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Coherent Diversification Measures in Portfolio Theory: An Axiomatic Foundation

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Abstract: We provide an axiomatic foundation for the measurement of correlation diversification in a one-period portfolio model. We propose a set of eight desirable axioms for this class of diversification measures. We name the measures satisfying these axioms *coherent correlation diversification measures*. We study the compatibility of our axioms with rank-dependent expected utility theory. We also test them against the two most frequently used methods for measuring correlation diversification in portfolio theory: portfolio variance and the diversification ratio. Lastly, we provide an example of a functional representation of a coherent correlation diversification measure.

Keywords: portfolio theory; diversification measurement; correlation diversification; diversification ratio; portfolio variance; rank-dependent expected utility theory

JEL Classification: D81; G1; G11

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1. Introduction

This paper is about diversification measurement in portfolio theory. For our purposes we focus on *correlation diversification*. The term 'correlation' refers to any dependence measure including similarity measures.

Correlation diversification is a diversification concept as old as the concept of *naive diversification*. It can be traced back to long before the birth of portfolio theory, marked by the works of Roy (1952) and Markowitz (1952). However, as a term, it was only recently introduced in the literature by Koumou (2020a). Its core element is asset dependence structure. More precisely, correlation diversification is a risk diversification strategy that exploits assets' dependence structure to reduce risk. It is based on the principle that, all other things being equal, a portfolio with low positively dependent assets will be less risky than a portfolio with high positively dependent assets. The intuition is that the less assets are positively dependent, the lower the likelihood they will do poorly simultaneously in the same proportion at the same time. Thus, the less positively dependent the assets of a portfolio are, the more correlation diversified this portfolio is.

Correlation diversification plays a central role in portfolio selection. As highlighted by Koumou (2020a), it is at the core of mean-variance models and its effect is covered through the covariance matrix. More generally, it drives the diversification in expected utility (EU) theory. Any risk-averse EU investor is *correlation averse*; as a consequence, he/she exhibits a preference for correlation diversification.[1]

Correlation diversification is also at the core of non-expected utility theories. For example, in Yaari's (1987) dual (DU) theory, a DU risk-averse investor exhibits a preference for correlation diversification (Hadar and Seo 1995, Corollary 1, p. 1176). As demonstrated by Andersen et al. (2018, Corollary 1, p. 1176), rank-dependent expected utility (RDEU) theory

(Quiggin 1982) (excluding EU and DU theories) exhibits correlation aversion, and implies therefore a preference for correlation diversification. In sum, correlation diversification can be viewed as the *rational* diversification principle for risk-averse EU, DU or RDEU investors in the absence of a lack of information.[2]

Despite the central role of correlation diversification in economic theories under risk and uncertainty, especially in portfolio diversification, conceptual problems involved in its measurement have been overlooked. None of the existing measures designed to capture its effect has theoretical foundations (axiomatic or decision-theoretic foundations).[3] Almost all studies on theoretical foundations of risk management have been devoted to risk measurement (see Artzner et al. 1999; Föllmer and Schied 2002; Frittelli and Gianin 2002, 2005; Rockafellar et al. 2006). Even though many proposed risk measurement theories, such as monetary risk measurement theories, take into account correlation diversification, they remain incomplete for quantifying its effect on decision makers.[4] Indeed, risk reduction is not equivalent to diversification. Diversification implies risk reduction, but the reverse is not true, because risk can also be reduced by concentration.[5] Consequently, standard risk measurement frameworks fail to adequately quantify and manage correlation diversification, except in the extreme case where all assets have the same risk.[6]

The aim of this paper is to bridge this gap in the literature. We provide an axiomatic foundation for correlation diversification measurement in a one-period portfolio model. We assume that the investor is risk-averse depending on his/her preferences and has complete information about the marginal and the joint distributions of asset future returns. We also assume that there are no short sales, but our results remain valid when short sales in the sense of Lintner (1965) are allowed.

Axiomatic approaches have proven to be useful in modern economics, in particular in economics of risk and uncertainty (Gilboa et al. 2019), and in risk (Föllmer and Schied 2002), poverty (Zheng 1997) and inequality (Chakravarty 1999) measurement. When applied to risk, poverty and inequality measurement theories, axiomatization helped to solve problems such as misunderstanding of the concept being measured, development of incoherent measures and the choice and comparison of measures. Due to the lack of theory, correlation diversification measurement is not spared from these problems. For example, the concept of diversification in general and correlation diversification in particular is misunderstood as revealed by the 2007–2009 financial crisis (Ilmanen and Kizer 2012; Miccolis and Goodman 2012; Statman 2013). An example of an inadequate correlation diversification measure is *diversification delta*, which was introduced by Vermorken et al. (2012) and revised by Salazar Flores et al. (2017).

Our paper contributes to the literature of correlation diversification measurement in several ways. First, we present and discuss a set of minimum desirable axioms that a measure of correlation diversification must satisfy in order to be considered coherent (Section 2). In the literature, some studies have presented and discussed some properties to support their proposed measures of diversification (Carmichael et al. 2022; Choueifaty et al. 2013; Evans and Archer 1968; Rudin and Morgan 2006; Vermorken et al. 2012). Our axioms build on the body of such properties. We generalize, supplement, and rationalize them to obtain an axiomatic system for coherent correlation diversification measures. Our axioms also draw on the literature of dependence measurement theory. Two of our axioms (Axioms 6 and 7) are inspired by the axiom of *invariance* of dependence measures, with respect to all strictly increasing and continuous transformations (Rényi 1959; Schmid et al. 2010).

Second, to ensure that our axioms do not violate investors' preference for diversification, we examine (Section 3) their compatibility with economic theories. We focus on RDEU theory (Quiggin 1982, 1993), one of the most accepted non-expected utility theories in the literature. We proceed by taking two steps. First, we identify the measure of correlation diversification at the core of RDEU theory as the differential between the weighted average of each asset risk and portfolio risk, similar to the study by Embrechts et al. (2009), where risk is measured by the standard certainty equivalent or risk premium. We assume strong risk aversion[7], and we rely on the notion of preference for diversification (Chateauneuf and Lakhnati 2007; Chateauneuf and Tallon 2002; Dekel 1989) for our identification. Next, we test the identified measure against our axioms to establish the conditions of their compatibility with RDEU theory focusing on two cases. Specifically, we test whether there exists a concave utility function and a concave distortion function such that the identified measure satisfies our axioms. This two-step strategy was used in the literature of risk measurement theories to examine the compatibility of Artzner et al. (1999)'s axioms with economic theories (Denuit et al. 2006; Tsanakas and Desli 2003).

Three main arguments justify our choice of RDEU. First, it covers the effect of correlation diversification. Second, it embeds EU theory and Yaari's (1987) DU theory, which means our results also cover these two economic theories. Third, it successfully explains many phenomena and paradoxes that are puzzling within the framework of expected utility theory, such as the Allais paradox (Quiggin 1993; Starmer 2000), the simultaneous risk-averse and risk-seeking behavior (Quiggin 1993), and the poor diversification funds and stock market participation of risk-averse households (Polkovnichenko 2005).

Third, in Section 4, we test our axioms against the two most frequently used methods for measuring correlation diversification in portfolio theory: *portfolio variance* (Markowitz 1952, 1959; Frahm and Wiechers 2013; Sharpe 1964) and the *diversification ratio* of Choueifaty and Coignard (2008).

Our axioms are not restrictive enough to specify a unique family of correlation diversification measures. This incompleteness is intentional, because it allows our axioms to be used for all families of correlation diversification measures. Thus, fourth, in Section 5, we provide an example of a functional representation of our axioms and not a representation theorem.

Section 6 concludes the paper. Proofs are given in the appendix. Throughout the paper, vectors and matrices have bold style.

2. Axioms

Let us first introduce the general definition of the correlation diversification measure we consider. Given that the core of correlation diversification is assets' dependence structure, any measure designed to capture its effect needs to depend not only on the vector of asset weights, $\mathbf{w}= (w_1, \dots, w_N)^\top$ with w_i the weight of asset i, but also on the vector of future return on assets, $\mathbf{R} = (R_1, \dots, R_N)^\top$ with $R_i \in \mathcal{R}$ the future return on asset i, where N is the number of assets, $\top$ the operator of transpose and $\mathcal{R}$ the space of bounded real-valued random variables.[8] Therefore, in the case where the investor is risk-averse and has complete information about the marginal and the joint distributions of assets future returns, it is natural to define a correlation diversification measure as follows.

