

Article

Dynamic Liability-Driven Investment under Sponsor's Loss Aversion

Dong-Hwa Lee ¹  and Joo-Ho Sung ^{2,*}

¹ Pension Research Division, National Pension Research Institute, Sejong 30116, Republic of Korea; ldhc9@khu.ac.kr

² School of Management, Kyung Hee University, Seoul 02447, Republic of Korea

* Correspondence: jhsung@khu.ac.kr; Tel.: +82-2-961-0675

Abstract: This paper investigates a dynamic liability-driven investment policy for defined-benefit (DB) plans by incorporating the loss aversion of a sponsor, who is assumed to be more sensitive to underfunding than overfunding. Through the lens of prospect theory, we first set up a loss-aversion utility function for a sponsor whose utility depends on the funding ratio in each period, obtained from stochastic processes of pension assets and liabilities. We then construct a multi-horizon dynamic control optimization problem to find the optimal investment strategy that maximizes the expected utility of the plan sponsor. A genetic algorithm is employed to provide a numerical solution for our nonlinear dynamic optimization problem. Our results suggest that the overall paths of the optimal equity allocation decline as the age of a plan participant reaches retirement. We also find that the equity portion of the portfolio increases when a sponsor is less loss-averse or the contribution rate is lower.

Keywords: liability-driven investment; defined-benefit pension plan; loss aversion; prospect theory



Citation: Lee, Dong-Hwa, and Joo-Ho Sung. 2024. Dynamic Liability-Driven Investment under Sponsor's Loss Aversion. *Risks* 12: 38. <https://doi.org/10.3390/risks12020038>

Academic Editors: Pierre Devolder and Massimiliano Menzietti

Received: 29 December 2023

Revised: 5 February 2024

Accepted: 9 February 2024

Published: 13 February 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The realization of optimal investment policies in defined-benefit (DB) pension plans has attracted a great deal of attention in view of pension fund risk management. Regarding DB pension funding, most studies have strongly focused on establishing optimal strategies derived from the well-known mean–variance (MV) framework motivated by risk aversion (e.g., Sharpe and Tint 1990; Ezra 1991; Leibowitz et al. 1992; Frauendorfer et al. 2007; Xie 2009; Ang et al. 2013; Jung et al. 2022) as well as the control theory framework (e.g., Cairns 2000; Haberman and Vigna 2002; Miao and Wang 2006; Ngwira and Gerrard 2007; Josa-Fombellida et al. 2018). However, as mentioned in Siegmann (2007), these approaches to optimizing investment policies do not take into account the sponsors' tendency to be more sensitive to losses than to gains. Hence, the previous studies, which only consider a risk-averse agent, would not be sufficient to explain the asymmetric behavioral attitude towards investment outcomes such as gain or loss.

In general, DB sponsors are very likely to have this asymmetric attitude of loss aversion with respect to the investment performance of a pension plan, in the sense that the underfunded status may lead to additional business risk or adversely impact the company's equity value and credit rating. Furthermore, the sponsor can be required to fill the funding gap (i.e., actuarial deficit) over a period specified by the supervising authority. Hence, we argue that, for a DB plan sponsor, a utility loss from a deficit is larger than a utility gain from a surplus when the absolute values of the gain and loss are the same. Put differently, it is reasonable to assume that a plan sponsor prefers to avoid underfunding than achieve overfunding. From this point of view, the financial behavior of DB sponsors could be transformed into a loss-aversion utility function based on the prospect theory of Tversky and Kahneman (1992).

However, the asymmetric behavioral attitudes of DB sponsors under prospect theory have yet to be fully incorporated into DB investment strategies. The primary purpose of this paper is to demonstrate how the sponsor's loss-aversion can be incorporated into the optimal investment strategy of DB pension plans. To achieve this, we introduce a loss-aversion utility function and formulate a dynamic multi-period optimization problem.

The remainder of this paper is organized as follows. Section 2 presents a literature review, and in Section 3, we make assumptions for modeling the loss-aversion utility function and present the dynamic optimization problem. Section 4 specifies input parameters for numerical illustrations, and Section 5 provides the numerical results. Section 6 presents the results of sensitivity analysis. Lastly, Section 7 presents our conclusions.

2. Literature Review

The investment strategies for DB pension plans differ from general asset-only investment strategies in terms of pension liability-driven investment. As DB sponsors have an obligation to provide the benefits promised in the event of numerous contingencies to a group of members, pension liability serves as an important reference point in investment. Thus, the quest for improved asset–liability management in DB pension plans has been ongoing for decades. The concept of ‘surplus management for DB plan’ was first proposed by Sharpe and Tint (1990) within the framework of mean–variance analysis as developed by Markowitz (1952). Drawing on the toolkit of mean–variance, researchers have evaluated investment policies for DB plans, as seen in Ezra (1991), Leibowitz et al. (1992), Peskin (1997), Waring (2004), and Hoevenaars et al. (2008).

Furthermore, to focus on the risk, traditional studies on managing DB pensions have concerned two types of risk: contribution rate risk¹ and solvency risk² (Haberman and Sung 1994; Haberman 1997; Cairns 2000; Haberman et al. 2000, 2003; and Ngwira and Gerrard 2007). Managing downside risk has become a more critical issue since the global financial crisis. During the financial crisis, the average funding status of the top 100 U.S. corporate plans declined from more than 100% in 2007 to less than 80% in 2010 (Deutsche Bank 2010). Under these circumstances, numerous studies have attempted to find and explore liability-driven investment (LDI), which could improve the funding status by managing the contribution rate risk and solvency risk. The framework of LDI generally distinguishes two different levels of asset allocation decision: a liability hedge portfolio and a portfolio based on various investment criteria (Amenc et al. 2010; Qian 2012; and Ang et al. 2013).

