



Article On the Stochastic Volatility in the Generalized Black-Scholes-Merton Model

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Abstract: This paper discusses the generalized Black-Scholes-Merton model, where the volatility coefficient, the drift coefficient of stocks, and the interest rate are time-dependent deterministic functions. Together with it, we make the assumption that the volatility, the drift, and the interest rate depend on a gamma or inverse-gamma random variable. This model includes the models of skew Student's t- and variance-gamma-distributed stock log-returns. The price of the European forward-start call option is derived from the considered models in closed form. The obtained formulas are compared with the Black-Scholes formula through examples.

Keywords: Black-Scholes formula; time-dependence; gamma distribution; inverse-gamma distribution; special function

1. Introduction

The standard Black-Scholes-Merton model (see Black and Scholes 1973; Merton 1973) assumes that the volatility coefficient is constant over the contract time. However, when the parameters of the model are calibrated with respect to the market option prices, one may find out that the volatility is a function of the exercise time and the strike coefficient. The dependence curve of the implied volatility of stock prices has the form of a convex function. This effect is called the "volatility smile". Taking it into account, Merton (1973) proposed to regard the drift and the volatility coefficients as functions of time. The problem of detecting the term structure of volatility is widely discussed in the literature.

Derman and Kani (1994) and Dupire (1994) suggested a general diffusion model for volatility under the hypothesis that it is a function of state and time. This model is called the local volatility model. Its multidimensional extensions were introduced and considered in the papers by Brigo and Mercurio (2002) and Brigo et al. (2003). Results in the local volatility model that are close to the Black-Scholes formula are given in the monograph by Oksendal (2003).

Diffusion stochastic volatility models infer that the stock price and the volatility coefficient are driven by correlated Brownian motions. A survey of the early papers in this direction is given in Section 7.4.2 of the monograph by Musiela and Rutkowski (2005). Among more recent papers, let us mention those where combined models are considered. The Heston-Hull-White models are studied by Grzelak and Oosterlee (2011), Levendis and Maré (2022) and Liu et al. (2023). The Heston-Cox-Ingersoll-Ross model is discussed in Cao et al. (2016) and Mao et al. (2022).

Modern research uncovers a more complicated term structure for volatility. Nonparametric methods of volatility assessment are given in Alghalith et al. (2020), Cuchiero et al. (2020), and Liu et al. (2019) (neural network estimation). The fat-tailedness of the volatility is detected by Mahieu and Schotman (1998) and Kim et al. (1998). Jumps of finite activity of the stock and the stock index volatilities are discovered by Bates (1996), Andersen et al. (2007), Eraker et al. (2003), Taylor et al. (2018). The stochastic volatility modeled by the inverse-gamma distribution is investigated in Nakajima and Omori (2012), Liu (2021) and



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Copyright: © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). Men et al. (2021). Nakakita and Nakatsuma (2021) modeled the stochastic volatility by the both gamma and inverse-gamma distributions. Processes with Lévy jumps of infinite activity are used for the stochastic volatility modeling as well. Namely, the gamma process is considered in this direction by Nzokem (2023). The parameters of the distributions are estimated in these papers for a variety of financial indices using the maximum likelihood technique and the methods of Bayesian statistics.

Together with the volatility coefficient, the Black-Scholes-Merton model implies that the drift coefficient of the stock log-return is also constant. However, the financial data often suggests the presence of jumps in the log-return. Andersen et al. (2001) include in the model of stock log-returns jumps of the general form. Several sources discuss Lévy's jumps of finite activity. Shackleton et al. (2010) and Hong et al. (2023) augment the Heston and the lognormal Ho-Lee models with compound Poisson jumps. Chib et al. (2002) discuss stochastic volatility models with jumps whose distribution is generated by the Bernoulli distribution. Nakajima and Omori (2012) and Nakakita and Nakatsuma (2021) study the skew Student's t-distribution as a model for stock prices. Being the normal-inverse mixture with the inverse-gamma mixing density, the skew Student's t-distribution has the stochastic mean, which is modeled by the inverse-gamma distribution.

Moreover, processes with jumps of infinite activity are also discussed for asset drift modeling. The skew Student's t-process is investigated as a model for financial index dynamics in the form of subordinated Brownian motion with drift by Aas and Haff (2006), Bibby and Sørensen (2003), Finlay and Seneta (2006), and Finlay and Seneta (2008). Aas and Haff (2006) study the asymptotic properties of the skew Student's t-distribution and estimate its parameters in their application to currency exchange rate modeling. Bibby and Sørensen (2003) compare the properties of various generalized hyperbolic distributions, including the skew Student one, and propose again that the skew Student's t-process is a good fit for the exchange rate simulation. Finlay and Seneta (2006) approve the skew Student's t-process for the modeling of the S&P500 data. Finlay and Seneta (2008) develop different techniques for parameter estimation in the skew Student's t-model and discuss the modeling of the S&P500 index and the oil prices. The variance-gamma process (the variance-gamma distribution is the normal-inverse mixture with the gamma mixing density, and hence the variance-gamma process has the stochastic drift modeled by the gamma process) is considered in Madan et al. (1998), Seneta (2004), Daal and Madan (2005), Ivanov (2018), Ivanov (2022), Linders and Stassen (2016), and Mozumder et al. (2015), among others. Madan et al. (1998) summarize the basic properties of the variance-gamma distribution and suggest the method of receiving analytical results in the variance-gamma model. Daal and Madan (2005) confirm the use of the variance-gamma model for the exchange rate simulation. Linders and Stassen (2016) model with the variance-gamma process the Dow Jones index dynamics. Mozumder et al. (2015) study the S&P500 index options in the variance-gamma model. Ivanov (2022) proceeds from the ideas of Madan et al. (1998) and obtains closed-form results for an extension of the variance-gamma model.

The model of this paper relates to the generalized Black-Scholes-Merton model of Section 7.1.10 of Musiela and Rutkowski (2005) and Chapter 12.3 of Oksendal (2003). We assume that the drift and volatility of the stock depend on the gamma or the inverse-gamma distribution, which are independent of the Brownian motion. It appears that it is possible in this model to obtain the price of a forward-start call option in closed form. We exploit the methodology of integral transformations of special functions in their application to mathematical finance, which was introduced in Madan et al. (1998) and then developed in particular in Ano and Ivanov (2016). A comparison of the results with the Black-Scholes formula is given in the section on numerical examples.

2. Materials and Methods

In this section, we present a mathematical model in which we formulate our results. Together with it, necessary definitions and a theoretical background are given. Used designations are also included.

2.1. Model

Throughout the paper, we designate as $(S_t)_{t \le T}$ and $(R_t)_{t \le T}$ the stock price and the bank account value dynamics over a period of time [0, T]. Set $0 \le T_0 < T$. A forward-start option (see Section 6.2 of the monograph by Musiela and Rutkowski (2005)) is a contract in which the holder receives at time T_0 an option with expiry date T and exercise price KS_{T_0} with some strike coefficient K > 0. On the other hand, the holder must pay at time 0 an up-front fee, the price of the option. Thus, a forward-start call option has payoffs.

$$(S_T - KS_{T_0})^{-}$$

at the maturity *T*.

Next, we give a mathematical specification of the discussed model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)$ be the probability space with filtration $(\mathcal{F}_t)_{t \leq T}$. We assume that the Brownian motion $(B_t)_{t \leq T}$, $B_0 = 0$, and the random variable *z* are defined on this probability space and the both processes $(B_t)_{t < T}$ and $(z_t)_{t < T}$ with

$$z_t = I_{\{t < t_0\}} + zI_{\{t > t_0\}}, \qquad t_0 \ge 0, \tag{1}$$

are adapted to the filtration $(\mathcal{F}_t)_{t \leq T}$. It is supposed that the processes are independent with each other.

If $t_0 = 0$, the additional uncertainty that is generated by the random variable *z* relates to the "volatility smile" effect. When $t_0 > 0$, the additional uncertainty appears because of supplemental factors. There can be the economic annual reporting, the political shifts, the microeconomic merger decisions among those factors. Then t_0 is suggested to be larger than the expected time of the factor forthcoming. If, for example, we discuss the annual reporting factor, it is obviously fair to assume that the time t_0 is not random but actually fixed.