Definition 1 (Correlation diversification measure)**.** *A correlation diversification measure is a conditional mapping given* $\mathbf{R}$*,* $D(\cdot|\mathbf{R})$*, from* $\mathbb{W}$ *into* $\mathbb{R}$ *assigning to* $\mathbf{w}$ *a positive real value* $D(\mathbf{w}|\mathbf{R}) \in \mathbb{R}_+$*, where* $\mathbb{W}$ *is the set of long-only portfolios,* $\mathbb{R}$ *is the set of real numbers and* $\mathbb{R}_+$ *is the set of positive real numbers.*

Without loss of generality, we assume that the well-diversified portfolio of $D(\mathbf{w}|\mathbf{R})$, denoted $\mathbf{w}^*$, is obtained by maximization. Formally

$$\mathbf{w}^* \in \underset{\mathbf{w}\in\mathbb{W}}{\operatorname{argMax}}\ D(\mathbf{w}|\mathbf{R}). \tag{1}$$

Therefore, given a measure $D(\cdot|\mathbf{R})$, we say that "portfolio $\mathbf{w}_1$ is more correlation diversified than portfolio $\mathbf{w}_2$" if and only if $D(\mathbf{w}_1|\mathbf{R}) \geq D(\mathbf{w}_2|\mathbf{R})$.

Note that $D(\mathbf{w}|\mathbf{R})$ can be explicitly or implicitly conditional on $\mathbf{R}$. In the case where short sales in the sense of Lintner (1965) are allowed, $D(\cdot|\mathbf{R})$ must be defined from $\mathbb{W}^- = \{\mathbf{w} = (w_1, \dots, w_N)^\top \in \mathbb{R}^N : \sum_{i=1}^N |w_i| = 1\}$ into $\mathbb{R}$ assigning to $\mathbf{w}$ a positive real value $D(|\mathbf{w}| \mid \operatorname{sign}(\mathbf{w})\mathbf{R}) \in \mathbb{R}_+$, where $|\mathbf{w}| = (|w_1|, \dots, |w_N|)^\top$ and $\operatorname{sign}(\mathbf{w})\mathbf{R} = (\operatorname{sign}(w_1)R_1, \dots, \operatorname{sign}(w_N)R_N)^\top$ with $|\cdot|$ the absolute value operator.

We now introduce our set of desirable axioms that a measure of correlation diversification must satisfy in order to be considered coherent. Our first two axioms formalize investors' preference for diversification over $\mathbb{W}$. These axioms are relevant for any diversification measure, and were first formulated in Carmichael et al. (2022). The first axiom is

Axiom 1 (Concavity). *For each* $\mathbf{w}_1$ *and* $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0,1]$ *and* $\mathbf{R} \in \mathcal{R}^N$,

$$D(\alpha\,\mathbf{w}_1 + (1-\alpha)\,\mathbf{w}_2|\mathbf{R}) \geq \alpha\, D(\mathbf{w}_1|\mathbf{R}) + (1-\alpha)\, D(\mathbf{w}_2|\mathbf{R}) \tag{2}$$

with strict inequality for at least one triplet $(\alpha, \mathbf{w}_1, \mathbf{w}_2)$.

Axiom 1 requires that $D(\mathbf{w}|\mathbf{R})$ is a nonlinear and concave function on $\mathbb{W}$. This guarantees that holding different assets can increase total diversification. It also ensures that diversification can be decomposed across asset classes.

Note that Axiom 1 can be replaced by a less restrictive axiom defined as follows.

Axiom 1' (Quasi-concavity). *For each* $\mathbf{w}_1$ *and* $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0,1]$ *and* $\mathbf{R} \in \mathcal{R}^N$,

$$D(\alpha\,\mathbf{w}_1 + (1-\alpha)\,\mathbf{w}_2|\mathbf{R}) \geq \min(D(\mathbf{w}_1|\mathbf{R}), D(\mathbf{w}_2|\mathbf{R})) \tag{3}$$

with strict inequality for at least one triplet $(\alpha, \mathbf{w}_1, \mathbf{w}_2)$.

Let the single-asset i portfolio be denoted by $\mathbf{e}_i = (e_{i1}, \ldots, e_{iN})^\top$, where $e_{ii} = 1$ for each $i = 1, \ldots, N$ and $e_{ij} = 0$ for $i \neq j$, $i, j = 1, \ldots, N$. The second axiom is expressed in

Axiom 2 (Size Degeneracy). *There is a constant (for a normalization)* $\underline{D} \in \mathbb{R}_+$ *such that for each* $\mathbf{R} \in \mathcal{R}^N$,

$$D(\mathbf{e}_i|\mathbf{R}) = \underline{D} \text{ for each } i = 1, \ldots, N. \tag{4}$$

Axiom 2 imposes that all single-asset portfolios are equally desirable in terms of diversification.

Axiom 2 together with Axiom 1 imply that diversification is always better than full concentration or specialization; formally for each $\mathbf{R} \in \mathcal{R}^N$ and for each $\mathbf{w} \in \mathbb{W}$

$$D(\mathbf{w}|\mathbf{R}) \geq D(\mathbf{e}_i|\mathbf{R}) \text{ for each } i = 1, \ldots, N. \tag{5}$$

It thus strengthens investors' preference for diversification materialized by Axiom 1. It guarantees that the size of $\mathbf{w}^*$, a well-diversified portfolio of $D(\mathbf{w}|\mathbf{R})$, is strictly greater than 1 when $D(\mathbf{w}|\mathbf{R})$ satisfies Axiom 1, except in the case where assets are identical as we show in Example 1. Clearly, the two axioms are necessary to prevent portfolio concentration on one asset from going undetected.

Let us now denote $\mathcal{A} = \{A_i\}_{i=1}^N$ our universe of N assets (risky or not), where A_i is for asset i in $\mathcal{A}$. Our next axiom is the formalization of the property of *duplication invariance* of Choueifaty et al. (2013). It is expressed in

Axiom 3 (Duplication invariance). *Let* $\mathcal{B} = \mathcal{A} \cup \{B_{N+1}\}$ *be a universe of assets such that* $B_{N+1} = A_k$, *with* $k \in \{1, \ldots, N\}$. *Then*

$$D\big(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\big) = D\big(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}\big) \tag{6}$$

$$\mathbf{w}^*_{A_i} = \mathbf{w}^*_{B_i} \text{ for each } i \neq k,\ i = 1, \ldots, N \tag{7}$$

$$\mathbf{w}^*_{A_k} = \mathbf{w}^*_{B_k} + \mathbf{w}^*_{B_{N+1}}. \tag{8}$$

The reasonableness and relevance of Axiom 3 is evident. It allows us to avoid risk concentration by ensuring that the well-diversified portfolio is not biased toward multiple representative assets. It is necessary to prevent risk concentration from going undetected. In Examples 1 and 2, we illustrate its importance.

Example 1 (Identical assets)**.** *Consider the case where assets are identical. It is straightforward to verify that Axiom 3 implies*

$$D(\mathbf{w}|\mathbf{R}) = D(\mathbf{e}_i|\mathbf{R}) \text{ for each } i = 1, \dots, N. \tag{9}$$

Therefore Axiom 3 ensures that there is no benefit to diversifying across identical assets. Such diversification is equivalent to full concentration.