On the other hand, the asymmetric attitude toward surplus and shortfall has hardly been considered in the literature when determining investment strategies for DB pension plans, even though loss aversion has been considered one of the crucial factors when individuals determine asset allocations in a defined contribution (DC) pension (Cairns et al. 2006; Blake et al. 2013). As mentioned in Siegmann (2007), DB sponsors are also more sensitive to losses than to gains, and Hwang and Satchell (2010) indicate that investors of pension funds are loss-averse in the UK and US markets.

The primary contribution of this paper is to recommend optimal asset allocations within a DB plan, considering loss aversion as introduced by Tversky and Kahneman (1992) within the framework of prospect theory. Loss aversion, which refers to people's tendency to be more sensitive to losses than gains, has been inadequately addressed in existing papers on DB pension plans. To achieve this, we explain the loss-aversion utility function and formulate a dynamic multi-period optimization problem, setting the pension liability as the reference point.

3. Assumption and Methodology

3.1. Assumption

As with mathematical models, it is essential to make some simple assumptions to effectively focus on the key features of the control optimization problem to be analyzed later.

- (A1) Only one member aged x joins at time $t = 0$ and retires with a lump-sum equivalent to the actuarial liability accrued up to age $x + n$. Also, the sponsor pays an annual fixed rate of members' labor income, and then, winds up through a pension buy-out. Note that in our numerical analysis, we set values of $x = 20$ and $n = 45$, i.e., a unique member joins the DB plan at age 20 and retires at age 65.³
- (A2) The sponsor is more loss-averse to underfunding than to overfunding as members close in on retirement. In other words, the sponsor is influenced by [Tversky and Kahneman's \(1992\)](#) prospect theory utility. In this context, the sponsor's reference point is pension liability.
- (A3) The actuarial pension liability is annually valued by the projected unit credit method⁴ (i.e., PUC method), which is used as a target liability of asset management. We performed the calculation using [Maurer et al. \(2009\)](#)'s notation.
- (A4) In line with the assumptions of key papers (e.g., [Albrecht and Maurer 2002](#); [Blake et al. 2013](#)), pension fund assets are invested in two underlying assets: bonds (representing a less risky asset) and equities (representing a risky asset).

Our starting point is to integrate all of these assumptions within an objective function designed to optimize the investment strategy over the control horizon $(0, n)$. In particular, the assumption (A2) reflects the members' perspective; for example, older members tend to be concerned more directly with ensuring their own accrued benefit security. On the other hand, the PUC method provides individual liability for each member and calculates it independently from other members. Here, we deal with one active member on the grounds that the overall investment optimality for total members could be obtained from the aggregation of individual investment optimality. Although the debate on whether the above proposal is true continues, it would be meaningful to provide a new investment strategy in this paper.

3.2. Sponsor's Loss-Aversion Utility Function

Given the actual pension assets A_t and the actuarial pension liability L_t , the financial status would be measured using the funding gap, FG_t , which is defined as follows: for each valuation date $t = 1, \dots, n$,

$$FG_t = A_t - L_t. \quad (1)$$

In general, if the funding gap moves into surplus, then the sponsor might have a chance of taking contribution holidays. Otherwise, the sponsor would likely pay additional contributions to alleviate the funding gap when the pension fund has a deficit. In this respect, we set up the pension liability as a baseline. Considering the assumption (A2), we can set up the following loss-aversion utility function. For each time $t = 1, \dots, n$,

$$U_t(FG_t) = \begin{cases} \frac{(FG_t)^{v_1}}{v_1} & \text{if } A_t \geq L_t \\ -\lambda_t \frac{(-FG_t)^{v_2}}{v_2} & \text{if } A_t < L_t. \end{cases} \quad (2)$$

And $\lambda_t = \lambda_0 + \delta \cdot t$, where v_1 and v_2 are the curvature parameters for the expected surplus and expected shortfall, respectively, and $0 < v_1 \leq v_2 < 1$; λ_t is the time-path loss-aversion parameter with the initially given $\lambda_0 (> 1)$ and $\delta (> 0)$. It is worth noting that the loss-aversion model (2) is designed to be S-shaped and to represent the notion that sponsors are more sensitive to underfunding as time goes on, because $0 < v_1 \leq v_2 < 1$ and $1 < \lambda_t < \lambda_{t+1}$. Accordingly, the DB sponsor would likely optimize the investment policy by maximizing their own utility function, which is quite different to the traditional approach, which aims to minimize the expected shortfall risk, such as $E[\max(L_t - A_t, 0)]$.

3.3. Stochastic Asset and Liability Model

In general, actuarial pension liabilities are utilized as a target value of asset management in DB pension plans. According to the assumptions (A1) and (A3), the liability L_t is a projected benefit obligation accrued up to age $x + t$: for each $t = 1, \dots, n$,

$$L_t = \frac{\alpha \cdot t \cdot W_{x+n-1} \cdot \ddot{a}_{x+n, i_t}}{(1 + i_t)^{n-t}} \quad \text{with initially given } L_0 = 0, \tag{3}$$

where α is the pension accrual rate, t is service years up to age $x + t$, W_{x+n-1} is the projected final salary, and \ddot{a}_{x+n, i_t} is the immediate life annuity factor at the normal retirement age $x + n$ discounted by the current valuation rate i_t .

To proceed with the mark-to-market liability valuation recommended by the International Financial Reporting Standards (IFRS 19), we assume that the valuation rate in model (3) follows the discrete-time Vasicek model as given below:

$$i_{t+1} = i_t + \kappa \cdot (\gamma - i_t) + \sigma \cdot \varepsilon_{i,t}, \tag{4}$$

in which κ is the mean-reverting property, γ is the long-term average, σ is the standard deviation of interest rates, and ε_t is a random variable which follows an independent and identically distributed standard normal distribution (hereafter denoted as $\varepsilon_t \sim iid N(0, 1)$).

Next, W_{x+n-1} is assumed to be determined by $W_{x+t} = W_{x+t-1} \times (1 + I_t)$, with a value of $W_x = 1$ initially given, in which the growth rate I_t relies on the model proposed in [Blake et al. \(2013\)](#) (for more details, see Appendix A).