The generalized Black-Scholes-Merton model implies that the stock volatility is random and even not necessarily adapted to the filtration generated by the underlying Brownian motion (see Section 7.1.10 of Musiela and Rutkowski (2005)). We consider the generalized Black-Scholes-Merton model under the assumption that the logarithm of the stock price $(S_t)_{t < T}$ follows an equation.

$$\log(S_t/S_0) = \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\theta}_s z_s ds + \int_0^t \tilde{\sigma}_s \sqrt{z_s} dB_s,$$
(2)

where $(\tilde{\mu}_t)_{t \leq T}$, $(\tilde{\theta}_t)_{t \leq T}$ and $(\tilde{\sigma}_t)_{t \leq T}$ are time-dependent deterministic processes and $\tilde{\sigma}_t > 0$. From the economic point of view, the independence between z_t and B_t relates to the situation when the volatility randomness is induced by macroeconomic factors which are independent with the firm microstructure development.

The logarithm of the bank account value is suggested to evaluate via an equation

$$\log R_t = \int_0^t \tilde{r}_s ds + \int_0^t \tilde{\rho}_s z_s ds \tag{3}$$

with time-dependent deterministic $(\tilde{r}_t)_{t \leq T}$, $\tilde{r}_t \geq 0$, and $(\tilde{\rho}_t)_{t \leq T}$, $\tilde{\rho}_t \geq 0$. The new terms after the time t_0 take into account both the ordinary Black-Merton-Scholes uncertainty and the supplemental uncertainty, which is indicated by the random variable z. The model concerns a the small or average additional indeterminacy of financial markets.

Let the initial probability measure P be a martingale measure for the discounted stock price process $(S_t/R_t)_{t < T}$. That is, we assume that

$$\tilde{\mu}_t + \tilde{\theta}_t = \tilde{r}_t + \tilde{\rho}_t - \frac{\tilde{\sigma}_t^2}{2}, \qquad t \ge 0,$$
(4)

and additionally

$$\tilde{\mu}_t = \tilde{r}_t, \quad \tilde{\theta}_t = \tilde{\rho}_t - \frac{\tilde{\sigma}_t^2}{2}, \qquad t \ge t_0.$$
 (5)

Indeed, then

$$S_t/R_t = S_0 \exp\left(\int_0^t \tilde{\sigma}_s \sqrt{z_s} dB_s - \frac{1}{2} \int_0^t \tilde{\sigma}_s^2 z_s ds\right)$$

and hence the martingale property is satisfied. The problem of equivalent martingale change of measure for semimartingales is widely discussed in the literature. We refer, in particular, to Subsection III.3 of the monograph by Jacod and Shiryaev (1987) and to the papers by Kallsen and Shiryaev (2002) and Eberlein et al. (2009).

Example 1. If we set $t_0 = \infty$, then the model (2) and (3) becomes the diffusion Black-Scholes-Merton model (see Chapter 12.3 of Oksendal (2003) and Chapter VIII.1 of Shiryaev (1999)) with

$$\log\left(S_t^{BS}/S_0^{BS}\right) = \int_0^t \left(\tilde{\mu}_s + \tilde{\theta}_s\right) ds + \int_0^t \tilde{\sigma}_s dB_s \tag{6}$$

and

$$\log R_t^{BS} = \int_0^t (\tilde{r}_s + \tilde{\rho}_s) ds.$$
(7)

Example 2. Let $t_0 = 0$. Then

$$\log(S_1/S_0) = \int_0^1 \tilde{\mu}_s ds + z \int_0^1 \tilde{\theta}_s ds + \sqrt{z} \int_0^1 \tilde{\sigma}_s dB_s$$
(8)

and therefore the model (2) can be discussed as an extension of the variance-gamma and the skew Student's t-models in the sense of the stock log-return behavior. Indeed, if we set $\tilde{\mu}_t \equiv \mu$, $\tilde{\theta}_t \equiv \theta$ and $\tilde{\sigma}_t \equiv \sigma$ in (8), then

$$\log(S_1/S_0) = \mu + \theta z + \sigma \sqrt{z}N,\tag{9}$$

where N = N(0, 1) is the standard normal random variable. Hence the log-return (9) is variancegamma distributed if z is gamma distributed and skew Student's t-distributed if z is inverse-gamma distributed.

Due to the initial probability measure is assumed to be martingale for the discounted stock price, the risk-neutral forward-start call option price \mathbb{C}^{FS} can be computed as

$$\mathbb{C}^{FS} = \mathbb{E}\Big(R_T^{-1}(S_T - KS_{T_0})^+\Big).$$

We use in the subsequent sections the designations S_t^G and S_t^{IG} for the stock prices in the gamma and inverse-gamma models. If $T_0 = 0$, then a forward-start call option becomes a standard European call one with the price

$$\mathbb{C} = \mathrm{E}\Big(R_T^{-1}(S_T - KS_0)^+\Big).$$

2.2. Special Functions

Let us define a complementary function $\mathfrak{I} = \mathfrak{I}(a, b, u_1, u_2, u_3)$ for a > 0, b > 0, $u_1, u_2, u_3 \in \mathbb{R}$ by the identity

$$\begin{split} \Im(a,b,u_{1},u_{2},u_{3}) &= \frac{\Gamma\left(a+\frac{1}{2}\right)}{\sqrt{2\pi}(b+u_{1})^{a}} \left[\frac{B\left(\frac{1}{2},a\right)}{\sqrt{2}} + \right. \\ &+ \frac{u_{2}}{\sqrt{b+u_{1}}} F\left(a+\frac{1}{2},\frac{1}{2},\frac{3}{2};-\frac{u_{2}^{2}}{2(b+u_{1})}\right) \right] I_{\{u_{3}=0\}} + \\ &+ \frac{|s|^{a-\frac{1}{2}}e^{s}(1+q)^{a}}{\sqrt{2\pi}(b+u_{1})^{a}} \left[B(a,1)\left(|s|K_{a+\frac{1}{2}}(|s|)+sK_{a-\frac{1}{2}}(|s|)\right) \times \right. \\ &\times \Phi_{1}\left(a,1-a,a+1;\frac{1+q}{2},-s(1+q)\right) - (1+q)sB(a+1,1) \times \\ &\times K_{a-\frac{1}{2}}(|s|)\Phi_{1}\left(a+1,1-a,a+2;\frac{1+q}{2},-s(1+q)\right) \right] I_{\{u_{3}\neq0\}} \end{split}$$
(10)

with

$$q = q(u_1, u_2) = \frac{u_2}{\sqrt{u_2^2 + 2(b + u_1)}}$$

and

$$s = s(u_1, u_2, u_3) = u_3 \sqrt{u_2^2 + 2(b + u_1)}.$$

As it is shown further in the paper, this function determines the forward-start option price in the both gamma and inverse-gamma volatility models. In (10), we denote as

$$\Gamma(u)$$
, $B(u_1, u_2)$, $K_{u_1}(u_2)$

the gamma function, the beta function and the MacDonald function (the modified Bessel function of the second kind), respectively. Furthermore, we use in (10) the designation

$$F(u_1, u_2, u_3; u_4)$$

for the hypergeometric Gauss function and the denotation

$$\Phi_1(u_1, u_2, u_3; u_4, u_5)$$

for the degenerate Appell function (or the Humbert series) which is the double sum

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(u_1)_{m+n} (u_2)_m}{m! n! (u_3)_{m+n}} u_4^m u_5^n$$

for $|u_4| < 1$, where $(u)_l, l \in \mathbb{N} \cup \{0\}$, is the Pochhammer's symbol. For more information on these special mathematical functions and their integral representations we refer to the monographs by Bateman and Erdélyi (1953), Srivastava and Karlsson (1985) and the handbooks by Erdélyi et al. (1954), Gradshteyn and Ryzhik (2007).

Together with it, we set the deterministic functions c_1 , c_2 , c_3 , c_4 by the identities

$$c_{1} = c_{1}(T_{0}) = \frac{\int_{T_{0}}^{T} \left(\frac{\tilde{\sigma}_{s}^{2}}{2} + \tilde{\rho}_{s}\right) ds}{\sqrt{\int_{T_{0}}^{T} \tilde{\sigma}_{s}^{2} ds}},$$
(11)

$$c_{2} = c_{2}(T_{0}) = \frac{\int_{T_{0}}^{T} \tilde{r}_{s} ds - \log K}{\sqrt{\int_{T_{0}}^{T} \tilde{\sigma}_{s}^{2} ds}},$$
(12)

$$c_{3} = c_{3}(T_{0}) = \frac{\int_{T_{0}}^{T} \left(\tilde{\rho}_{s} - \frac{\tilde{\sigma}_{s}^{2}}{2}\right) ds}{\sqrt{\int_{T_{0}}^{T} \tilde{\sigma}_{s}^{2} ds}},$$
(13)

$$c_4 = c_4(T_0) = \int_{T_0}^T \tilde{\rho}_s ds,$$
 (14)

$$c_5 = c_5(T_0) = \frac{1}{2} \sqrt{\int_{T_0}^T \tilde{\sigma}_s^2 ds}.$$
 (15)

3. Results

The two subsections below introduce formulas for the prices of forward-start European call options in the models with gamma and inverse-gamma volatilities. As corollaries, the prices of standard European calls are derived. The prices of the related put options could be computed via the call-put parity. For details, see Eberlein et al. (2008) and Herdegen and Schweizer (2018).