Example 2 (Case $N = 2$)**.** *Consider the case where $N = 2$. In this case, $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{A_1, A_2, B_3\}$ such that $B_3 = A_1$ for example. Axiom 3 states that*

- *(i) The optimal degree of diversification of $\mathcal{A}$ and $\mathcal{B}$ must be equal;*
- *(ii) The optimal weight of A_1 in $\mathcal{A}$ must be equal to the optimal weight of A_1 in $\mathcal{B}$;*
- *(iii) The optimal weight of A_2 in $\mathcal{A}$ must be equal to the sum of the optimal weights of A_2 and B_3 in $\mathcal{B}$.*

Our next axiom is complementary to Axiom 3 in the case where assets are identical. It is expressed in

Axiom 4 (Reverse Risk Degeneracy)**.** *Suppose that $N > 1$ and, without loss of generality, that $w_i > 0$ for each $i = 1, \dots, N$. If a random vector $\mathbf{R}^*$ solves for $\mathbf{R}$ in the following equation*

$$D(\mathbf{w}|\mathbf{R}) = \underline{D}, \tag{10}$$

then $\mathbf{R}^*$ must be lower comonotonic.

Before discussing Axiom 4, we recall the definition of comonotonicity, upper comonotonicity and lower comonotonicity.

Definition 2 (Comonotonicity, Upper comonotonicity and Lower comonotonicity)**.**

- *(a) Comonotonicity (Dhaene et al. 2002a): A random vector $\mathbf{R} = (R_1, \dots, R_N)^\top$ is comonotonic if and only $F_{\mathbf{R}}(\mathbf{r}) = \min_{1 \le i \le N} F_{R_i}(r_i)$, for all $\mathbf{r} = (r_1, \dots, r_N)^\top$.*
- *(b) Upper comonotonicity (Cheung 2009): A random vector $\mathbf{R} = (R_1, \dots, R_N)^\top$ is upper comonotonic if and only there is a point $\bar{\mathbf{r}} = (\bar{r}_1, \dots, \bar{r}_N)^\top \in \mathbb{R}^N \cup (-\infty, \dots, -\infty)$, called the comonotonic threshold of $\mathbf{R}$, such that the following are true:*
  - *(i) $F_R(\bar{\mathbf{r}}) < 1$*
  - *(ii) if $\mathbf{r} \in (\bar{r}_1, \infty) \times \cdots \times (\bar{r}_N, \infty)$, then $F_{\mathbf{R}}(\mathbf{r}) = \min_{1 \le i \le N} F_{R_i}(r_i)$.*
  - *(iii) if $\mathbf{r} \notin (\bar{r}_1, \infty) \times \cdots \times (\bar{r}_N, \infty)$, then $F_{\mathbf{R}}(\mathbf{r}) = F_{\mathbf{R}}(\min(r_1, \bar{r}_1), \dots, \min(r_N, \bar{r}_N))$*
- *(c) Lower comonotonicity (Cheung 2010): A random vector $\mathbf{R} = (R_1, \dots, R_N)^\top$ is lower comonotonic if and only if $-\mathbf{R} = (-R_1, \dots, -R_N)^\top$ is upper comonotonic.*

Intuitively, comonotonicity corresponds to an extreme form of positive dependency including perfect linear and nonlinear positive dependence, whereas upper (lower) comonotonicity corresponds to comonotonicity behavior in the upper (lower) tail. For more details on upper and lower comonotonicity see also Dong et al. (2010), Nam et al. (2011) and Hua and Joe (2012).

Axiom 4 is also necessary to prevent undetected portfolio risk concentration. Diversification is only desirable in the downside not in the upside. Thus, as stated by Axiom 4, it is desirable that if correlation diversification of a non-single asset portfolio reaches its lower bound then the vector of asset returns $\mathbf{R}$ is lower comonotonic.

In Example 3, we illustrate the importance of Axiom 4.

Example 3 (Embrechts et al.'s (2009) class of measures)**.**
Consider Embrechts et al.'s (2009) class of correlation diversification measures defined as

$$D(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} \varrho(w_i R_i) - \varrho\Big(\mathbf{w}^\top \mathbf{R}\Big), \tag{11}$$

where $\varrho(\cdot)$ is a risk measure. Assume that the risk measure $\varrho(\cdot)$ is additive for independence.[9] *Then, according to $D(\mathbf{w}|\mathbf{R})$ in (11), any portfolio with independent asset returns and single-asset portfolios would have the same degree of correlation diversification, which is counterintuitive. Axiom 4 rules out the sub-class of Embrechts et al.'s (2009) correlation diversification measures with risk measures additive for independence.*

Note that the solution $\mathbf{R}^*$ can be different from $\mathbf{R}$. Example 4 provides an illustration.

Example 4 (Diversification ratio)**.** *Consider the diversification ratio,* $\mathrm{DR}(\mathbf{w}|\mathbf{R})$*, defined in (33) in Section 4. It is straightforward to verify that* $\mathrm{DR}(\mathbf{e}_i|\mathbf{R}) = 1$ *for each* $i = 1, \ldots, N$*. This implies that* $\underline{D} = 1$*. Now let us resolve the equation* $\mathrm{DR}(\mathbf{w}|\mathbf{R}) = 1$ *for* $\mathbf{R}$ *with* $w_i > 0$ *for each* $i = 1, \ldots, N$

$$\mathrm{DR}(\mathbf{w}|\mathbf{R}) = 1 \Longleftrightarrow \frac{\mathbf{w}^\top \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\, \mathbf{w}}} = 1 \tag{12}$$

$$\Longleftrightarrow \left(\sum_{i=1}^{N} w_i \sigma_i\right)^2 = \sum_{i,j=1}^{N} \rho_{ij}\sigma_i\sigma_j w_i w_j \tag{13}$$

$$\Longleftrightarrow \sum_{i,j=1}^{N} (\rho_{ij} - 1)\sigma_i\sigma_j w_i w_j = 0 \tag{14}$$

$$\mathrm{DR}(\mathbf{w}|\mathbf{R}) = 1 \Longleftrightarrow \rho_{ij} = 1, \quad \textit{for each}\; i,j = 1, \ldots, N. \tag{15}$$

Then the solution of the equation $\mathrm{DR}(\mathbf{w}|\mathbf{R}) = 1$ *is* $\mathbf{R}^* = (R^*, \ldots, R^*)$ *with* $R^* = R_i$ *or* $R^* = \frac{R_i - \mu_i}{\sigma_i}$ *for each* $i = 1, \ldots, N$*, where* $\mu_i = \mathrm{E}(R_i)$*,* $\sigma_i = \sqrt{\mathrm{Var}(R_i)}$ *and* ρ_{ij} *is the correlation between* R_i *and* R_j *with* $\mathrm{E}(\cdot)$ *the expectation operator and* $\mathrm{Var}(\cdot)$ *that of the variance.*

Note also that Axiom 4 is formulated in the weakest possible form. For instance, it may be intuitive to replace "$\mathbf{R}^*$ must be lower comonotonic" by "$\mathbf{R}^*$ must be perfectly positively linearly dependent (Pearson correlation matrix of $\mathbf{R}^*$ is a matrix of ones)". This change will make Axiom 4 stronger and harder to satisfy. It will rule out the use of asymmetric dependence measures to construct correlation diversification measures. Given that our intention is to provide a minimum set of axioms, we have chosen to present Axiom 4 in the current weakest form.

Our next axiom formalizes the relationship between diversification and portfolio size at the *optimum*. It is relevant for any diversification measure. It is expressed in

Axiom 5 (Size Monotonicity)**.** *Let $\mathcal{B}$ be a universe of assets such that $\mathcal{A} \subseteq \mathcal{B}$. Then*

$$D\big(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}\big) \geq D\big(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\big). \tag{16}$$

Axiom 5 is natural in the portfolio diversification literature (see Carmichael et al. 2022; Evans and Archer 1968; Rudin and Morgan 2006; Vermorken et al. 2012). It reveals that, at the optimum, diversification is a non-decreasing function of portfolio size.

Our next two axioms are inspired by the *invariance* property of multivariate dependence measure.[10] The first axiom is expressed in

Axiom 6 (Translation invariance)**.** *Let* $\mathbf{a} = (a, \ldots, a)^\top \in \mathbb{R}^N$*. Then for each* $\mathbf{w} \in \mathbb{W}$*,*

$$D(\mathbf{w}|\mathbf{R}) = D(\mathbf{w}|\mathbf{R} + \mathbf{a}). \tag{17}$$

Axiom 6 states that adding a constant to each asset return does not change the degree of correlation diversification. The desirability of Axiom 6 comes from the fact that assets' dependence structure does not change when you add a constant to asset returns.