On the other hand, applying the assumptions (A1) and (A4), the pension fund growth can be modeled into the following recursive equation: for each $t = 1, 2, \dots, n$,

$$A_t = \exp(r_t) \times (A_{t-1} + \pi \times W_{x+t-1}) \quad \text{with initially given } A_0 = 0, \tag{5}$$

in which π is the fixed contribution rate and r_t is the investment rate return over age $(x + t - 1, x + t)$.

Here, the return r_t is proposed to follow a discrete-time geometric Brownian motion, as formulated below⁵ (it is worth noting that θ_t , representing an equity portion of investment, is introduced as a controlling variable that will be optimally determined in later sections):

$$r_t = \left[\theta_t \left(\left(\mu_e - \frac{1}{2} \sigma_e^2 \right) + \sigma_e \cdot \varepsilon_{e,t} \right) + (1 - \theta_t) \left(\left(\mu_b - \frac{1}{2} \sigma_b^2 \right) + \sigma_b \cdot \varepsilon_{b,t} \right) \right], \tag{6}$$

where $0 \leq \theta_t \leq 1$ (i.e., no short-selling and borrowing); μ_e and σ_e (μ_b and σ_b) are the expected rate and standard deviation of returns on the equity fund (on the bond fund), respectively; $\varepsilon_{e,t} = Z_{1,t}$, $\varepsilon_{b,t} = \rho Z_{1,t} + \sqrt{1 - \rho^2} Z_{2,t}$ and $Z_{1,t}, Z_{2,t} \sim iid N(0, 1)$, in which ρ is the cross-correlation coefficient between equity and bond.

3.4. Control Optimization Problem

As specified in Sections 3.2 and 3.3, the scheme sponsor is assumed to have their own loss-aversion utility such as function (4). Thus, the control performance index (CPI) can be designed to give a discounted loss-aversion utility caused by a sequence of investment control actions $\{\theta_s : s = t, t + 1, \dots, n\}$. For each control time $t = 1, 2, \dots, n$, we then define

$$CPI(t) = \sum_{s=t}^n \beta^{s+1-t} \cdot E[U_s(FG_s)], \tag{7}$$

where $\beta (> 0)$ represents the sponsor's discount factor to reflect the relative importance of the near-future funding gap against the distant-future funding gap. From the viewpoint of achieving our investment purpose, an optimal sequence of asset allocations will be derived

by maximizing the above $CPI(t)$ simultaneously with respect to $\theta_t, \theta_{t+1}, \dots, \theta_n$. Hence, the investment control optimization is in the form

$$\max_{\{\theta_t, \theta_{t+1}, \dots, \theta_n\}} CPI(t), \quad (8)$$

subject to the models given by Equations (3)–(6).

As with optimization problems, it is not easy to overcome the algebraic complexities. Hence, we adopt the principle of optimality which is proposed by Bellman (1957). As described in Haberman and Sung (1994), the principle of optimality enables the multi-stage optimizing problem to be transformed into a problem of making optimal controlled decisions one at a time, but sequentially. We can then solve the finite horizon dynamic optimization problem (8) by proceeding with a backward induction at time t . To achieve this, we define

$$V(FG_t, t) = \max_{\{\theta_t, \theta_{t+1}, \dots, \theta_n\}} CPI(t). \quad (9)$$

Applying Bellman's principle of optimality, we can transform the above (9) into the sequential control optimization problem below: for each control time t ,

$$V(FG_t, t) = \max_{\theta_t} \left\{ \beta \cdot E[U_t(FG_t)] + \max_{\theta_{t+1}} \left\{ \beta^2 \cdot E[U_{t+1}(FG_{t+1})] + \dots + \max_{\theta_n} \left\{ \beta^{n+1-t} \cdot E[U_n(FG_n)] \right\} \right\} \dots \right\}. \quad (10)$$

Then, this can be reformulated in the form

$$V(FG_t, t) = \max_{\theta_t} \left\{ \beta \cdot E[U_t(FG_t)] + \max_{\theta_{t+1}, \theta_{t+2}, \dots, \theta_n} \left\{ \beta^2 \cdot E[U_{t+1}(FG_{t+1})] + \dots + \beta^{n+1-t} \cdot E[U_n(FG_n)] \right\} \right\}. \quad (11)$$

Consequently, we obtain the Bellman equation below to provide a sequence of optimal equity portions $\{\theta_t : t = 1, 2, \dots, n\}$ by means of repeating backward recursions of $V(FG_t, t)$ from times $t = n$ to $t = 1$:

$$V(FG_t, t) = \max_{\theta_t} \{ \beta \cdot E[U_t(FG_t)] + V(FG_{t+1}, t+1) \}, \quad (12)$$

in which the boundary condition is $V(FG_n, n) = \beta \cdot E[U_n(FG_n)]$.

However, one issue is that an analytically explicit solution does not exist for the problem (12), largely due to the functional feature of dynamic nonlinearity.⁶ Alternatively, we design a genetic algorithm (GA) to obtain numerical solutions, which was first developed by Holland (1975). The GA is a heuristic and stochastic method that is useful in optimization problems where various local optima exist. Through a crossover and mutation, GA takes a more global view and provides a more comprehensive search than the computations behind the traditional methods (Judd 1998). The GA has been widely used as a powerful tool in portfolio optimization processes for complex (constrained or unconstrained) problems where traditional techniques are not applicable (e.g., Yang 2006; Zhang and Zhang 2009; Soleimani et al. 2009; Gupta et al. 2013; Liu and Zhang 2013; Huang and Zhao 2014; Jin et al. 2019). Although Liu and Zhang (2013) point out that the premature convergence of GA would sometimes lead to a local optimal solution, Soleimani et al. (2009) show that GA is also applicable to large-scale optimization problems and the GA solution is reasonably close to the global optima. The GA package in R is used to implement the optimization problem.⁷ The GA procedure for our control problem is described in detail in Appendix B.