3.1. Gamma Volatility

It is assumed throughout this subsection that the random variable *z* in (1) is gamma distributed, that is $z = \gamma = \gamma(a, b)$, where the parameters a > 0 and b > 0. The properties of gamma distribution are considered in the Section 5.3.3 of the monograph by Schoutens (2003) and in the papers by Madan et al. (1998), Rathgeber et al. (2016), among others. The gamma distribution has the probability density function

$$f_{\gamma}(x) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)}, \quad x > 0.$$
 (16)

The characteristic function of the gamma random variable is

$$\varphi_{\gamma}(u) = \mathrm{E}e^{iu\gamma} = \left(\frac{b}{b-iu}\right)^a.$$
 (17)

It has the mean and the variance

$$E\gamma = \frac{a}{b}$$
 and $V\gamma = E(\gamma - E\gamma)^2 = \frac{a}{b^2}$, (18)

respectively.

The next theorem gives us the price of the European forward-start call option in the gamma volatility model.

Theorem 1. Assume that $T_0 \ge t_0$. Then the forward-start call option price in the gamma model

$$\mathbb{C}_{G}^{FS} =$$

$$= \frac{S_{0}b^{a}}{\Gamma(a)} \left(\Im(a, b, 0, c_{1}, c_{2}) - K \exp\left(-\int_{T_{0}}^{T} \tilde{r}_{s} ds\right) \Im(a, b, c_{4}, c_{3}, c_{2}) \right),$$
(19)

where the function \Im is defined in (10) and c_1 , c_2 , c_3 , c_4 are set by (11), (12), (13), (14), respectively.

Remark 1. In the classical Black-Scholes-Merton framework, the forward-start call option price is computed through the conditional expectation with respect to S_{T_0} , that is

$$\mathbb{C}_{BS}^{FS} = R_T^{-1} \mathbb{E} (S_T - KS_{T_0})^+ = R_T^{-1} \mathbb{E} \Big(\mathbb{E} \Big[(S_T - KS_{T_0})^+ |S_{T_0} \Big] \Big) =$$

= $\frac{R_{T_0}}{R_T} \mathbb{E} \left(\frac{S_{T_0}}{R_{T_0}} \mathbb{E} \Big[\left(\frac{S_T}{S_{T_0}} - K \right)^+ |S_{T_0} \Big] \right) = S_0 R_{T-T_0} \mathbb{E} (S_{T-T_0} - K)^+$

since $S_T S_{T_0}^{-1}$ and S_{T_0} are independent with each other, where

$$R_{T-T_0} \mathbf{E} (S_{T-T_0} - K)^{-}$$

is again a Black-Scholes price. If $T_0 > t_0$, then this direct approach does not work in our model because of $S_T S_{T_0}^{-1}$ and S_{T_0} both depend on z. Hence we pass to the conditional expectations with respect to z in the proof of Theorem, see (A2) in Appendix A.

As a straightforward corollary of Theorem 1, we get the standard European call price.

Corollary 1. Let $\tilde{r}_t \equiv r$, $\tilde{\rho}_t \equiv \rho$, $\tilde{\sigma}_t \equiv \sigma$. Then the standard European call price in the gamma model

$$\mathbb{C}_G = \frac{S_0 b^a}{\Gamma(a)} \left(\Im(a, b, 0, c_1, c_2) - K e^{rT} \Im(a, b, c_4, c_3, c_2) \right)$$
(20)

with

$$c_1 = \left(\frac{\rho}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T}, \qquad c_2 = \frac{rT - \log K}{\sigma\sqrt{T}},$$
$$c_3 = \left(\frac{\rho}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}, \qquad c_4 = \rho T.$$

3.2. Inverse Gamma Volatility

We suggest in this subsection that *z* in (1) has the inverse-gamma distribution $\zeta = \zeta(a, b), a > 0, b > 0$. This distribution may be used, in particular, for the modeling of the skew Student's t-distribution through a normal mean-variance mixture (see McNeil et al. (2005)). The probability density function of the inverse-gamma distribution is

$$f_{\varsigma}(x) = \frac{b^{a}}{\Gamma(a)} x^{-a-1} e^{-\frac{b}{x}}, \qquad x > 0.$$
(21)

with a, b > 0. It has the characteristic function (see, for example, Witkovský (2001))

$$\varphi_{\varsigma}(u) = \frac{2(-iub)^{\frac{a}{2}}}{\Gamma(a)} K_a \left(2\sqrt{-iub}\right).$$
(22)

The inverse-gamma distribution has the mean

$$E\varsigma = \frac{b}{a-1}I_{\{a>1\}} + \infty I_{\{a\le1\}}$$
(23)

and the variance

$$V\varsigma = \frac{b^2}{(a-1)^2(a-2)} I_{\{a>2\}} + \infty I_{\{a\leq 2\}}.$$
(24)

The next theorem computes the forward-start call option price in the model with inverse-gamma volatility.

Theorem 2. Let $T_0 \ge t_0$ and $\tilde{\rho}_t \equiv 0$ for $t \ge T_0$. Then the forward-start call option price in the inverse-gamma model

(25)

$$\mathbb{C}_{IG}^{FS} = = \frac{S_0 b^a}{\Gamma(a)} \left(\Im(a, b, 0, c_2, c_5) - K \exp\left(-\int_{T_0}^T \tilde{r}_s ds\right) \Im(a, b, 0, c_2, -c_5) \right),$$

where \Im , c_2 and c_5 are defined in (10), (12) and (15).

A corollary below determines the standard European call price.

Corollary 2. Assume that $\tilde{r}_t \equiv r$, $\tilde{\sigma}_t \equiv \sigma$. Then the standard European call price in the inversegamma model

$$\mathbb{C}_{IG} = \frac{S_0 b^a}{\Gamma(a)} \left(\Im(a, b, 0, c_2, c_5) - K e^{-rT} \Im(a, b, 0, c_2, -c_5) \right)$$
(26)

with

$$c_2 = \frac{rT - \log K}{\sigma \sqrt{T}}, \qquad \qquad c_5 = \frac{\sigma \sqrt{T}}{2}.$$

3.3. Discussion

We have established the new analytical expressions that give us the formulas for the forward-start call option prices in the generalized Black-Scholes-Merton model, additionally driven by the gamma and the inverse-gamma distributions. These distributions are widely regarded in the literature for their aim of modeling different financial indices. One may mention the papers by Luciano and Schoutens (2016), Moosbrucker (2006), Wallmeier and Diethelm (2012), Göncü et al. (2016) on exploiting the gamma distribution, and the works by Fung and Seneta (2010), Takahashi et al. (2021), and Nakajima (2020) on using the inverse-gamma one in the stock market simulation.

The considered models extend the generalized Black-Scholes-Merton, the variancegamma and the skew Student's t-models. The obtained formulas depend on the values of special mathematical functions but can be computed over 0.5 s on modern software. The prices of forward-start put options can be calculated by exploiting the duality principle (Eberlein et al. 2008; Herdegen and Schweizer 2018). Future research may relate to the computation of the prices of exotic stock options and options on bonds in the discussed models. Furthermore, developments of these models based on the assumption of the linear drift jump (see Ivanov 2022) could be considered. Since the Bermudan options can be viewed as a spread of the forward-start contracts (see Schweizer 2002), the results of the paper may be used for the American-style options pricing as well.

4. Numerical Examples

In this section, we compare the prices of standard European call options in the models with the gamma and the inverse-gamma volatilities with the Black-Scholes formula. The idea is to pick the parameters of the gamma and the inverse-gamma distributions so that they even out the maximal number of the characteristics of the underlying processes. In fact, it has become possible to get the identities

$$\mathbf{E}S_t^{IG} = \mathbf{E}S_t^G = \mathbf{E}S_t^{BS},$$

$$E\left(\log S_t^{IG}\right) = E\left(\log S_t^G\right) = E\left(\log S_t^{BS}\right)$$
$$V\left(\log S_t^G\right) = V\left(\log S_t^{BS}\right).$$

and

Furthermore, the parameters of the distributions are suggested, and the option prices are computed for the three models in separate subsections.