Note that when risk is defined as a capital requirement (for example Expected Shortfall or Conditional Value-at-Risk), Axiom 6 can be seen as counterintuitive. As an illustration, consider $D(\mathbf{w}|\mathbf{R})$ defined in Example 3 with $\varrho(\cdot)$ the capital requirement verifying the property of translation invariance defined as follows (Sereda et al. 2010).

Definition 3 (Translation invariant risk measure)**.** *A risk measure* $\varrho(\cdot)$ *is translation invariant if for all* $a \in \mathbb{R}$ *and* $X \in \mathcal{R}$*,* $\varrho(X + a) = \varrho(X) - \eta\, a$ *with* $\eta \geq 0$*.*

Given $\mathbf{w}$, assume that $a = \frac{\varrho(\mathbf{w}^\top \mathbf{R})}{\eta}$ with $\eta > 0$. Then $\varrho(\mathbf{w}^\top \mathbf{R} + \mathbf{a}) = 0$, but $D(\mathbf{w}|\mathbf{R}) = D(\mathbf{w}|\mathbf{R} + \mathbf{a}) \geq 0$. This counterintuitive result can be viewed as over-diversification, and can be interpreted as an extreme precaution against extreme risk.

The second axiom is expressed in

Axiom 7 (Homogeneity)**.** *Let* $b{>}0$*. Then there exists* $\kappa \in \mathbb{R}$ *such that for each* $\mathbf{w} \in \mathbb{W}$

$$D(\mathbf{w}|b\mathbf{R}) = b^\kappa D(\mathbf{w}|\mathbf{R}). \tag{18}$$

When $\kappa = 0$, the desirability of Axiom 7 comes naturally from the *invariance* property of the multivariate dependence measure. In this case, Axiom 7 ensures that $D(\mathbf{w}|\mathbf{R})$ must not depend on the scale of $\mathbf{R}$.

When $\kappa \neq 0$, the desirability of Axiom 7 comes naturally from the fact that there are dependence measures which are not scale-invariant. An example is *relatively expectation dependence* of Wright (1987).

Note that Axiom 7 is also formulated in the weakest possible form, again to provide a minimum set of axioms. For instance, inspired by the *symmetry* axiom of dependence measures, it may be intuitive to replace "$b \geq 0$" by "$b \in \mathbb{R}$" and "b^κ" by "$|b|^\kappa$". These changes will make Axiom 7 stronger and harder to satisfy. It will rule out the use of asymmetric dependence measure to construct correlation diversification measures.[11]

Our last axiom presents the behavior of $D(\mathbf{w}|\mathbf{R})$ when the sequence $R_1, \ldots, R_N$ are exchangeable random variables.[12] It is expressed in

Axiom 8 (Symmetry)**.** *If* $R_1, \ldots, R_N$ *are exchangeable, then* $D(\mathbf{w}|\mathbf{R})$ *is symmetric in* $\mathbf{w}$*.*

Axiom 8 states that a correlation diversification measure must be symmetric in $\mathbf{w}$ if $R_1, \ldots, R_N$ are exchangeable. The idea behind Axiom 8 is that exchangeable random variables imply homogeneous risks. Thus, the decision-maker must be indifferent in terms of diversification between $\mathbf{w}$ and $\mathbf{\Pi w}$, where $\mathbf{\Pi}$ is a permutation matrix.

From Marshall et al. (2011, C.2. and C.3. Propositions, pp. 97–98), Axiom 1 (or 1′) and 8 taken together imply that $D(\mathbf{w}|\mathbf{R})$ is Schur-concave in $\mathbf{w}$ when $R_1, \ldots, R_N$ are exchangeable. As a result, when $R_1, \ldots, R_N$ are exchangeable, $D(\mathbf{w}|\mathbf{R})$ must be a measure of naive diversification, isotonic with majorization diversification (Ortobelli Lozza et al. 2018) and its optimal diversified portfolio $\mathbf{w}^*$ must be the naive portfolio $(1/N, \ldots, 1/N)^\top$. This result is consistent with the principle that the exchangeability assumption on $R_1, \ldots, R_N$ is equivalent to the assumption that the decision-maker has no information about asset risk characteristics $\mathbf{R}$.

3. Compatibility With Economic Theories

In the literature of risk measurement theories, it was demonstrated that risk measures derived from an axiomatic approach (i.e., risk measures which are not based on economic preferences) can be incompatible with economic theories (Denuit et al. 2006; Dhaene et al. 2003; Rothschild and Stiglitz 1970; Tsanakas and Desli 2003). For example, Dhaene et al. (2003) and Denuit et al. (2006) point out the incompatibility of Artzner et al.'s (1999) coherent risk measures with expected utility theory. Thus, in this section, we examine the compatibility of our axioms with economic theories.

We focus on rank-dependent expected utility (RDEU) theory, a non-expected utility theory first proposed by Quiggin (1982) under the name of *anticipation utility theory* and further studied by many economists (see Abdellaoui 2002; Chew and Epstein 1989; Nakamura 1995; Quiggin 1993; Quiggin and Wakker 1994; Segal 1993). Our choice of RDEU is motivated by three considerations. First, it covers the effect of correlation diversification. Second, it embeds expected utility theory and Yaari's (1987) dual theory. Third, it successfully explains many phenomena and paradoxes that are puzzling within the framework of expected utility theory, such as the Allais paradox (Quiggin 1993; Starmer 2000), the simultaneous risk-averse and risk-seeking behavior (Quiggin 1993), and the poor diversification funds and stock market participation of risk-averse households (Polkovnichenko 2005).

We follow a commonly used two-step strategy in the literature of risk measurement theories to control for the compatibility of Artzner et al. (1999)'s axioms with economic theories (Denuit et al. 2006; Tsanakas and Desli 2003). First, we identify the correlation diversification measure at the core of RDEU. Second, we test this identified measure against our axioms to establish the conditions of their compatibility with RDEU theory.

3.1. Identification

Consider a preference relation $\succeq$ on $\mathcal{R}$. We denote by $\sim$ its symmetric part. Assume that the preference relation $\succeq$ has RDEU representation. Then, from Konrad (1993) (see also Denuit et al. 2006; Tsanakas and Desli 2003),

$$R_i \succeq R_j \Longleftrightarrow U(w_0(1+R_i)) \geq U(w_0(1+R_j)), \tag{19}$$

where

$$U(R) = \int_{-\infty}^{\infty} u(r)dh(F_R(r)), \tag{20}$$

with w_0 the investor's initial wealth, $F_R(r)$ the cumulative function of $R \in \mathcal{R}$, $u(\cdot)$ being a continuous, strictly increasing utility function and unique up to positive affine transformations, and $h(\cdot)$ being a unique, continuous and strictly increasing distortion function from $[0,1]$ into $[0,1]$ satisfying $h(0) = 0$ and $h(1) = 1$. In the rest of the section, we assume that $w_0 = 1$.

Our identification procedure is based on the notion of preference for diversification. There are several notions of preference for diversification (see De Giorgi and Mahmoud 2016). We consider the one introduced by Dekel (1989) and extended later by Chateauneuf and Tallon (2002) and Chateauneuf and Lakhnati (2007) to the space of random variables. Its definition is as follows.

Definition 4 (Preference for diversification)**.** *The preference relation $\succeq$ exhibits preference for diversification if for any $R_i \in \mathcal{R}$ and $\alpha_i \in [0,1]$, $i = 1, \ldots, N$ such that $\sum_{i=1}^N \alpha_i = 1$,*

$$R_1 \sim R_2 \sim \ldots \sim R_N \Longrightarrow \sum_{i=1}^N \alpha_i R_i \succeq R_j \quad \textit{for each } j = 1, \ldots, N. \tag{21}$$

Definition 4 states that if assets are equally desirable, then the investor will want to diversify. A RDEU investor exhibits preference for diversification as defined in Definition 4 if and only if $u(\cdot)$ and $h(\cdot)$ are both concave (see De Giorgi and Mahmoud 2016, Proposition 13). Thus, we limit ourselves to the case where $u(\cdot)$ and $h(\cdot)$ are concave.