4. Model Parameters

We need to set up the parameters for the Equations (2)–(6) for the simulation-based optimization using the GA method. Among these parameters, it is somewhat challenging and even beyond our scope to estimate for loss-aversion and income growth parameters. For this reason, we refer to previous studies for the parameters used in this paper.

4.1. Baseline of Parameters in Equation (2)

The parameter set in model (2) is $\{v_1, v_2, \lambda_0, \delta\}$, which specifies the sponsor’s loss-aversion propensity. Note that the first three parameters are typical ones used in the previous literature, while we additionally develop δ to make the degree of loss aversion time-dependent. Tversky and Kahneman (1992) determined that $\{v_1 = 0.88, v_2 = 0.88, \lambda_0 = 2.25\}$ from an experimental survey conducted on graduate students, whereas Blake et al. (2013) proposed that $\{v_1 = 0.44, v_2 = 0.88, \lambda_0 = 4.50\}$ in an experiment conducted on DC plan members in the UK. Berkelaar et al. (2004) also estimated the implied loss-aversion level in the U.S. stock market data and reported that $\lambda_0 = 2.711$, which is a similar level to the loss-aversion parameter in Tversky and Kahneman (1992). Although the assumptions of our model are not perfectly fit to these studies, the values $\{v_1 = 0.44, v_2 = 0.88, \lambda_0 = 2.00\}$ will be used as a set of loss-aversion baseline parameters, and we additionally assume that $\delta = 0.1$. In the sensitivity analysis, we will check the robustness of our numerical results by changing the set of loss-aversion parameters.

For the labor income model $W_{x+t} = W_{x+t-1} \times (1 + I_t)$, the growth rate I_t is specified by three parameters $\{r_I, \sigma_1, \sigma_2\}$ as described in Appendix A. The estimates $\{r_I = 0.02, \sigma_1 = 0.05, \sigma_2 = 0.02\}$ by Blake et al. (2013) are fully adopted as an income-growth baseline.

4.2. Baseline of Parameters in Equations (4) and (6)

To estimate the parameters for the discrete-time Vasicek model (4), we use the 10-year Treasury yield over 1985–2018. Also, we utilize the S&P 500 Index and Barclays US Treasury Total Return Index over 1985–2018 in order to estimate the parameters in the investment model (6). The parameters are summarized in Table 1, and are used as a baseline.

Table 1. Estimation results.

	Asset Return		Valuation Interest Rate	
	Equity	Bond		
μ	0.0796	0.0615	κ	0.1658
σ	0.1620	0.0565	γ	0.0383
ρ	−0.2434 (p-value < 0.1)		σ	0.0113

Source: S&P 500 Index (monthly equity index from 1985 to 2018); Barclays U.S. Treasury Total Return Index (monthly bond index from 1985 to 2018); 10-year U.S. Treasury yield (monthly interest rate from 1985 to 2018).

Lastly, Table 2 presents the overall parameters used in our main analysis. In the sensitivity analysis, we also check whether our result holds consistently in different dimensions by changing the parameters.

Table 2. Parameters set for method.

Loss Aversion	
Initial loss aversion to shortfall, λ_0	2.0
Loss-aversion property to shortfall, δ	0.1
Curvature parameter for surplus, v_1	0.44
Curvature parameter for shortfall, v_2	0.88
Asset	
Expected return on equity fund, μ_e	0.0796
Expected return on bond fund, μ_b	0.0615
Volatility of return on equity fund, σ_e	0.1620
Volatility of return on bond fund, σ_b	0.0565

Table 2. Cont.

Valuation Interest rate	
Speed of mean reversion, κ	0.1658
Long-term mean, γ	0.0383
Volatility of interest rate, σ	0.0113
Others	
Initial salary, W_x	1.0
Long-term average of real salary growth rate, r_I	0.02
Volatility of shock correlated with equity returns, σ_1	0.05
Volatility of annual rate of salary growth rate, σ_2	0.02
Fixed contribution rate, π	0.0833
Pension accrual rate, α	0.0167
Sponsor's discount factor, β	0.98
Initial age, x	20
Control-time horizon, n	45

Note: Unlike DC pension plans, where individuals determine asset allocation, DB pension plans have the plan sponsor determining the asset allocation. Therefore, it is necessary to assess the loss-aversion parameters of the plan sponsors through surveys. However, there is no prior research, including this study, that has been conducted based on surveys targeting plan sponsors. Therefore, we conducted this analysis based on assumptions regarding loss-aversion parameters. To overcome these limitations, we conducted various sensitivity analyses. Source: authors' assumptions.

5. Numerical Results

This section illustrates, by means of numerical examples, the time-path optimal asset allocations. The GA method helps to search and clarify simulation-based optimization, which would not otherwise be possible.

The numerical solutions are presented in Figure 1. The solid line shows the time-path optimal equity allocation maximizing the loss-aversion utility model set out in a previous section, whereas the dashed line represents the time-path optimal equity allocation for the traditional model which aims to minimize the expected shortfall risk. Here, the expected shortfall risk is defined as $\sum_{t=1}^n E(L_t - A_t | A_t < L_t) \times Pr(A_t < L_t)$. Overall, the optimal investment under our loss-aversion model maintains a much higher equity portion over the control period. As given in Table 3 below, we analyze the funding ratio at terminal time ($t = 45$) to compare the performance of each approach. While the downside risk is greater under the utility maximization model compared to the risk-minimization model, the utility model recommends asset allocations that are anticipated to achieve a higher funding ratio for the sponsor, as assumed in the baseline parameters.