4.1. Black-Scholes Price

We discuss in this subsection the Black-Scholes-Merton model (6) and (7) under the condition (4) assuming that $S_0^{BS} = 1$, $\tilde{r}_t = \tilde{\rho}_t \equiv 0$ and $\tilde{\sigma}_t \equiv 1$. Then

$$\log S_t^{BS} = B_t - \frac{t}{2} \tag{27}$$

and the variance of the stock price

$$\mathbf{V}S_t^{BS} = \mathbf{E}\left(S_t^{BS}\right)^2 - 1 = e^t - 1.$$

The logarithm of the stock price (27) has the mean

$$\mathrm{E}\left(\log S_t^{BS}\right) = -\frac{t}{2} \tag{28}$$

and the variance

$$V\left(\log S_t^{BS}\right) = t. \tag{29}$$

Let $T_0 = 0$ and T = 1. Then the standard European call option price is

$$\mathbb{C}_{BS} = \mathbb{E}\left(S_1^{BS} - K\right)^+ = \mathbb{N}\left(\frac{1}{2} - \log K\right) - K\mathbb{N}\left(-\frac{1}{2} - \log K\right),\tag{30}$$

where N(u) is the normal distribution function.

4.2. Gamma Volatility Price

It is assumed that in the gamma volatility model of Section 3.1 $S_0^G = 1$, $\tilde{r}_t = \tilde{\rho}_t \equiv 0$ and $\tilde{\sigma}_t \equiv 1$. Then we have that

$$\log S_t^G = \sqrt{\gamma} B_t - \frac{\gamma t}{2}$$

in this case and

$$\mathbf{V}S_t^G = \mathbf{E}e^{\gamma t} - 1 = \left(\left(\frac{b}{b-t}\right)^a - 1\right)I_{\{t < b\}} + \infty I_{\{t \ge b\}}$$

from (17). The logarithm of the stock price has the mean

$$\mathrm{E}\left(\log S_{t}^{G}\right) = -\frac{t\mathrm{E}\gamma}{2} = -\frac{at}{2b} \tag{31}$$

and the variance

$$V(\log S_t^G) = E\left(\sqrt{\gamma}B_t - \frac{\gamma t}{2}\right)^2 - \frac{a^2t^2}{4b^2} = \frac{at}{b} + \frac{t^2E\gamma^2}{4} - \frac{a^2t^2}{4b^2} = \frac{at}{b} + \frac{t^2a}{4b^2}$$

in point of (18). We have that

$$\mathbf{E}S_t^G = \mathbf{E}S_t^{BS} = 1$$

and if we set a = b = 1, then it follows from (28) and (31) that

$$\operatorname{E}\left(\log S_{t}^{G}\right) = \operatorname{E}\left(\log S_{t}^{BS}\right) = -\frac{t}{2}.$$

Let $t_0 = T_0 = 0$ and T = 1. Then according to (20)

$$\mathbb{C}_G = \Im(a, b, 0, c_1, c_2) - K\Im(a, b, 0, c_3, c_2)$$

(32)

with $c_1 = \frac{1}{2}$, $c_2 = -\log K$, $c_3 = -\frac{1}{2}$. Since a = b = 1, we get that $\mathbb{C}_G = \Im\left(1, 1, 0, \frac{1}{2}, -k\right) - K\Im\left(1, 1, 0, -\frac{1}{2}, -k\right),$

where $k = \log K$.

It follows immediately from (10) that

$$\begin{split} \Im\left(1,1,0,\frac{1}{2},-k\right) &= \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2\pi}} \left(\frac{B\left(\frac{1}{2},1\right)}{\sqrt{2}} + \frac{1}{2}F\left(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\right)\right) I_{\{K=1\}} + \\ &+ \frac{|s|^{\frac{1}{2}}e^{s}(1+q)}{\sqrt{2\pi}} \left[B(1,1)\left(|s|K_{\frac{3}{2}}(|s|) + sK_{\frac{1}{2}}(|s|)\right) \times \\ &\times \Phi\left(1,0,2;\frac{1+q}{2},-s(1+q)\right) - (1+q)sB(2,1) \times \\ &\times K_{\frac{1}{2}}(|s|)\Phi\left(2,0,3;\frac{1+q}{2},-s(1+q)\right) \right] I_{\{K\neq1\}} \end{split}$$
(33)

with

$$q = \frac{c_1}{\sqrt{c_1^2 + 2}} = \frac{1}{3}$$
 and $s = c_2\sqrt{c_1^2 + 2} = -\frac{3k}{2}$

and

$$\begin{split} \Im\Big(1,1,0,-\frac{1}{2},-k\Big) &= \frac{\Gamma(\frac{3}{2})}{\sqrt{2\pi}} \left(\frac{B\left(\frac{1}{2},1\right)}{\sqrt{2}} - \frac{1}{2}F\left(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\right)\right)I_{\{K=1\}} + \\ &+ \frac{|s|^{\frac{1}{2}}e^{s}(1+q)}{\sqrt{2\pi}} \left[B(1,1)\left(|s|K_{\frac{3}{2}}(|s|) + sK_{\frac{1}{2}}(|s|)\right) \times \\ &\times \Phi\Big(1,0,2;\frac{1+q}{2},-s(1+q)\Big) - (1+q)sB(2,1) \times \\ &\times K_{\frac{1}{2}}(|s|)\Phi\Big(2,0,3;\frac{1+q}{2},-s(1+q)\Big)\right]I_{\{K\neq1\}} \end{split}$$
(34)

•

with

$$q = \frac{c_3}{\sqrt{c_3^2 + 2}} = -\frac{1}{3}$$
 and $s = c_2\sqrt{c_3^2 + 2} = -\frac{3k}{2}$

We get from (33) that

$$\Im\left(1,1,0,\frac{1}{2},-k\right) = \left(\frac{1}{2} + \frac{1}{4\sqrt{2}}F\left(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\right)\right)I_{\{K=1\}} + \frac{|k|^{\frac{1}{2}}\sqrt{3}}{K^{\frac{3}{2}}\sqrt{\pi}}\left[\left\{|k|K_{\frac{3}{2}}\left(\frac{3|k|}{2}\right) - kK_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)\right\}\Phi\left(1,0,2;\frac{2}{3},2k\right) + \frac{2k}{3}K_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)\Phi\left(2,0,3;\frac{2}{3},2k\right)\right]I_{\{K\neq1\}}$$

and from (34) that

$$\begin{split} \Im\Big(1,1,0,-\frac{1}{2},-k\Big) &= \left(\frac{1}{2}-\frac{1}{4\sqrt{2}}F\Big(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\Big)\Big)I_{\{K=1\}} + \\ &+\frac{|k|^{\frac{1}{2}}\sqrt{3}}{2K^{\frac{3}{2}}\sqrt{\pi}}\bigg[\bigg\{|k|K_{\frac{3}{2}}\Big(\frac{3|k|}{2}\Big)-kK_{\frac{1}{2}}\Big(\frac{3|k|}{2}\Big)\bigg\}\Phi\Big(1,0,2;\frac{1}{3},k\Big) + \\ &+\frac{k}{3}K_{\frac{1}{2}}\Big(\frac{3|k|}{2}\Big)\Phi\Big(2,0,3;\frac{1}{3},k\Big)\bigg]I_{\{K\neq1\}}. \end{split}$$

Since

$$\Phi(u_1, u_2, u_3; u_4, u_5) =$$

$$= \frac{1}{B(u_1, u_3 - u_1)} \int_0^1 x^{u_1 - 1} (1 - x)^{u_3 - u_1 - 1} (1 - u_4 x)^{-u_2} e^{u_4 x} dx$$
(35)

for $u_1 > 0$, $u_3 > u_1$ and $u_4 < 1$ according to formula 4.3.24 of Erdélyi et al. (1954), we get that

$$\Phi(1,0,2;A,B) = J_0(B)$$

and

$$\Phi(2,0,3;A,B) = 2J_1(B),$$

where

$$J_0(B) = \int_0^1 e^{Bx} dx$$
 and $J_1(B) = \int_0^1 x e^{Bx} dx$.