Let $\varrho_C(R) = -C(R)$ be a risk measure induced by the certainty equivalent $C(\cdot)$ of $U(\cdot)$. Definition 4 implies that the following equivalent conditions are satisfied

$$R_1 \sim R_2 \sim \ldots \sim R_N \Longrightarrow U\left(1 + \sum_{i=1}^{N} \alpha_i\, R_i\right) \geq U(1 + R_j) \quad \text{for each } j = 1, \ldots, N; \tag{22}$$

$$R_1 \sim R_2 \sim \ldots \sim R_N \Longrightarrow \varrho_C\left(1 + \sum_{i=1}^{N} \alpha_i\, R_i\right) \leq \varrho_C(1 + R_j) \quad \text{for each } j = 1, \ldots, N; \tag{23}$$

Because diversification is a risk reduction tool, we focus on (23). Multiplying inequalities in (23) by α_j and summing over j, we obtain

$$\varrho_C\left(1 + \sum_{i=1}^{N} \alpha_i\, R_i\right) \leq \sum_{i=1}^{N} \alpha_j \varrho_l(1 + R_j). \tag{24}$$

From (24), in general (including when assets are not equally desirable) and in line with Embrechts et al. (2009) (see Equation (11)), the gain of diversification in RDEU theory can be measured by the difference

$$D_C(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} w_i \varrho_C(1 + R_i) - \varrho_C\left(1 + \sum_{i=1}^{N} w_i\, R_i\right). \tag{25}$$

The correlation diversification measure induced by RDEU is therefore $D_C(\mathbf{w}|\mathbf{R})$.

3.2. Test

We now test the identified measure $D_C(\mathbf{w}|\mathbf{R})$ against our axioms. Specifically, we test whether there exists a concave utility function and a concave distortion function such that $D_C(\mathbf{w}|\mathbf{R})$ satisfies our axioms. If this is the case, we consider that our axioms are compatible with RDEU theory. We analyze two cases:

(1) $u(\cdot)$ is nonlinear and the distribution of asset returns belongs to the location-scale family. In this case, RDEU theory reduces to mean-variance models including Markowitz's (1952) specification.

(2) $u(\cdot)$ is linear and $h(\cdot)$ is nonlinear. In this case, RDEU theory reduces to Yaari's (1987) dual theory.

3.2.1. $u(\cdot)$ Nonlinear and Location-Scale Family of Distributions

We assume that $u(\cdot)$ is nonlinear and the distribution of asset returns belongs to the location-scale family. This family of distributions includes the normal distribution, the student's t-distribution and all other elliptical distributions. For more details see Meyer (1987). Thus, from Konrad (1993),

$$U(1 + R) = \int_{-\infty}^{\infty} u(1 + \mu + \sigma\hat{r}) dh\big(F_{\widehat{R}}(\hat{r})\big), \tag{26}$$

where $\mu = \mathrm{E}(R)$, $\sigma = \sqrt{\mathrm{Var}(R)}$ and $\widehat{R} = \frac{R-\mu}{\sigma}$. The certainty equivalent $C(\cdot)$ of $U(\cdot)$ is $C(\mu, \sigma) \equiv C(1 + R) = u^{-1}(U(1 + R))$, where $u^{-1}(\cdot)$ is the inverse of $u(\cdot)$. The induced correlation diversification measure becomes

$$D_C(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} w_i C(\mu_i, \sigma_i) - C(\mu(\mathbf{w}), \sigma(\mathbf{w})), \tag{27}$$

where $\mu(\mathbf{w})$ is the expected return on the portfolio and $\sigma(\mathbf{w})$ its volatility.

In Proposition 1, we test $D_C(\mathbf{w}|\mathbf{R})$ in (27) against our axioms.

Proposition 1 (Test: $u(\cdot)$ nonlinear and location-scale family of distributions)**.** *If $u(\cdot)$ is nonlinear and the distribution of asset returns belongs to the location-scale family, then $D_C(\mathbf{w}|\mathbf{R})$ satisfies Axioms 1–8 if the certainty equivalent of $U(\cdot)$ has the following additive separable form in μ and σ*

$$C(\mu,\sigma) = a + b\left(\mu - \tau\sigma^2\right) \tag{28}$$

with $\tau > 0$, $a, b \in \mathbb{R}$ and $b \neq 0$.

From Proposition 1, we therefore have the following result in terms of compatibility with RDEU theory.

Corollary 1 (Compatibility: $u(\cdot)$ nonlinear, location-scale family of distributions, and additive separable certainty equivalent)**.** *Axioms 1–8 are compatible with RDEU theory if $u(\cdot)$ is nonlinear, the distribution of asset returns belongs to the location-scale family and the certainty equivalent of $U(\cdot)$ has the additive separable form in (28).*

Below, we present an example of location-scale distribution, $u(\cdot)$ and $h(\cdot)$ for which Proposition 1 is valid.

Example 5 ($h(x) = \frac{\int_0^x \exp(-\beta F_R^{-1}(t))dt}{\int_0^1 \exp(-\beta F_R^{-1}(t))dt}$, normal distribution and $u(\cdot)$ negative exponential utility)**.** *Assume that*

(i) $u(\cdot)$, $h(\cdot)$ and $F_R(\cdot)$ are twice continuously differentiable. Then, $U(1+R)$ can be written as an expected utility

$$U(1+R) = \int_{-\infty}^{\infty} u(1+r)h'(F_R(r))d(F_R(r)) = \mathrm{E}(v(1+R)) \tag{29}$$

with $v(1+r) = u(1+r)h'(F_R(r))$;

(ii) $F_R(\cdot)$ is strictly increasing;

(iii) $u(\cdot)$ is the negative exponential utility function: $u(x) = -\exp(-\lambda x)$ with $\lambda > 0$ the coefficient of risk aversion;

(iv) $h(x) = \frac{\int_0^x \exp(-\beta F_R^{-1}(t))dt}{\int_0^1 \exp(-\beta F_R^{-1}(t))dt}$ with $\beta \geq 0$ a parameter capturing the degree of concavity of $h(x)$;

(v) $\mathbf{w}^\top\mathbf{R}$ is a continuous normal random variable.

In the case where $\beta = 0$, $h(x) = x$ and we obtain the standard expected utility theory with

$$C(1+\mathbf{w}^\top\mathbf{R}) = 1 + \left(\mathbf{w}^\top\boldsymbol{\mu} - \frac{\lambda}{2}\sigma^2(\mathbf{w})\right), \tag{30}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)^\top$ is the vector of expected return on assets.

Note that Proposition 1 also implies that Axioms 1–8 are compatible with Markowitz's (1952) mean-variance model.

3.2.2. $u(\cdot)$ Linear and $h(\cdot)$ Nonlinear: Yaari's (1987) Dual Utility Theory

Now, we assume that $u(\cdot)$ linear and $h(\cdot)$ nonlinear. In this case, RDEU theory reduces to Yaari's (1987) dual utility theory and the certainty equivalent $C(\cdot)$ is $U(\cdot)$ itself (see Yaari 1987, p. 101). The induced correlation diversification measure becomes

$$D_C(\mathbf{w}|\mathbf{R}) = U(1+\mathbf{w}^\top\mathbf{R}) - \sum_{i=1}^{N} w_i U(1+R_i). \tag{31}$$

In Proposition 2, we test $D_C(\mathbf{w}|\mathbf{R})$ in (31) against our axioms.

Proposition 2 (Test: $u(\cdot)$ linear and $h(\cdot)$ nonlinear)**.** *If $u(\cdot)$ linear and $h(\cdot)$ nonlinear, then $D_C(\mathbf{w}|\mathbf{R})$ satisfies Axioms 1–8 if $h(\cdot)$ is concave.*

From Proposition 2, we therefore have the following result in terms of compatibility with RDEU theory.