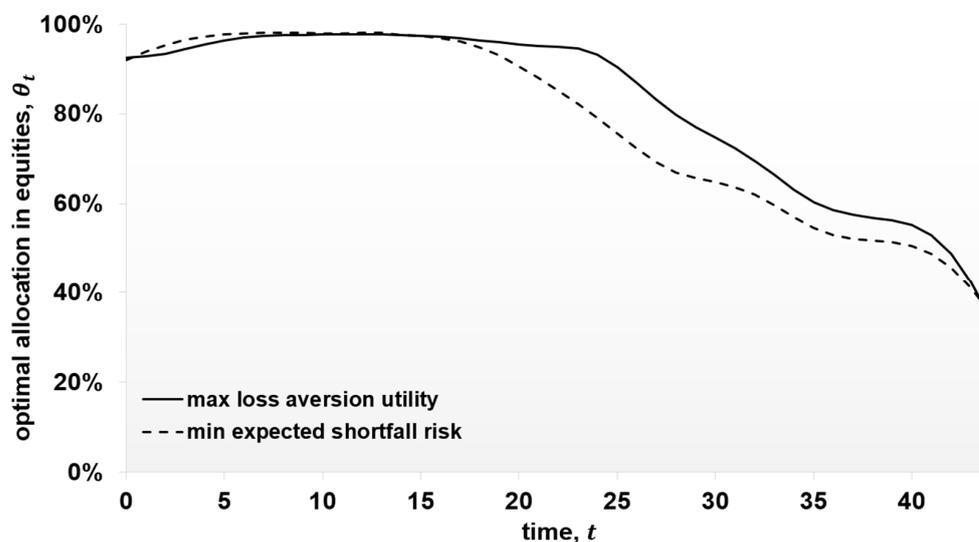


Figure 1. Time-path optimal equity allocation.

Table 3. Funding ratio at normal retirement age of 65.

	Maximizing the Loss-Aversion Utility	Minimizing the Expected Shortfall Risk
Mean funding ratio	1.38	1.34
5th percentile	0.64	0.67
25th percentile	0.96	0.97
Median	1.26	1.24
75th percentile	1.66	1.60
95th percentile	2.49	2.33

Note: This table reports funding ratios at terminal time ($t = 45$) under approaches of maximizing the loss-aversion utility and minimizing expected shortfall risk. Here, the funding ratio is defined as an asset/liability and we obtain funding ratio distributions by generating 10,000 Monte Carlo simulations for equity return, bond return, labor income, and interest rates. Source: authors' calculations.

Hence, we would like to highlight the generality of our loss-aversion model, allowing for the sponsors' diverse attitudes of loss aversion by fixing the loss aversion and sensitivity to the expected shortfall risk.

6. Sensitivity Analysis

In this section, we change the values of each of the loss-aversion ratios, curvature parameters, and contribution rates, and compare them with each other.

6.1. Loss-Aversion Ratio

The model applied in this paper is heavily influenced by the loss-aversion ratio. For example, if the loss-aversion ratio increases sharply, the sponsor mainly focuses on minimizing the expected underfunding. In contrast, a strategy maximizing the expected overfunding is preferred if the loss-aversion ratio is close to 0. Here, we compare the baseline case with two cases: $\delta = 0.3, 0.5$.

Figure 2 shows optimal asset allocations for different speeds of loss aversion. We can identify that sponsors with a higher speed of increase in the loss-aversion ratio tend to switch out of the equity and switch even earlier into the bond because they are more intensely affected by the expected shortfall risk. And also, we obtained higher loss-aversion results for lower expected underfunding and overfunding, as shown in Table 4.

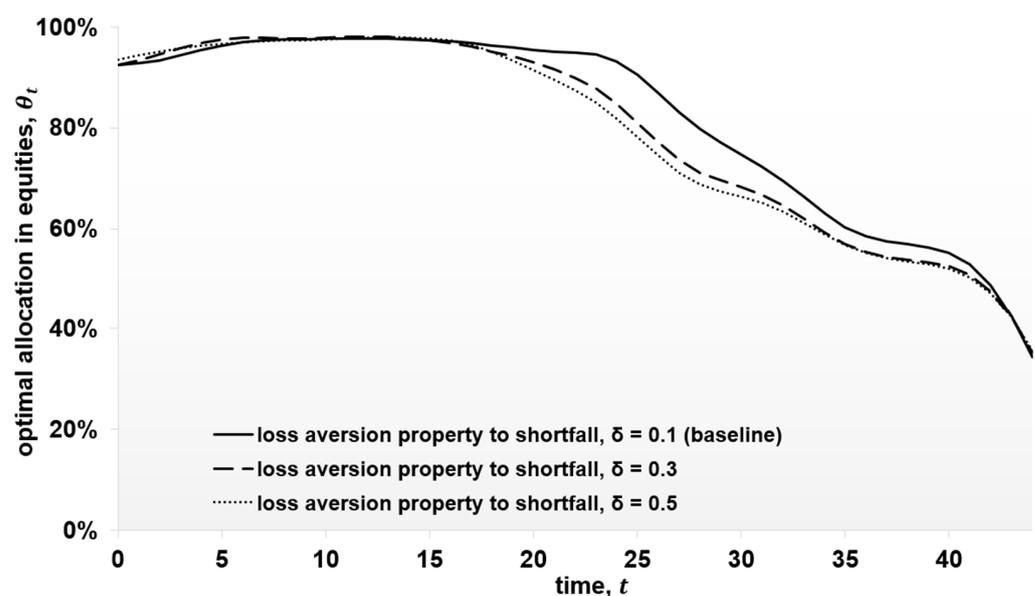
**Figure 2.** Time-path optimal equity allocation depending on the loss-aversion ratio.

Table 4. Expected value of overfunding and underfunding at normal retirement age of 65.

	Loss-Aversion Property to Shortfall			Minimizing the Expected Shortfall Risk
	0.1	0.3	0.5	
Expected value of overfunding	28.07	25.93	25.35	24.66
Expected value of underfunding	3.17	3.07	3.05	3.03

Note: This table reports expected values of overfunding and underfunding at normal retirement age of 65 for each approach, maximizing the loss-aversion utility and minimizing the expected shortfall risk. Here, the expected value of overfunding is defined as $\sum_{t=1}^n E(A_t - L_t | A_t \geq L_t) \times Pr(A_t \geq L_t)$ and the expected value of underfunding is defined as $\sum_{t=1}^n E(L_t - A_t | A_t < L_t) \times Pr(A_t < L_t)$. The expected shortfall risks are also given. Source: authors' calculations.