Hence we obtain that

$$\Im\left(1,1,0,\frac{1}{2},-k\right) =$$

$$= \left(\frac{1}{2} + \frac{1}{4\sqrt{2}}F\left(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\right)\right)I_{\{K=1\}} + \frac{|k|^{\frac{1}{2}}\sqrt{3}}{K^{\frac{3}{2}}\sqrt{\pi}} \times \\ \times \left[\left\{|k|K_{\frac{3}{2}}\left(\frac{3|k|}{2}\right) - kK_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)\right\}J_{0}(2k) + \frac{4k}{3}K_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)J_{1}(2k)\right]I_{\{K\neq1\}}$$
(36)

and

$$\Im\left(1,1,0,-\frac{1}{2},-k\right) =$$

$$= \left(\frac{1}{2} - \frac{1}{4\sqrt{2}}F\left(\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{1}{8}\right)\right)I_{\{K=1\}} + \frac{|k|^{\frac{1}{2}}\sqrt{3}}{2K^{\frac{3}{2}}\sqrt{\pi}} \times \\ \times \left[\left\{|k|K_{\frac{3}{2}}\left(\frac{3|k|}{2}\right) - kK_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)\right\}J_{0}(k) + \frac{2k}{3}K_{\frac{1}{2}}\left(\frac{3|k|}{2}\right)J_{1}(k)\right]I_{\{K\neq1\}}.$$
(37)

4.3. Inverse-Gamma Volatility Price

We set in the model of Section 3.2 $S_0^{IG} = 1$, $\tilde{r}_t = \tilde{\rho}_t \equiv 0$ and $\tilde{\sigma}_t \equiv 1$. Then $\log S_t^{IG} = \sqrt{\zeta}B_t - \frac{\zeta t}{2}$

and it issues from (22) that

$$\mathbf{V}S_t^{IG} = \mathbf{E}e^{\varsigma t} - 1 = \infty.$$

With respect to (23),

$$\mathbb{E}\left(\log S_t^{IG}\right) = -\frac{t\mathbb{E}\zeta}{2} = -\frac{tb}{2(a-1)}I_{\{a>1\}} - \infty I_{\{a\le1\}}.$$
(38)

It follows from (24) that

$$\begin{aligned} \mathbf{V} \Big(\log S_t^{IG} \Big) &= \mathbf{E} \Big(\sqrt{\varsigma} B_t - \frac{\varsigma t}{2} + \frac{t \mathbf{E} \varsigma}{2} \Big)^2 = \mathbf{E} \Big(\sqrt{\varsigma} B_t - \frac{t}{2} (\varsigma - \mathbf{E} \varsigma) \Big)^2 = \\ &= t \mathbf{E} \varsigma + \frac{t^2}{4} \mathbf{V} \varsigma = \frac{t b}{a - 1} I_{\{a > 1\}} + \infty I_{\{a \le 1\}} + \\ &+ \frac{b^2 t^2}{4(a - 1)^2 (a - 2)} I_{\{a > 2\}} + \infty I_{\{a \le 2\}} = \\ &= \frac{t b}{a - 1} \Big(1 + \frac{b t}{4(a - 1)(a - 2)} \Big) I_{\{a > 2\}} + \infty I_{\{a \le 2\}}. \end{aligned}$$

We have that

$$\mathbf{E}S_t^{IG} = \mathbf{E}S_t^G = \mathbf{E}S_t^{BS} = 1.$$

If we set a = 3 and b = 2, then it results from (28), (31) and (38) that

$$\operatorname{E}\left(\log S_{t}^{IG}\right) = \operatorname{E}\left(\log S_{t}^{G}\right) = \operatorname{E}\left(\log S_{t}^{BS}\right) = -\frac{t}{2}.$$

Moreover, then

$$\mathcal{V}\left(\log S_t^{IG}\right) = \mathcal{V}\left(\log S_t^G\right) = t\left(1 + \frac{t}{4}\right).$$

Set $t_0 = T_0 = 0$, T = 1, a = 3, b = 2 and $k = \log K$. Then in accordance with (26)

$$\mathbb{C}_{IG} = 4\left(\Im\left(3, 2, 0, -k, \frac{1}{2}\right) - K\Im\left(3, 2, 0, -k, -\frac{1}{2}\right)\right).$$
(39)

Furthermore, we have immediately from (10) that

$$\Im\left(3,2,0,-\log K,\frac{1}{2}\right) = \frac{|s|^{\frac{5}{2}}e^{s}(1+q)^{3}}{8\sqrt{2\pi}} \left[B(3,1)\left(|s|K_{\frac{7}{2}}(|s|)+sK_{\frac{5}{2}}(|s|)\right) \times \Phi\left(3,-2,4;\frac{1+q}{2},-s(1+q)\right) - (1+q)sB(4,1) \times K_{\frac{5}{2}}(|s|)\Phi\left(4,-2,5;\frac{1+q}{2},-s(1+q)\right)\right]$$
(40)

with

$$q = \frac{c_2}{\sqrt{c_2^2 + 4}} = -\frac{k}{\sqrt{k^2 + 4}}$$
 and $s = c_1\sqrt{c_2^2 + 4} = \frac{\sqrt{k^2 + 4}}{2}$

and

$$\Im\left(3,2,0,-\log K,-\frac{1}{2}\right) = \frac{|s|^{\frac{5}{2}}e^{s}(1+q)^{3}}{8\sqrt{2\pi}} \left[B(3,1)\left(|s|K_{\frac{7}{2}}(|s|)+sK_{\frac{5}{2}}(|s|)\right) \times \Phi\left(3,-2,4;\frac{1+q}{2},-s(1+q)\right) - (1+q)sB(4,1) \times K_{\frac{5}{2}}(|s|)\Phi\left(4,-2,5;\frac{1+q}{2},-s(1+q)\right)\right]$$
(41)

with

$$q = \frac{u_2}{\sqrt{c_2^2 + 4}} = -\frac{k}{\sqrt{k^2 + 4}}$$
 and $s = c_3\sqrt{c_2^2 + 4} = -\frac{\sqrt{k^2 + 4}}{2}$

It results from (40) that

$$\Im\left(3,2,0,-k,\frac{1}{2}\right) = \frac{e^{\frac{\sqrt{k^2+4}}{2}}\left(\sqrt{k^2+4}-k\right)^3\left(k^2+4\right)^{\frac{1}{4}}}{128\sqrt{\pi}} \times \left[\frac{1}{3}\left\{K_{\frac{7}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)+K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)\right\} \times \Phi\left(3,-2,4;\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}},\frac{k-\sqrt{k^2+4}}{2}\right)-\frac{\sqrt{k^2+4}-k}{4\sqrt{k^2+4}} \times K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)\Phi\left(4,-2,5;\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}},\frac{k-\sqrt{k^2+4}}{2}\right)\right]$$

and from (41) that

$$\Im\left(3,2,0,-k,-\frac{1}{2}\right) = \frac{e^{-\frac{\sqrt{k^2+4}}{2}}\left(\sqrt{k^2+4}-k\right)^3 (k^2+4)^{\frac{1}{4}}}{128\sqrt{\pi}} \times \left[\frac{1}{3}\left\{K_{\frac{7}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) - K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)\right\} \times \right. \\ \left. \times \Phi\left(3,-2,4;\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}},\frac{\sqrt{k^2+4}-k}{2}\right) + \frac{\sqrt{k^2+4}-k}{4\sqrt{k^2+4}} \times \left. \times K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) \Phi\left(4,-2,5;\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}},\frac{\sqrt{k^2+4}-k}{2}\right)\right].$$

Let

$$J_2(A,B) = \int_0^1 x^2 (1-Ax)^2 e^{Bx} dx$$

and

$$J_3(A,B) = \int_0^1 x^3 (1-Ax)^2 e^{Bx} dx.$$

Then we have with respect to (35) that

$$\Phi(3, -2, 4; A, B) = 3J_2(A, B)$$

and

$$\Phi(4, -2, 5; A, B) = 4J_3(A, B).$$

Therefore,

$$\Im\left(3,2,0,-k,\frac{1}{2}\right) = \frac{e^{\frac{\sqrt{k^2+4}}{2}}\left(\sqrt{k^2+4}-k\right)^3\left(k^2+4\right)^{\frac{1}{4}}}{128\sqrt{\pi}} \times \\ \times \left[\left\{K_{\frac{7}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) + K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)\right\} \times \\ \times J_2\left(\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}}, \frac{k-\sqrt{k^2+4}}{2}\right) - \frac{\sqrt{k^2+4}-k}{\sqrt{k^2+4}} \times \\ \times K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) J_3\left(\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}}, \frac{k-\sqrt{k^2+4}}{2}\right)\right]$$

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(42)

and

$$\Im\left(3,2,0,-k,-\frac{1}{2}\right) = \frac{e^{-\frac{\sqrt{k^2+4}}{2}}\left(\sqrt{k^2+4}-k\right)^3\left(k^2+4\right)^{\frac{1}{4}}}{128\sqrt{\pi}} \times \left[\left\{K_{\frac{7}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) - K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right)\right\} \times J_2\left(\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}}, \frac{\sqrt{k^2+4}-k}{2}\right) + \frac{\sqrt{k^2+4}-k}{\sqrt{k^2+4}} \times K_{\frac{5}{2}}\left(\frac{1}{2}\sqrt{k^2+4}\right) J_3\left(\frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}}, \frac{\sqrt{k^2+4}-k}{2}\right)\right].$$
(43)

4.4. Comparison of the Prices

The Black-Scholes price is calculated according to (30). The gamma volatility price is computed by (32) taking into account (36) and (37). The inverse-gamma volatility price is determined with respect to (39) using (42) and (43).