Corollary 2 (Compatibility: $u(\cdot)$ linear and $h(\cdot)$ nonlinear concave)**.** *Axioms 1–8 are compatible with RDEU theory if $u(\cdot)$ is linear and $h(\cdot)$ is nonlinear and concave.*

Corollaries 1 and 2 demonstrate the compatibility of our axioms with RDEU theory under some conditions. The conditions in Corollary 1 might be thought to be restrictive, thereby considerably weakening the desirability of Axioms 1–8, because they require that risk must be measured by variance. However, this is not the case because the majority of axioms remain compatible with expected utility theory ($u(\cdot)$ nonlinear and $h(\cdot)$ linear) when we consider other standard utility functions and/or a non-location-scale family of distributions. For example, consider the negative exponential utility with a non-location-scale family of distributions and assume that $h(\cdot)$ is nonlinear concave and the risk premium is measured using the distortion-exponential principle. The results in the study by Tsanakas and Desli (2003, Section 5, p. 978) show that, in general, $D_C(\mathbf{w}|\mathbf{R})$ fails only Axioms 4 and 7, and will satisfy Axioms 1–8 for small portfolios (small in terms of size) of risks. In sum, our axioms capture important aspects of RDEU investors' preference for diversification.

4. Existing Diversification Measures

In this section, we explore whether some useful methods of measuring correlation diversification satisfy our axioms. We consider the two most frequently used methods on the marketplace and by academic researchers in portfolio theory:

(i) Portfolio variance

$$\sigma^2(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \boldsymbol{\Sigma}\, \mathbf{w}, \tag{32}$$

where $\boldsymbol{\Sigma} = (\sigma_i \sigma_j \rho_{ij})_{i,j=1}^N$ is the variance-covariance matrix.

(ii) Diversification ratio (DR)

$$\mathrm{DR}(\mathbf{w}|\mathbf{R}) = \frac{\mathbf{w}^\top \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma}\, \mathbf{w}}}, \tag{33}$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)^\top$ is the vector of asset volatility.

Portfolio variance, $\sigma^2(\mathbf{w}|\mathbf{R})$, is the risk measure in the mean-variance model. It is usually used to quantify the benefit of diversification (Markowitz 1952, 1959; Sharpe 1964), and is formally analyzed as a correlation diversification measure by Frahm and Wiechers (2013).

The diversification ratio, $\mathrm{DR}(\mathbf{w}|\mathbf{R})$, is a normalized correlation diversification measure introduced by Choueifaty and Coignard (2008); see also Choueifaty et al. (2013). An intuitive interpretation of the DR is the Sharpe ratio when each asset's volatility is proportional to its expected premium, i.e., $\mu_i - \mu_{N+1} = \delta\sigma_i$, for each $i = 1, \ldots, N$ where $\delta > 0$ and μ_{N+1} is the return on the risk-free asset. We remark that a general version of $\mathrm{DR}(\mathbf{w}|\mathbf{R})$, in terms of risk measure, was analyzed by Tasche (2006) and recently by Han et al. (2022).

Proposition 3 presents the results of our exploration.

Proposition 3 (Test: Existing diversification measures)**.** *The following statements hold.*

(i) Portfolio variance satisfies Axioms 1–8 if and only if all assets have the same volatility.
(ii) The diversification ratio satisfies Axioms 1′–8.

Part (i) of Proposition 3 shows that portfolio variance is a coherent correlation diversification measure only under the very restrictive (if not impossible) condition that all assets have identical variances. This result, rather than weakening our axioms, reveals the limits of portfolio variance as an adequate measure of correlation diversification in the mean-variance model.[13]

Part (ii) of Proposition 3 shows that the diversification ratio is a coherent correlation diversification measure unconditionally. Since the diversification ratio coincides with the Sharpe ratio if and only if each asset's volatility is proportional to its expected premium, part (ii) also establishes that the Sharpe ratio is a coherent correlation diversification measure if and only if each asset's volatility is proportional to its expected premium. This result is summarized in Corollary 3.

Corollary 3 (Sharpe ratio)**.** *The Sharpe ratio is a coherent correlation diversification measure if and only if each asset's volatility is proportional to its expected premium.*

In sum, Proposition 3 strengthens our axioms. They are relevant to two measures with which we have considerable experience. In turn, several desirability properties of these two measures can be defended by our axioms. This is also true for their popular use.

We remark that Embrechts et al.'s (2009) class of correlation diversification measures defined in Example 3 as

$$D(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} \varrho(w_i R_i) - \varrho\left(\mathbf{w}^\top \mathbf{R}\right) \tag{34}$$

also satisfies most of our axioms depending on its underlying risk measure $\varrho(\cdot)$. For example, in the case where $\varrho(\cdot)$ is the Value at Risk, one can verify that $D(\mathbf{w}|\mathbf{R})$ satisfies Axioms 2, 3 and 5 to 8 and fails Axiom 1 and its weak version 1′ because the Value at Risk is not a subadditive risk measure. In the case where $\varrho(\cdot)$ is the Conditional Value-at-Risk or the semi-variance, one can verify that $D(\mathbf{w}|\mathbf{R})$ satisfies Axioms 1 to 3 and 5 to 8. In addition, if $D(\mathbf{w}|\mathbf{R})$ satisfies Axiom 4, which is to be demonstrated, then $D(\mathbf{w}|\mathbf{R})$ is coherent. Embrechts et al.'s (2009) class of correlation diversification measures is therefore a way to construct coherent correlation diversification measures. The challenge is the choice of its underlying risk measure. In the next section, throughout an example of a functional representation of our axioms, we provide an alternative way to construct coherent correlation diversification measures.

5. Example of a Functional Representation

We close this paper by providing an example of a functional representation of our axioms. Indeed, our axioms are not restrictive enough, as materialized by the weakest form of Axioms 1′, 4 and 7, to specify a unique family of correlation diversification measures. This incompleteness is intentional. It allows our axioms to be used for a large family of correlation diversification measures.

Consider the function $d : \mathbb{R}^{2N} \to \mathbb{R}$ that satisfies the following properties:

(P1) $d(\mathbf{w}, \mathbf{R})$ is concave in $\mathbf{w}$ for each fixed $\mathbf{R} \in \mathcal{R}^N$;
(P2) $d(\mathbf{\Pi w}, \mathbf{\Pi R}) = d(\mathbf{w}, \mathbf{R})$ for all permutation matrices $\mathbf{\Pi}$;
(P3) $d(\mathbf{w}, \mathbf{R})$ is Borel-measurable in $\mathbf{R}$ for each fixed $\mathbf{w}$.

The result in Marshall et al. (2011, B.1. Proposition, p. 393) implies that when the sequence $R_1, \ldots, R_N$ is exchangeable, the expectation of $d(\cdot,\cdot)$, $\mathrm{E}(d(\mathbf{w}, \mathbf{R}))$, satisfies Axioms 1 and 8. If in addition $d(\mathbf{w}, \mathbf{R})$ satisfies Axioms 2–7, then $\mathrm{E}(d(\mathbf{w}, \mathbf{R}))$ satisfies Axioms 1–8. As a result, a possible functional representation of our axioms is

$$D(\mathbf{w}|\mathbf{R}) = \mathrm{E}(d(\mathbf{w}, \mathbf{R})), \tag{35}$$

where $d(\mathbf{w}, \mathbf{R})$ satisfies Axioms 2–7 and properties (P1) to (P3).

The functional representation in (35) is an alternative way to construct new coherent correlation diversification measures. The challenge is the definition of $d(\mathbf{w}, \mathbf{R})$. In Example 6, we present an example of $d(\mathbf{w}, \mathbf{R})$.

Example 6 (Diversification returns)**.** *Consider $d(\mathbf{w}, \mathbf{R})$ such that*

$$d(\mathbf{w}, \mathbf{R}) = \sum_{i=1}^{N} w_i (R_i - \mu_i)^2 - \sum_{i,j=1}^{N} w_i w_j (R_i - \mu_i)(R_j - \mu_j). \tag{36}$$

Obviously, $d(\mathbf{w}, \mathbf{R})$ satisfies properties (P1) to (P3), and Axioms 2–7. As a consequence, $D(\mathbf{w}|\mathbf{R}) = \mathrm{E}(d(\mathbf{w}, \mathbf{R}))$ is coherent (see Appendix A.1 for the formal proof) and known as the diversification returns, a popular correlation diversification measure analyzed by Willenbrock (2011), Chambers and Zdanowicz (2014), Bouchey et al. (2012), Qian (2012), and by Fernholz (2010) but under the name excess growth rate.