6.2. Curvature Parameters

Parameters ν_1 and ν_2 in the loss-aversion model denote the curvature parameters for expected overfunding and underfunding, respectively. By adjusting the curvature parameters, we can more closely reflect the financial behavior of the sponsors. Here, we only consider the effect of changing the curvature parameter ν_1 , as the sponsor is assumed to be more sensitive to underfunding.

Figure 3 shows the optimal asset allocations in equity for numerous curvature parameters. When the parameter ν_1 increases from $\nu_1 = 0.44$ to $\nu_1 = 0.88$, the sponsor will focus on generating overfunding. Accordingly, they would keep holding a relatively higher equity portion over the control period. One peculiarity is the relatively lower equity weighting at the earlier stage. It could be interpreted that a higher ν_1 requires relatively lower equity investment at the beginning stage to provide a build-up of assets relatively sufficient for aggressive investments.

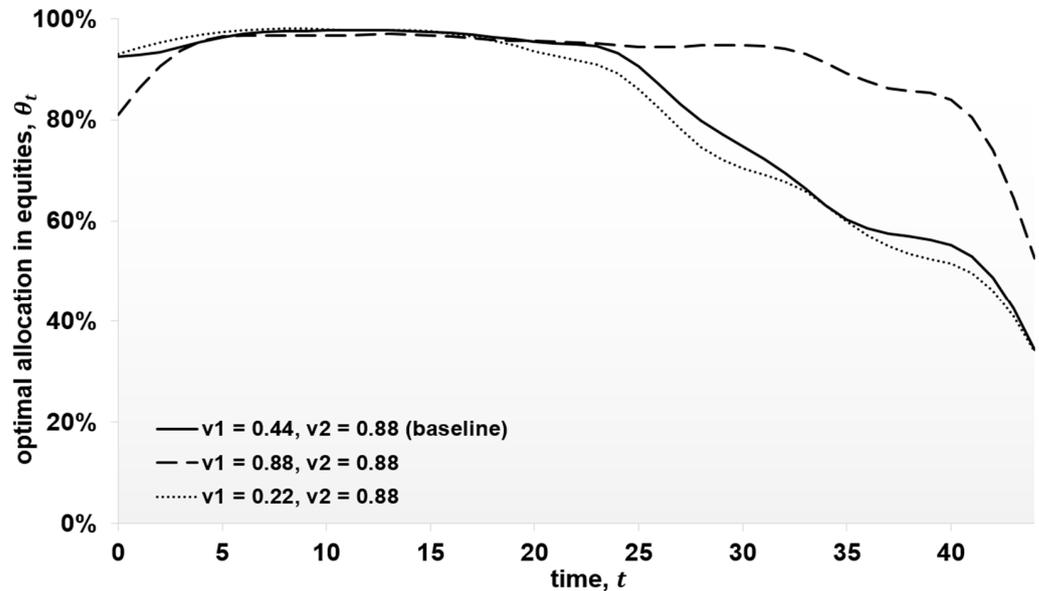


Figure 3. Time-path optimal equity allocation depending on curvature parameters.

6.3. Contribution Rate

We next turn our attention to the contribution rate, which is a significant variable in DB plans. Even though the contribution rate is adjusted by spreading the funding gap, this paper assumes a constant contribution rate to focus on the investment strategy. Instead, we apply numerous constant contribution rate scenarios to identify the contribution rate effect.

Figure 4 depicts the optimal equity portions for contribution rates, showing that the optimal equity portion declines as the contribution rate increases and is relatively higher at an earlier stage regardless of contribution levels. Overall, a higher contribution rate

leads to a more conservative investment, despite a relatively higher chance of accumulating sufficient buffer assets for risk taking.

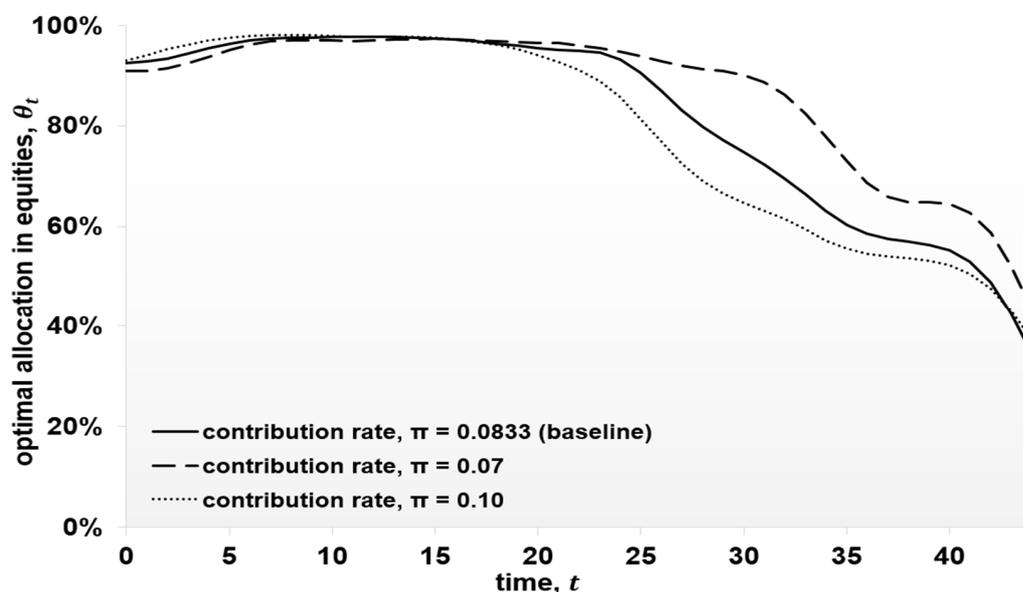


Figure 4. Time-path optimal equity allocation depending on contribution rate.