Table 1 shows the dynamics of the standard call option prices in dependence on the increase in the strike coefficient *K* from 0.05 to 10. The numbers around K = 1 in Table 1 relate to the skewness of the probability densities in the gamma and the inverse-gamma models. Since the drift coefficients are negative, the probability density of the logarithm of stock price in the gamma model has a larger weight of extremal events than the probability density of the logarithm of stock price in the inverse-gamma one. This fact explains the inequality $\mathbb{C}_G > \mathbb{C}_{IG}$ for large strike coefficients.

$\mathbb{C} \setminus K$	0.05	0.1	0.5	0.75	1	1.25	1.5	2	10
\mathbb{C}_{BS}	0.950	0.901	0.595	0.473	0.383	0.316	0.264	0.191	0.010
\mathbb{C}_{G}	0.951	0.903	0.583	0.438	0.334	0.267	0.222	0.167	0.033
\mathbb{C}_{IG}	0.951	0.902	0.586	0.452	0.356	0.288	0.237	0.171	0.025

Table 1. The comparison of the standard European call option prices.

One may notice that the simplest call-put parity identity

$$S_1 - K = (S_1 - K)I_{\{S_1 > K\}} - (K - S_1)I_{\{S_1 \le K\}}$$
(44)

allows us to obtain the standard European put option prices in our examples directly from Table 1. Indeed, the put option price

$$\mathbb{P} = \mathrm{E}(K - S_1)^+ = \mathbb{C} + K - 1$$

from (44).

5. Conclusions

We have discussed the generalized Black-Scholes-Merton model, in which the stock volatility is modeled using the gamma and the inverse-gamma distributions. The idea for this research is produced by the papers on the statistical analysis of stock market data by many authors. We refer, in particular, to Seneta (2004), Daal and Madan (2005), Nakakita and Nakatsuma (2021), Mozumder et al. (2015), Luciano and Schoutens (2016) on the gamma distribution, and to Nakajima and Omori (2012), Aas and Haff (2006), Finlay and Seneta (2008), and Takahashi et al. (2021) on the inverse-gamma one in the stochastic volatility modeling.

We have selected for the numerical examples the parameters of the models so that the characteristics of the underlying processes are as close to each other as possible. It is clear from the examples that it can be expected that for large strike coefficients the call option prices should be vastly different for the all three models. At the same time, the difference between the put option prices does not exceed 20%. It should be also noticed that the calculation time of our formulas for the gamma and the inverse-gamma models is comparable with the time of the Black-Scholes formula computation.

The obtained results extend the results of Madan et al. (1998) and Ano and Ivanov (2016), which are derived for the variance-gamma model. Furthermore, the price of the forward-start call option is computed in a model of the skew Student's t-type. The idea of research is confirmed by the variety of works that have approved the use of the gamma and the inverse-gamma distributions for financial modeling. Numerical examples have shown that the results for the standard call option price may differ substantially from the Black-Scholes formula.

Future studies can relate to the modeling and computation of more complicated derivatives, including American-style ones. Moreover, the theoretical developments of the studied models can also be processed for the potential of obtaining closed-form results in option pricing.

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Appendix A

Proof of Theorem 1. Set

$$g(z_T) = \mathbf{E}\Big[R_T^{-1}(S_T - KS_{T_0})^+ | z_T\Big].$$

Then

$$g(z_T) = g(\gamma) = \mathbf{E} \Big[R_T^{-1} (S_T - K S_{T_0})^+ |\gamma \Big] = \\ = \mathbf{E} \Big[\frac{S_{T_0}}{R_{T_0}} \frac{R_{T_0}}{R_T} \Big(\frac{S_T}{S_{T_0}} - K \Big)^+ \Big| \gamma \Big].$$
(A1)

Since the random variable $S_{T_0}R_{T_0}^{-1} = S_{T_0}(\gamma)R_{T_0}^{-1}(\gamma)$ is independent with the random variables $R_{T_0}R_T^{-1} = R_{T_0}(\gamma)R_T^{-1}(\gamma)$ and $S_TS_{T_0}^{-1} = S_T(\gamma)S_{T_0}^{-1}(\gamma)$ for any particular realization of γ , we get from (A1) that

$$g(z_T) = \mathbb{E}\left[\frac{S_{T_0}}{R_{T_0}}\Big|\gamma\right] \mathbb{E}\left[\frac{R_{T_0}}{R_T}\left(\frac{S_T}{S_{T_0}} - K\right)^+\Big|\gamma\right] = S_0(g_1(\gamma) - g_2(\gamma))$$
(A2)

with

$$g_1(\gamma) = \mathbf{E}\left(\frac{R_{T_0}S_T}{R_T S_{T_0}} I_{\left\{\frac{S_T}{S_{T_0}} > K\right\}} \middle| \gamma\right)$$

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and

$$g_2(\gamma) = K \mathbb{E}\left(\frac{R_{T_0}}{R_T} I_{\left\{\frac{S_T}{S_{T_0}} > K\right\}} \middle| \gamma\right).$$

Because of

$$\mathbb{C}_G = \mathrm{E}(g(z_T)),$$

we get from (A2) that

$$\mathbb{C}_G = S_0[\mathrm{E}(g_1(\gamma)) - \mathrm{E}(g_2(\gamma))]. \tag{A3}$$

Next, one may observe that

 $g_1(\gamma) = \mathbf{E} \bigg[\exp \bigg(\sqrt{\gamma} \int_{T_0}^T \tilde{\sigma}_s dB_s - \frac{\gamma}{2} \int_{T_0}^T \tilde{\sigma}_s^2 ds \bigg) I_{\left\{ \sqrt{\gamma} \int_{T_0}^T \tilde{\sigma}_s dB_s > \xi_1(\gamma) \right\}} \Big| \gamma \bigg].$

with

$$\tilde{\zeta}_1(\gamma) = \log K - \int_{T_0}^T \tilde{r}_s ds + \gamma \int_{T_0}^T \left(\frac{\tilde{\sigma}_s^2}{2} - \tilde{\rho}_s\right) ds.$$

Let

$$\overline{\sigma}(\gamma) = \sqrt{\gamma \int_{T_0}^T \tilde{\sigma}_s^2 ds}.$$

Since

$$Law\left(\sqrt{\gamma}\int_{T_0}^T \tilde{\sigma}_s dB_s \middle| \gamma\right) = Law\left(\mathcal{N}\left(0,\overline{\sigma}^2(\gamma)\right) \middle| \gamma\right),$$

where $\ensuremath{\mathcal{N}}$ is the normally distributed random variable, we get that

$$\begin{split} g_{1}(\gamma) &= \exp\left(-\frac{\overline{\sigma}^{2}(\gamma)}{2}\right) \mathbb{E}\left[\exp\left(\mathcal{N}\left(0,\overline{\sigma}^{2}(\gamma)\right)\right) I_{\left\{\mathcal{N}\left(0,\overline{\sigma}^{2}(\gamma)\right) > \xi_{1}(\gamma)\right\}} \middle| \gamma\right] = \\ &= \exp\left(-\frac{\overline{\sigma}^{2}(\gamma)}{2}\right) \int_{\xi_{1}(\gamma)}^{\infty} \frac{1}{\overline{\sigma}(\gamma)\sqrt{2\pi}} \exp\left(x - \frac{x^{2}}{2\overline{\sigma}^{2}(\gamma)}\right) dx = \\ &= \int_{\xi_{1}(\gamma)}^{\infty} \frac{1}{\overline{\sigma}(\gamma)\sqrt{2\pi}} \exp\left(-\frac{\left(x - \overline{\sigma}^{2}(\gamma)\right)^{2}}{2\overline{\sigma}^{2}(\gamma)}\right) dx = \\ &= \mathbb{N}\left(\overline{\sigma}(\gamma) - \frac{\xi_{1}(\gamma)}{\overline{\sigma}(\gamma)}\right). \end{split}$$