Now, in Counterexample 1, we present a counterexample of the functional representation in (35) to prove its non-uniqueness.

Counterexample 1 (Diversification Ratio)**.** *Consider $D(\mathbf{w}|\mathbf{R})$ such that*

$$D(\mathbf{w}|\mathbf{R}) = \mathrm{DR}(\mathbf{w}|\mathbf{R}). \tag{37}$$

Part (ii) of Proposition 3 shows that $D(\mathbf{w}|\mathbf{R})$ in (37) satisfies our axioms. However, it is straightforward to verify that it does not have the form in (35).

6. Concluding Remarks and Future Research

In this paper, we have developed an axiomatic system of eight axioms for correlation diversification measures in a one-period portfolio theory under the assumption of complete information about the marginal and the joint distributions of assets' future returns. We have considered as coherent any correlation diversification measure satisfying these axioms.

Using rank-dependent expected utility theory, we have demonstrated the compatibility of our axioms with economic investors' preference for diversification under unrestricted conditions related to investors' risk aversion. These results strengthen the economic desirability, reasonableness and relevance of our axioms.

We have also explored whether correlation diversification measures such as portfolio variance and diversification ratio, which are used in the academic literature and on the marketplace, satisfy those axioms. We have shown that this is the case under unrestricted conditions, except for portfolio variance. These results strengthen both the axioms and the diversification ratio. However, they reveal the limit of portfolio variance as an adequate measure of correlation diversification in mean-variance models.

We have also extended the result of the diversification ratio to the Sharpe ratio under very restrictive conditions. This reveals the limit of the Sharpe ratio as an adequate measure of correlation diversification in the mean-variance model.

Finally, due to the intentional incompleteness of our axioms, we have provided an example of a functional representation of our axioms.

Our objective was to offer a first step toward a theory of correlation diversification measures. We believe that, with our axiomatic system, research is going in the right direction for better understanding portfolio diversification. Feasible and desirable extensions for future research are:

(i) To re-examine the compatibility of our axioms considering other risk measures in rank-dependent expected utility theory, such as the optimal expected utility risk measures in the study by Geissel et al. (2018) and the extreme risk aggregation approach in the study by Chen and Hu (2019);
(ii) To extend the compatibility of our axioms to *cumulative prospect theory*;

(iii) To investigate what axioms could be added or strengthened in order to provide a unique family of representations, given that our axiomatic system does not do this;
(iv) To develop more empirical research on portfolio correlation diversification.

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Appendix A. Proofs

Appendix A.1. Proposition 1

Suppose that the certainty equivalent of $U(\cdot)$ has the following additive separable form

$$C(\mu,\sigma) = a + b\left(\mu - \tau\sigma^2\right) \tag{A1}$$

with $\tau > 0$, $a, b \in \mathbb{R}$ and $b \neq 0$. Then

$$D_C(\mathbf{w}|\mathbf{R}) = b\tau \underbrace{\left(\sum_{i=1}^{N} w_i\sigma_i^2 - \sigma^2(\mathbf{w})\right)}_{\text{Diversification returns}}. \tag{A2}$$

Without a loss of generality we assume that $\tau = b = 1$.

Axiom 1: Since $\sigma^2(\mathbf{w})$ is convex on $\mathbb{W}$, $D_C(\mathbf{w}|\mathbf{R})$ is concave on $\mathbb{W}$.

Axiom 2: It is straightforward to verify that $D_C(\mathbf{e}_i|\mathbf{R}) = \sigma_i^2 - \sigma_i^2 = 0 = \underline{D}$, for each $i = 1,\ldots,N$.

Axiom 3: Since $B_{N+1} = A_k$ with $k \in \{1,\ldots,N\}$ and $B_i = A_i$ for each $i = 1,\ldots,N$,

$$\begin{aligned} D_C(\mathbf{w}_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}) &= \sum_{i=1}^{N+1} w_{B_i}\sigma_{B_i}^2 - \sum_{i,j=1}^{N+1} w_{B_i}w_{B_j}\rho_{B_iB_j}\sigma_{B_i}\sigma_{B_j} \\ &= \sum_{i=1,i\neq k}^{N-1} w_{B_i}\sigma_{B_i}^2 + \left(w_{B_{N+1}} + w_{B_k}\right)\sigma_{B_k}^2 \\ &\quad - \sum_{i,j=1,i,j\neq k}^{N-2} w_{B_i}w_{B_j}\rho_{B_iB_j}\sigma_{B_i}\sigma_{B_j} - \sum_{i=1}^{N-2} w_{B_i}\left(w_{B_k} + w_{B_{N+1}}\right)\rho_{B_iB_k}\sigma_{B_i}\sigma_{B_k}. \end{aligned}$$

Let $\mathbf{w}_{\mathcal{A}}^{**} = \left(w_{B_1}^*,\ldots,w_{B_{k-1}}^*, w_{B_k}^* + w_{B_{N+1}}^*, w_{B_{k+1}}^*,\ldots,w_{B_N}^*\right)^\top$ and $\mathbf{w}_{\mathcal{B}}^{**} = \left(w_{A_1}^*,\ldots,w_{A_{k-1}}^*, \frac{w_{A_k}^*}{2}, w_{A_{k+1}}^*,\ldots,\frac{w_{A_k}^*}{2}\right)^\top$. It follows that

$$D_C(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}) = D_C(\mathbf{w}^{**}_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}) \leq D_C(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}),$$
$$D_C(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}) = D_C(\mathbf{w}^{**}_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}) \leq D_C(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}).$$

Then

$$\begin{aligned} D_C(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}) &= D_C(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}) \\ w^*_{A_i} &= w^*_{B_i}, \text{ for each } i \neq k,\ i = 1, \ldots, N \\ w^*_{A_k} &= w^*_{B_k} + w^*_{B_{N+1}}. \end{aligned}$$

Axiom 4: Since $w_i \geq 0$, for each $i = 1, \ldots, N$ and $D_C(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i \| R_i - \mathbf{w}^\top \mathbf{R}\|_2^2$, $D_C(\mathbf{w}|\mathbf{R}) = 0 \Leftrightarrow R_i = \mathbf{w}^\top \mathbf{R}$, for each $i = 1, \ldots, N$. The result follows.

Axiom 5: Consider a portfolio $\mathbf{w}_{\mathcal{B}} = \left(\mathbf{w}^*_{\mathcal{A}}, \mathbf{0}\right)^\top$ with $\mathbf{0} = (0, \ldots, 0)^\top$, where the length of $\mathbf{0}$ is equal to the cardinal of $\mathcal{B}$ minus that of $\mathcal{A}$. Since $D_C\left(\mathbf{w}^*_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}\right) \geq D_C\left(\mathbf{w}_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}\right)$ and $D_C\left(\mathbf{w}_{\mathcal{B}}|\mathbf{R}_{\mathcal{B}}\right) = D_C\left(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\right)$, $D_C\left(\mathbf{w}^*_{\mathcal{B}}|\ \mathbf{R}_{\mathcal{B}}\right) \geq D_C\left(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\right)$.

Axiom 6: Because covariance is translation invariant.

Axiom 7: Because covariance is homogeneous of degree two.

Axiom 8: Since $\sigma_i = \sigma_j = \underline{\sigma}$ for each $i, j = 1, \ldots, N$ and $\rho_{ij} = \underline{\rho}$ for each $i \neq j = 1, \ldots, N$ when $R_1, \ldots, R_N$ is exchangeable, $D_C(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \underline{\sigma}^2 \left(\sum_{i=1}^N w_i^2 + \underline{\rho} \sum_{i,j=1}^N w_i w_j\right)$. It is straightforward to verify that $D_C(\mathbf{w}|\mathbf{R})$ is symmetric.

Appendix A.2. Proposition 2

Suppose that $h(.)$ is concave and let us show that $D_C(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

Axiom 1: Since $h(.)$ is concave, $-U(.)$ is convex on $\mathcal{R}$ (Dhaene et al. 2006; Sereda et al. 2010; Tsanakas and Desli 2003) and consequently, $D_C(\mathbf{w}|\mathbf{R})$ is concave.