7. Conclusions

We established and applied a loss-aversion utility model to propose a dynamic liability-driven investment policy reflecting the financial behavior of DB sponsors, such as being more sensitive to underfunding than to overfunding; in particular, the degree of loss aversion is adjustable using the loss-aversion ratio and curvature parameters. Therefore, we believe that our approach would likely be employed as a powerful and practical tool for investment strategy on the grounds that the financial behavior of sponsors could be incorporated practically through experimental surveys for individual loss aversion.

The main findings under our investment optimization problem are that, firstly, the optimal path of equity allocations should be designed to decline as control time goes on to retirement; secondly, the optimal investment policy under lower loss aversion has to take a relatively higher equity portion than that under higher loss aversion; and finally, the optimal equity portion declines with increasing contribution rate.

The major limitation of this paper is that sponsors are assumed to have attitudes of loss aversion to investment performance without conducting an experimental survey. Although we conducted a sensitivity analysis instead, applying more realistic settings is an important next step. Also, further work for relaxing each of assumptions (A1)~(A4) is worthy of exploration.

The other limitation is that we only considered two underlying assets. In practice, sponsors can invest pension assets in alternative investment products such as real estate and commodities. If these are additionally considered as underlying assets, more diverse investment strategies could be suggested for sponsors. We would like to leave this topic for future development.

Additionally, we only consider the constant contribution rate in order to focus on the investment strategy. Although we conducted a sensitivity analysis of contribution rates, the sponsor can adjust their contribution rates based on funding status in reality. To consider changing contribution rates with regard to funding status, for example, the spreading method of Maurer et al. (2009) can be applied to the model. We would like to leave this topic for future development.

Lastly, this study assumes the presence of only one member within the DB pension scheme. To provide more realistic conclusions in future research, it is necessary to analyze a

diverse population structure. For instance, analyzing schemes with a steady-state population structure is possible, and considering schemes with gradually aging or younger population structures is also feasible. We would like to leave this topic for future development.

Author Contributions: Conceptualization, D.-H.L. and J.-H.S.; Methodology, D.-H.L. and J.-H.S.; Formal Analysis, D.-H.L.; Investigation, D.-H.L.; Writing, D.-H.L. and J.-H.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The views expressed in this paper are the authors' and do not represent those of the Korean National Pension Research Institute.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Stochastic Labor Income Process

In deriving the growth rate of labor income for the projected salary, the following is adopted in this paper, which was first modeled by Cairns et al. (2006) and used in Blake et al. (2013). The stochastic growth rate of labor income between ages x and $x + 1$, I_x is calculated according to

$$I_x = r_I + \frac{S_x - S_{x-1}}{S_{x-1}} + \sigma_1 Z_{1,x} + \sigma_2 Z_{2,x}. \quad (\text{A1})$$

where r_I is the long-term average of the real growth rate in national average earnings and S_x is the career salary profile at each age x ; σ_1 and σ_2 are the volatility of a shock from equity return and a shock to labor income growth, respectively; and $Z_{1,x}$ and $Z_{2,x} \sim iid N(0, 1)$ are mutually independent.

Appendix B. Genetic Algorithm Procedure

The GA is a heuristic and stochastic method that has long been recognized as a useful tool for complex optimization problems. We set the GA parameters, such as crossover probability, elitism, and mutation probability, by following Scrucca (2013). Our GA procedure is summarized as follows (See, Figure A1):

(Step 1: initialization) Our GA method starts with randomly generating an initial population of size 500, which consists of several chromosomes created by randomly assigning "0" and "1" to all genes; in this paper, each chromosome represents an equity portion in asset allocation at each control time t , $\{\theta_1, \theta_2, \dots, \theta_{45}\}$.

(Step 2: evaluation) Based on the evaluation criterion originating from the objective function (9), the performance $V(FG_t, t)$ of each chromosome is evaluated.

(Step 3: selection) Individual chromosomes with high $V(FG_t, t)$ are selected in order to generate the next generation. Selected chromosomes, which are called parents, produce the next generation (offspring).

(Step 4: crossover with probability of 0.8 and elitism of 25) To generate the new offspring chromosomes, pairs of parents are randomly selected from the parent population and two parents are interchanged at a crossover point.

(Step 5: mutation with probability of 0.1) The mutation operator randomly modifies genes of chromosomes by flipping "0" to "1" and vice versa. The mutation prevents local convergence of the algorithm.

(Step 6: repeat Steps 3~5) As in the flow chart below, the GA repeats the evaluation and reproduction steps until the termination criteria are met. In this process, the populations converge to one including only chromosomes with good fitness from reproduction, mutation, and crossover.

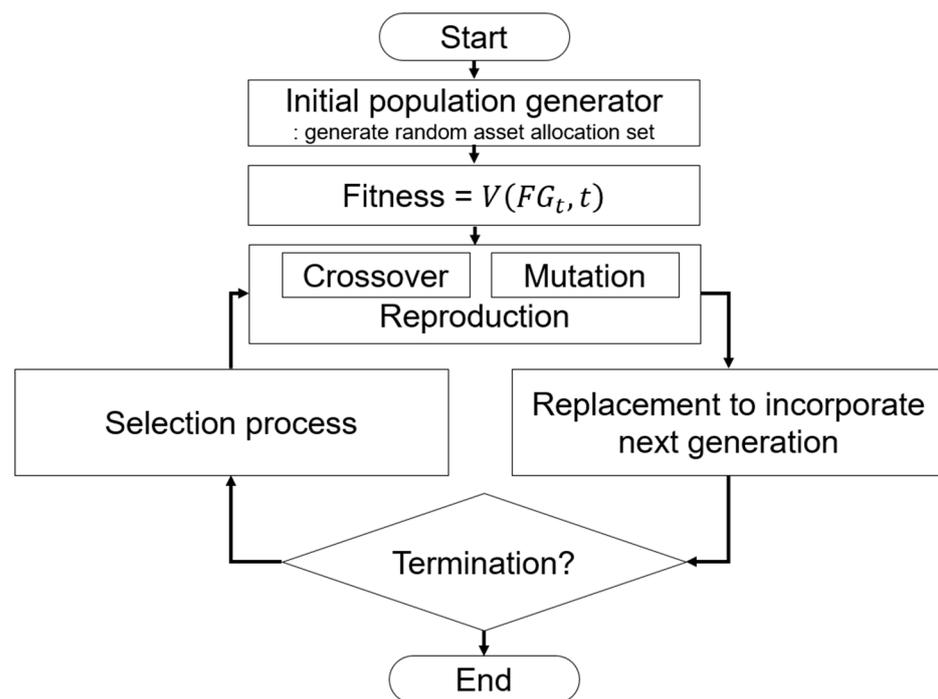


Figure A1. Flowchart of genetic algorithm.