Furthermore,

$$g_{2}(\gamma) = = K \exp\left(-\int_{T_{0}}^{T} (\tilde{r}_{s} + \gamma \tilde{\rho}_{s}) ds\right) \int_{\xi_{1}(\gamma)}^{\infty} \frac{1}{\overline{\sigma}(\gamma)\sqrt{2\pi}} \exp\left(-\frac{x^{2}}{2\overline{\sigma}(\gamma)^{2}}\right) dx = = K \exp\left(-\int_{T_{0}}^{T} (\tilde{r}_{s} + \gamma \tilde{\rho}_{s}) ds\right) N\left(-\frac{\xi_{1}(\gamma)}{\overline{\sigma}(\gamma)}\right).$$

One may notice that

$$\overline{\sigma}(\gamma) - \frac{\xi_1(\gamma)}{\overline{\sigma}(\gamma)} = \sqrt{\gamma \int_{T_0}^T \tilde{\sigma}_s^2 ds} - \frac{\log K - \int_{T_0}^T \tilde{r}_s ds + \gamma \int_{T_0}^T \left(\frac{\tilde{\sigma}_s^2}{2} - \tilde{\rho}_s\right) ds}{\sqrt{\gamma \int_{T_0}^T \tilde{\sigma}_s^2 ds}} = c_1 \sqrt{\gamma} + \frac{c_2}{\sqrt{\gamma}}$$

with

$$c_1 = \frac{\int_{T_0}^T \left(\frac{\tilde{\sigma}_s^2}{2} + \tilde{\rho}_s\right) ds}{\sqrt{\int_{T_0}^T \tilde{\sigma}_s^2 ds}}, \qquad \qquad c_2 = \frac{\int_{T_0}^T \tilde{r}_s ds - \log K}{\sqrt{\int_{T_0}^T \tilde{\sigma}_s^2 ds}}$$

and

$$-\frac{\xi_1(\gamma)}{\overline{\sigma}(\gamma)}=c_3\sqrt{\gamma}+\frac{c_2}{\sqrt{\gamma}},$$

where

 $c_3=rac{\int_{T_0}^T \left(ilde{
ho}_s-rac{ ilde{\sigma}_s^2}{2}
ight)ds}{\sqrt{\int_{T_0}^T ilde{\sigma}_s^2 ds}}.$

Hence

$$g_1(\gamma) = N\left(c_1\sqrt{\gamma} + \frac{c_2}{\sqrt{\gamma}}\right)$$
 (A4)

and

$$g_2(\gamma) = K \exp\left(-\int_{T_0}^T (\tilde{r}_s + \gamma \tilde{\rho}_s) ds\right) N\left(c_3 \sqrt{\gamma} + \frac{c_2}{\sqrt{\gamma}}\right).$$
(A5)

Set

$$c_4 = \int_{T_0}^T \tilde{\rho}_s ds.$$

Then we get from (A3)–(A5) that

$$\mathbb{C}_{G} = S_{0} \left(\mathbb{E} \left[\mathbb{N} \left(c_{1} \sqrt{\gamma} + \frac{c_{2}}{\sqrt{\gamma}} \right) \right] - (A6) - K \exp \left(-\int_{T_{0}}^{T} \tilde{r}_{s} ds \right) \mathbb{E} \left[e^{-c_{4} \gamma} \mathbb{N} \left(c_{3} \sqrt{\gamma} + \frac{c_{2}}{\sqrt{\gamma}} \right) \right] \right).$$

One may see from (A6) that we need to calculate the integral

$$I_G = I_G(v_1, v_2, v_3) = \frac{\Gamma(a)}{b^a} \int_0^\infty e^{-v_1 x} \mathbf{N}\left(v_2 \sqrt{x} + \frac{v_3}{\sqrt{x}}\right) f_\gamma(x) dx$$

for $v_1 \ge 0$ and $v_2, v_3 \in \mathbb{R}$. Then

$$\mathbb{C}_{G} = \frac{S_{0}b^{a}}{\Gamma(a)} \left(I_{G}(0, c_{1}, c_{2}) - K \exp\left(-\int_{T_{0}}^{T} \tilde{r}_{s} ds\right) I_{G}(c_{4}, c_{3}, c_{2}) \right).$$
(A7)

We have that

$$I_G = \int_0^\infty x^{a-1} e^{-(b+v_1)x} \mathbf{N}\left(v_2\sqrt{x} + \frac{v_3}{\sqrt{x}}\right) dx.$$

This integral is computed in point of the cases below.

Case 1. $v_3 = 0$. Then we have with respect to Case 2.2 at p. 208 of Ano and Ivanov (2016) that

$$I_{G}(v_{1}, v_{2}, v_{3}) =$$

$$= \frac{\Gamma\left(a + \frac{1}{2}\right)}{\sqrt{2\pi}(b + v_{1})^{a}} \left(\frac{B\left(\frac{1}{2}, a\right)}{\sqrt{2}} + \frac{v_{2}}{\sqrt{b + v_{1}}}F\left(a + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -\frac{v_{2}^{2}}{2(b + v_{1})}\right)\right).$$
(A8)

Case 2. $v_3 \neq 0$. We get in accordance with (21) of Ano and Ivanov (2016) that

$$\begin{split} I_{G}(v_{1}, v_{2}, v_{3}) &= \frac{|s|^{a-\frac{1}{2}} e^{s} (1+q)^{a}}{\sqrt{2\pi} (b+v_{1})^{a}} \bigg[\mathsf{B}(a,1) \bigg(|s| \mathsf{K}_{a+\frac{1}{2}}(|s|) + \\ &+ s \mathsf{K}_{a-\frac{1}{2}}(|s|) \bigg) \Phi \bigg(a, 1-a, a+1; \frac{1+q}{2}, -s(1+q) \bigg) - \\ &(1+q) s \mathsf{B}(a+1,1) \mathsf{K}_{a-\frac{1}{2}}(|s|) \Phi \bigg(a+1, 1-a, a+2; \frac{1+q}{2}, -s(1+q) \bigg) \bigg], \end{split}$$
(A9)

where

$$q = rac{v_2}{\sqrt{v_2^2 + 2(b + v_1)}}$$
 and $s = v_3 \sqrt{v_2^2 + 2(b + v_1)}$

Combining together (A7)–(A9) and using the auxiliary function (10), we establish that

$$\mathbb{C}_{G} = \frac{S_{0}b^{a}}{\Gamma(a)} \left(\Im(a, b, 0, c_{1}, c_{2}) - K \exp\left(-\int_{T_{0}}^{T} \tilde{r}_{s} ds\right) \Im(a, b, c_{4}, c_{3}, c_{2}) \right).$$

Proof of Theorem 2. Similarly to (A6), we get that in this case

$$\mathbb{C}_{IG} = S_0 \left(\mathbb{E} \left[\mathbb{N} \left(c_5 \sqrt{\varsigma} + \frac{c_2}{\sqrt{\varsigma}} \right) \right] -$$

$$-K \exp \left(-\int_{T_0}^T \tilde{r}_s ds \right) \mathbb{E} \left[\mathbb{N} \left(-c_5 \sqrt{\varsigma} + \frac{c_2}{\sqrt{\varsigma}} \right) \right] \right)$$
(A10)

with

$$c_5 = \frac{1}{2} \sqrt{\int_{T_0}^T \tilde{\sigma}_s^2 ds}$$
 and $c_2 = \frac{\int_{T_0}^T \tilde{r}_s ds - \log K}{\sqrt{\int_{T_0}^T \tilde{\sigma}_s^2 ds}}.$

Set

$$I_{IG} = I_{IG}(v_1, v_2) = \frac{\Gamma(a)}{b^a} \int_0^\infty N\left(v_1\sqrt{x} + \frac{v_2}{\sqrt{x}}\right) f_{\varsigma}(x) dx$$

for $v_1 \neq 0$ and $v_2 \in \mathbb{R}$. Then we have from (A10) that

$$\mathbb{C}_{IG} = \frac{S_0 b^a}{\Gamma(a)} \left(I_{IG}(c_5, c_2) - K \exp\left(-\int_{T_0}^T \tilde{r}_s ds\right) I_{IG}(-c_5, c_2) \right).$$
(A11)

One may observe that

$$I_{IG} = \int_0^\infty x^{-a-1} e^{-\frac{b}{x}} N\left(v_1\sqrt{x} + \frac{v_2}{\sqrt{x}}\right) dx.$$