Axiom 2: Let $\mathbf{e}_i \in \mathbb{W}$ be a single-asset i portfolio. It is straightforward to show that $D_C(\mathbf{e}_i|\mathbf{R}) = U(1+R_i) - U(1+R_i) = 0 = \underline{D}$.

Axiom 3: Follows the proof of Proposition 1.

Axiom 4: Since $-U(.)$ is coherent risk measure, comonotonic and non-independent additive (Dhaene et al. 2006; Sereda et al. 2010; Tsanakas and Desli 2003), $D_C(\mathbf{w}|\mathbf{R})$ satisfies Axiom 4.

Axiom 5: Follows the proof of Proposition 1.

Axiom 6: Since $h(.)$ is concave, $-U(.)$ is translation invariant (Dhaene et al. 2006; Sereda et al. 2010; Tsanakas and Desli 2003). Therefore $D_C(\mathbf{w}|\mathbf{R})$ is translation invariant.

Axiom 7: Since $h(.)$ is concave, $-U(.)$ is positive homogeneous of degree one (Dhaene et al. 2006; Sereda et al. 2010; Tsanakas and Desli 2003). Therefore $D_C(\mathbf{w}|\mathbf{R})$ is homogeneous of degree one.

Axiom 8: Suppose that $R_1, \ldots, R_N$ is exchangeable. From Marshall et al. (2011, B.2. Proposition, pp. 394), it is straightforward to verify that $D_C(\mathbf{w}|\mathbf{R})$ is symmetric.

Appendix A.3. Proposition 3

Appendix A.3.1. Portfolio Variance

Sufficiency

Suppose that assets have identical variances and let us show that portfolio variance satisfies our axioms. It is straightforward to verify that if assets have identical variances, i.e., $\sigma_i^2 = \underline{\sigma}^2$, then

$$\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}), \tag{A3}$$

where $\boldsymbol{\sigma}^2 = (\sigma_1^2, \ldots, \sigma_N^2)^\top$ is the vector of asset variance. From (A3) and Proposition 1, it follows that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

Necessity

For the converse, suppose that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms and let us show that assets have identical variances. To do so, we proceed by contradiction. Suppose that asset do not have identical variances and without the loss of generality that $N = 2$ such that $\sigma_1^2 < \sigma_2^2$. Then $\sigma^2(\mathbf{e}_1|\mathbf{R}) < \sigma^2(\mathbf{e}_2|\mathbf{R})$. Thus, $\sigma^2(\mathbf{w}|\mathbf{R})$ fails Axiom 2. This contradicts our hypothesis that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms. As a consequence, if $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then assets have identical variances.

Appendix A.3.2. Diversification Ratio

Axiom 1′: Since $\sigma(\mathbf{w})$ is convex and $\sum_{i=1}^{N} w_i\sigma_i$ is linear on $\mathbb{W}$, from Avriel et al. (2010), $\mathrm{DR}(\mathbf{w}|\mathbf{R})$ is quasi-concave.
Axiom 2: $\mathrm{DR}(\mathbf{e}_i|\mathbf{R}) = \frac{\sigma_i}{\sigma_i} = 1$, for each $i = 1, \dots, N$.
Axiom 3: Follows the proof of Proposition 1.
Axiom 4: See Example 4.
Axiom 5: Follows the proof of Proposition 1.
Axiom 6: Because volatility is translation invariant.
Axiom 7: Because volatility is homogeneous of degree one.
Axiom 8: Follows the proof of Proposition 1.

Notes

1. Correlation aversion was introduced as a term by Epstein and Tanny (1980) and as a concept by Richard (1975) but under the name *multivariate risk aversion*, and was popularized by Eeckhoudt et al. (2007). In two-attribute utility theory, Eeckhoudt et al. (2007) define correlation aversion as follows: a decision maker is correlation averse if he/she prefers the $50-50$ lottery $[(x-k, y); (x, y-c)]$ to the $50-50$ lottery $[(x, y); (x-k, y-c)]$ for all $(x, y) \in \mathbb{R}_+^2$ such that $x-k > 0$ and $y-c > 0$ with $k \geq 0$ and $c \geq 0$. Richard (1975)'s definition is based on the second-order mixed partial derivatives of the two-attribute utility function: a decision maker is correlation averse if the second-order mixed partial derivative of his/her two-attribute utility function is negative. The equivalence between the two definitions can be found in Eeckhoudt et al. (2007); see also Dorfleitner and Krapp (2007).
2. Several other studies also demonstrated the important role of asset dependence in portfolio diversification with expected utility theory; see Samuelson (1967), Scheffman (1973 1975), Brumelle (1974), MacMinn (1984) and Wright (1987).
3. We refer readers to Koumou (2020a) for a review of existing measures of correlation diversification.
4. For example, in Artzner et al.'s (1999) monetary risk measurement theory, correlation diversification is taken into account through the properties of *sub-additivity* and *homogeneity*. In Föllmer and Weber's (2015) monetary risk measurement theory, correlation diversification is taken into account through the property of *convexity*. In concave distortion risk measures, correlation diversification is taken into account through the properties of *comonotonic additivity* and *sub-additivity* (Dhaene et al. 2006).
5. For example, in Artzner et al.'s (1999) and Föllmer and Weber's (2015) monetary risk measurement theories, the possibility of reducing risks by concentration is taken into account through the property of *monotonicity*, and is as important as diversification.
6. To illustrate, without loss of generality, consider the variance risk measure and an universe of four assets A_1, A_2, A_3 and A_4. Assume that $(\sigma_1 + \sigma_2)^2 < \sigma_3^2 + \sigma_4^2$ with σ_i^2 the variance of A_i, that the correlation between assets A_1 and A_2 is equal to one ($\rho_{12} = 1$), and that the correlation between assets A_3 and A_4 is equal to zero ($\rho_{34} = 0$). It is easy to verify that the variance of the portfolio $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$ is lower than the variance of the portfolio $\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$. However, portfolio $\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$ is more diversified in terms of correlation diversification than portfolio $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$. Thus, the less risky portfolio is not the more correlation diversified portfolio. As a result, variance risk measure is not an adequate correlation diversification measure.
Now, assume that the four assets have the same variance σ^2. In this case, it is straightforward to verify that the variance of portfolio $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$ is greater than the variance of portfolio $\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$. Thus, the less risky portfolio is the more correlation diversified portfolio. As a result, variance risk measure is an adequate correlation diversification measure, in this example.
7. Strong risk aversion is equivalent to risk aversion in the sense of a mean-preserving spread as defined by De Giorgi and Mahmoud (2016, Definition 9, p. 152).
8. More precisely, $\mathcal{R} = \mathbb{L}^\infty(\Omega, \mathcal{E}, P)$ is the space of bounded real-valued random variables on a probability space $(\Omega, \mathcal{E}, P)$, where Ω is the set of states of nature, $\mathcal{E}$ is the sigma-algebra of events, and P is a sigma-additive probability measure on $(\Omega, \mathcal{E})$.
9. A risk measure $\varrho(.)$ is additive for independent risks if for independent $X, Y \in \mathcal{R}$, $\varrho(X+Y) = \varrho(X) + \varrho(Y)$ (Sereda et al. 2010). An example of a risk measure additive for independence is the *mixed Esscher premium* or the *mixed exponential premium* analyzed by Goovaerts et al. (2004).

[10] A dependence measure ρ_c is invariant if $\rho_c(X_1, \ldots, X_N) = \rho_c(I_1(X_1), \ldots, I_N(X_N))$ for strictly increasing and continuous transformations I_i (Schmid et al. 2010, p. 213).

[11] A dependence measure ρ_c is symmetric if $\rho_c(X_1, \ldots, X_N) = \rho_c(-X_1, \ldots, -X_N)$ (Schmid et al. 2010, p. 213, Equation 10.5).

[12] The random variables $R_1, \ldots, R_N$ are said to be exchangeable if and only if their joint distribution $F_{\mathbf{R}}(\mathbf{r})$ is symmetric. A well-known example of an exchangeable sequence of random variables is an independent and identically distributed sequence of random variables. For more details on exchangeable random variables, we refer readers to Aldous (1985).

[13] The measure of correlation diversification at the core of the mean-variance model was identified by Carmichael et al. (2022); see also Koumou (2020b).

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