, surface, inertial, and hybrid), which also coexist with their specific system of ligaments [28].

5. Discussion of the Results

Starting from the works of the late 60s of the last century [6] and up to the present time [8,13–16], the construction of solutions to problems of generation and propagation of internal waves in a viscous continuously stratified fluid based on the fundamental equations of fluid mechanics [9–11], is limited to defining only partial solutions that characterize waves.

Exceptions from the analysis of the problem in a linear or weakly nonlinear description of a large group of solutions are carried out on the basis of arbitrarily interpreted physical considerations [9] or even without explanations of the performed re-duction. At the same time, the methods of the modern theory of singular perturbations [25] have room to find complete solutions of the system of fundamental equations, supplemented by physically justified boundary conditions without involving addition-al parameters, in particular, sources of mass and momentum. Complete solutions of the linearized fundamental system

are consistent with data of high-resolution laboratory investigations of periodic internal wave generation [41], propagation internal waves in an arbitrarily stratified liquid [42], reflection from a critical layer where frequencies of buoyancy and the running waves equal [43].

In this paper, we study the properties of solutions to a problem of two-dimensional wave generation by a band oscillating along an inclined surface in an exponentially stratified viscous fluid. In the second section, a complete solution of the linearized system of equations of motion with no-slip boundary conditions for velocity and impermeability for matter is presented, which includes regularly perturbed functions describing beams of periodic internal waves and singular functions characterizing ligaments.

The wave beams form an oblique cross of St. Andrew in space, the slope of the rays of which is determined by the ratio of the wave frequency to the buoyancy frequency, and the beam width is determined by the width of the band and an additional viscous correction. The ratio of the band size to the viscous wave scale $L_\nu = \sqrt[3]{g\nu}/N$ characterizes the modality of the beam. The group of singular solutions describes ligaments that are systems of thin extended perturbations on the radiating surface and in the liquid bulk, where they outline the wave beams. The characteristic transverse scale of the ligaments is determined by the values of the kinetic coefficients and natural frequencies of the problem that are buoyancy or the source oscillation frequency. Both components of periodic disturbance flows, which in an infinite space are described by functions of the same type and differ in the relations between the real and imaginary parts of the solution, can interact with each other when taking into account the nonlinear properties of equations and boundary conditions.

In the third section, we present a solution to the generation of periodic perturbations by a band oscillating along an inclined plane, constructed by the methods of singular perturbation theory in the approximation of weak nonlinearity. When formulating the problem, both the intrinsic nonlinearities of the equations and of the boundary conditions due to the periodic replacement of a part of the fluid by a band near the body edge are taken into account. The solutions constructed in the second order of the perturbation theory describe the interaction of waves with waves, waves with ligaments, and ligaments with each other. Each act of interaction generates all components of the flow, both internal waves and accompanying ligaments, which explains the continuous evolution of the formed flows and the complication of their structure.

In the fourth section, the solution to the problem of calculating the wave component of flows by a rapid biharmonically oscillating vertical band in a continuously stratified viscous fluid is considered. In the considered problem, both characteristic frequencies ω_1 and ω_2 exceed the buoyancy frequency, $\omega_1, \omega_2 >> N$. If the difference in oscillation frequencies falls within the frequency range of the existence of propagating internal waves $(\omega_{1,} - \omega_2) < N$, the interacting ligaments on the band become a source of internal waves. The conditions are determined under which the waves are emitted efficiently enough and can be recorded in a laboratory experiment.

The inclusion of the action of other dissipative factors in the calculations provides additional attenuation of wave motions and a significant complication of the flow structure, in which interfaces and fibers with their own $\delta_N^\nu = \sqrt{\frac{\nu}{N}}$, $\delta_N^{\kappa_T} = \sqrt{\frac{\kappa_T}{N}}$ and combinatory $\delta_N^{\nu,\kappa_T} = \sqrt[4]{\frac{\nu\cdot\kappa_T}{N^2}}$, $\delta_N^{\nu,\kappa_S} = \sqrt[4]{\frac{\nu\cdot\kappa_S}{N^2}}$ length scales appear. Taking into account the registration criteria for both waves and ligaments determines the conditions for setting up a laboratory experiment and carrying out a numerical simulation of natural processes based on a system of fundamental equations. The observation area should be large enough to contain large-scale components, and the resolution of the instruments (or the numerical code) should be high enough to resolve the finest components. The results of calculating two-dimensional problems of flow around obstacles have room, in a unified formulation, to calculate the flow around a vertical band with a height h, moving horizontally at a constant velocity U in a wide range of parameters, including creeping flows at $\mathrm{Re} = \frac{hU}{\nu} \sim 1$ [44], and unsteady vortex flows at $\mathrm{Re} \sim 100,000$ [45], consistent with the experiment.

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