Let us define for $t \le v_2$ functions $\tilde{v}_2(t)$, $\tilde{v}_1(t)$ and u(t) as

$$ilde{v}_{2}(t) = t, \qquad ilde{v}_{1}(t) = rac{v_{1}\sqrt{v_{2}^{2}+2b}}{\sqrt{t^{2}+2b}}$$

and

$$u(t) = \tilde{v}_1(t)\sqrt{x} + \frac{\tilde{v}_2(t)}{\sqrt{x}}.$$

Then

$$N\left(v_{1}\sqrt{x} + \frac{v_{2}}{\sqrt{x}}\right) = \int_{-\infty}^{v_{1}\sqrt{x} + \frac{v_{2}}{\sqrt{x}}} N'_{u}(u)du = \int_{-\infty}^{v_{2}} N'_{u}(u)u'_{t}dt$$

and we have that

$$I_{IG} = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-a-1} e^{-\frac{b}{x}} \left(\int_{-\infty}^{v_2} e^{-\frac{(v_1(t)x + v_2(t))^2}{2x}} \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} v_1 \sqrt{v_2^2 + 2b} (t^2 + 2b)^{-\frac{3}{2}} t \right) dt \right) dx = \frac{1}{\sqrt{2\pi}} (I_{IG1} - I_{IG2}),$$
(A12)

where

$$I_{IG1} = \int_{-\infty}^{v_2} e^{-\tilde{v}_1(t)\tilde{v}_2(t)} \left(\int_0^\infty x^{-a-\frac{3}{2}} e^{-\frac{\tilde{v}_1^2(t)x}{2} - \frac{\tilde{v}_2^2(t)+2b}{2x}} dx \right) dt$$

and

$$I_{IG2} = s \int_{-\infty}^{v_2} e^{-\tilde{v}_1(t)\tilde{v}_2(t)} t(t^2 + 2b)^{-\frac{3}{2}} \left(\int_0^{\infty} x^{-a - \frac{1}{2}} e^{-\frac{\tilde{v}_1^2(t)x}{2} - \frac{\tilde{v}_2^2(t) + 2b}{2x}} dx \right) dt$$

with

$$s = v_1 \sqrt{v_2^2 + 2b}.$$

Formula 3.471.9 of Gradshteyn and Ryzhik (2007) comprises the identity

$$\int_{0}^{\infty} x^{u_{1}-1} e^{-\frac{u_{2}}{x}-u_{3}x} dx = 2\left(\frac{u_{2}}{u_{3}}\right)^{\frac{u_{1}}{2}} K_{u_{1}}(2\sqrt{u_{2}u_{3}}),$$
(A13)

where $u_1 \in \mathbb{R}$, $u_2 > 0$ and $u_3 > 0$. It results immediately from (A13) that

$$\begin{split} &\int_{0}^{\infty} x^{-a-\frac{1}{2}-j} e^{-\frac{\tilde{v}_{1}^{2}(t)x}{2} - \frac{\tilde{v}_{2}^{2}(t)+2b}{2x}} dx = 2 \left(\frac{\tilde{v}_{2}^{2}(t)+2b}{\tilde{v}_{1}^{2}(t)}\right)^{\frac{1}{4}-\frac{a+j}{2}} \times \\ &\times \mathbf{K}_{\frac{1}{2}-a-j} \left(|\tilde{v}_{1}(t)| \sqrt{\tilde{v}_{2}^{2}(t)+2b}\right) = 2 \left(\frac{t^{2}+2b}{|s|}\right)^{\frac{1}{2}-a-j} \mathbf{K}_{\frac{1}{2}-a-j}(|s|) \end{split}$$

for j = 0, 1. Hence we get that

$$I_{IG1} = 2|s|^{a+\frac{1}{2}} K_{a+\frac{1}{2}}(|s|) \int_{-\infty}^{v_2} (t^2 + 2b)^{-a-\frac{1}{2}} e^{-\frac{st}{\sqrt{t^2 + 2b}}} dt$$

and

$$I_{IG2} = 2s|s|^{a-\frac{1}{2}}K_{a-\frac{1}{2}}(|s|)\int_{-\infty}^{v_2} t(t^2+2b)^{-a-1}e^{-\frac{st}{\sqrt{t^2+2b}}}dt$$

Set

$$w = \frac{t}{\sqrt{t^2 + 2b}}.$$

Then

$$t = \frac{w\sqrt{2b}}{\sqrt{1-w^2}}, \qquad t^2 + 2b = \frac{2b}{1-w^2}, \qquad t'_w = \frac{\sqrt{2b}}{(1-w^2)^{\frac{3}{2}}}$$

and one may observe that

$$I_{IG1} = \frac{|s|^{a+\frac{1}{2}} K_{a+\frac{1}{2}}(|s|)}{2^{a-1} b^{a}} \int_{-1}^{\frac{v_{2}}{\sqrt{v_{2}^{2}+2b}}} \left(1-w^{2}\right)^{a-1} e^{-sw} dw$$

and

$$I_{IG2} = \frac{s|s|^{a-\frac{1}{2}} K_{a-\frac{1}{2}}(|s|)}{2^{a-1}b^a} \int_{-1}^{\frac{v_2}{\sqrt{v_2^2 + 2b}}} w \left(1 - w^2\right)^{a-1} e^{-sw} dw.$$

Let

$$q = \frac{v_2}{\sqrt{v_2^2 + 2b}}$$

and

$$y = \frac{1+w}{1+q}$$

Then

$$w = (1+q)y - 1,$$
 $w'_y = 1 + q$

and hence

$$I_{IG1} = \frac{(1+q)|s|^{a+\frac{1}{2}}e^{s}\mathbf{K}_{a+\frac{1}{2}}(|s|)}{2^{a-1}b^{\varkappa}} \int_{0}^{1} \left(1 - \left((1+q)y - 1\right)^{2}\right)^{a-1}e^{-s(1+q)y}dy = \\ = \frac{(1+q)^{a}|s|^{aa+\frac{1}{2}}e^{s}\mathbf{K}_{a+\frac{1}{2}}(|s|)}{b^{\varkappa}} \int_{0}^{1}y^{a-1}\left(1 - \frac{1+q}{2}y\right)^{a-1}e^{-s(1+q)y}dy$$
(A14)

and

$$\begin{split} I_{IG2} &= \frac{(1+q)s|s|^{a-\frac{1}{2}}e^{s}K_{a-\frac{1}{2}}(|s|)}{2^{a-1}b^{a}} \times \\ &\times \int_{0}^{1}((1+q)y-1)\left(1-((1+q)y-1)^{2}\right)^{a-1}e^{-s(1+q)y}dy = \\ &= \frac{(1+q)^{a}s|s|^{a-\frac{1}{2}}e^{c}K_{a-\frac{1}{2}}(|s|)}{b^{a}}\left((1+q)\int_{0}^{1}y^{a}\left(1-\frac{1+q}{2}y\right)^{a-1} \times \\ &\times e^{-s(1+q)y}dy - \int_{0}^{1}y^{a-1}\left(1-\frac{1+q}{2}y\right)^{a-1}e^{-s(1+q)y}dy \right). \end{split}$$
(A15)

Furthermore, we apply (35) to (A14) and (A15) and infer that

$$I_{IG1} = (A16)$$

$$= \frac{(1+q)^{a}|s|^{a+\frac{1}{2}}e^{s}K_{a+\frac{1}{2}}(|s|)}{b^{a}}B(a,1)\Phi\left(a,1-a,a+1;\frac{1+q}{2},-s(1+q)\right)$$

and

$$I_{IG2} = \frac{(1+q)^{a}s|s|^{a-\frac{1}{2}}e^{s}K_{a-\frac{1}{2}}(|s|)}{b^{a}}\left((1+q)B(a+1,1)\times\right)$$
$$\times \Phi\left(a+1,1-a,a+2;\frac{1+q}{2},-s(1+q)\right) - B(a,1)\times\right)$$
$$\times \Phi\left(a,1-a,a+1;\frac{1+q}{2},-s(1+q)\right).$$
(A17)

It follows from (A12), (A16) and (A17) that

$$I_{IG} = \frac{(1+q)^{a}|s|^{a-\frac{1}{2}}e^{s}}{b^{a}\sqrt{2\pi}} \left[B(a,1)\left(|s|K_{a+\frac{1}{2}}(|s|) + sK_{a-\frac{1}{2}}(|s|)\right) \times \Phi\left(a,1-a,a+1;\frac{1+q}{2},-s(1+q)\right) - s(1+q)B(a+1,1) \times K_{a-\frac{1}{2}}(|s|)\Phi\left(a+1,1-a,a+2;\frac{1+q}{2},-s(1+q)\right) \right].$$
(A18)

Finally, it results from (A11), (A18) and (10) that

$$\mathbb{C}_{IG} = \frac{S_0 b^a}{\Gamma(a)} \bigg(\Im(a, b, 0, c_2, c_5) - K \exp\bigg(-\int_{T_0}^T \tilde{r}_s ds\bigg) \Im(a, b, 0, c_2, -c_5) \bigg).$$

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