Notes

- ¹ Contributions are divided into regular contributions, arising from the labor services provided by employees in the current period, and adjustment contributions, paid additionally to ensure financial soundness based on the results of the pension fund financial verification. Here, the contribution rate risk refers to the variability in the total contribution or the variability in the adjustment contribution.
- ² In the DB pension scheme, it refers to the possibility that future accumulated pension assets may fall short of the pension obligation agreed upon with the member or the volatility of the surplus (=accumulated pension asset—pension liability).
- ³ This study assumes the existence of only one member, and at the initial time point ($t = 0$), the member is 20 years old ($x = 20$). As time (t) increases, the age (x) also increases. We focused on the implications of incorporating loss aversion into asset allocation from the perspective of asset liability management in a defined-benefit pension scheme. Accordingly, we made a simplifying assumption by considering a single member within the pension scheme. If the number of employees increases, we can calculate the total liability of the entire pension fund by summing up the liabilities for each employee. Similarly, for assets, we can calculate the total assets by summing up the assets for each employee. Then, with the overall assets and liabilities of the entire fund determined, we can apply them to Equation (2) to calculate the utility at time t from the sponsor's perspective based on the total assets and liabilities of the fund.
- ⁴ For a more detailed discussion of the PUC method, see Blake (2006, p. 194).
- ⁵ As in the study of Albrecht and Maurer (2002), we assume that the bond rate of return follows a geometric Brownian motion, which is generally used for stock return.
- ⁶ As mentioned in Haberman and Sung (1994), since an explicit closed-form solution for the nonlinear dynamic equation cannot be obtained, we employed the genetic algorithm to derive a numerical solution.
- ⁷ For a more detailed discussion of the R function GA, see Scrucca (2013).

References

- Albrecht, Peter, and Raimond Maurer. 2002. Self-annuitization, consumption shortfall in retirement and asset allocation: The annuity benchmark. *The Journal of Pension Economics and Finance* 1: 269–88. [CrossRef]
- Amenc, Noël, Lionel Matellini, Felix Goltz, and Vincent Milhu. 2010. *New Frontiers in Benchmarking and Liability-Driven Investing*. Nice: EDHEC-Risk Institute Publication.
- Ang, Andrew, Bingxu Chen, and Suresh Sundaresan. 2013. Liability-driven investment with downside risk. *The Journal of Portfolio Management* 40: 71–87. [CrossRef]
- Bellman, Richard. 1957. *Dynamic Programming*. Princeton: Princeton University Press.
- Berkelaar, Arjan B., Roy Kouwenberg, and Thierry Post. 2004. Optimal portfolio choice under loss aversion. *The Review of Economics and Statistics* 86: 973–87. [CrossRef]

- Blake, David. 2006. *Pension Finance*. Chichester: Wiley.
- Blake, David, Douglas Wright, and Yumeng Zhang. 2013. Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion. *Journal of Economic Dynamics and Control* 37: 195–209. [[CrossRef](#)]
- Cairns, Andrew. 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. *Astin Bulletin* 30: 19–55. [[CrossRef](#)]
- Cairns, Andrew J. G., David Blake, and Kevin Dowd. 2006. Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control* 30: 843–77. [[CrossRef](#)]
- Deutsche Bank. 2010. *Pension Risk and the Rise of LDI*. Frankfurt: Deutsche Bank Group.
- Ezra, D. Don. 1991. Asset allocation by surplus optimization. *Financial Analysts Journal* 47: 51–57. [[CrossRef](#)]
- Frauentorfer, Karl, Ulrich Jacoby, and Alvin Schwendener. 2007. Regime switching based portfolio selection for pension funds. *Journal of Banking and Finance* 31: 2265–80. [[CrossRef](#)]
- Gupta, Pankaj, Garima Mittal, and Mukesh Kumar Mehawat. 2013. Expected value multiobjective portfolio rebalancing model with fuzzy parameters. *Insurance: Mathematics and Economics* 52: 190–203. [[CrossRef](#)]
- Haberman, Steven. 1997. Stochastic investment returns and contribution rate risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics* 19: 127–39. [[CrossRef](#)]
- Haberman, Steven, and Elena Vigna. 2002. Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics* 31: 35–69. [[CrossRef](#)]
- Haberman, Steven, and Joo-Ho Sung. 1994. Dynamic approaches to pension funding. *Insurance: Mathematics and Economics* 15: 151–62. [[CrossRef](#)]
- Haberman, Steven, Christopher Day, David Fogarty, Zaki Khorasane, Martin Mcwhirter, Nicola Nash, Bernard Ngwira, Douglas Wright, and Yakoub Yakoubov. 2003. A stochastic approach to risk management and decision making in defined benefit pension schemes. *British Actuarial Journal* 9: 493–618. [[CrossRef](#)]
- Haberman, Steven, Zoltan Butt, and Chryssoula Megaloudi. 2000. Contribution and solvency risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics* 27: 237–59. [[CrossRef](#)]
- Hoevenaars, Roy P. M. M., Roderick D. J. Molenaar, Peter C. Schotman, and Tom B. M. Steenkamp. 2008. Strategic asset allocation with liabilities: Beyond stocks and bonds. *Journal of Economics Dynamics and Control* 32: 2939–70. [[CrossRef](#)]
- Holland, John H. 1975. *Adaptation in Natural and Artificial Systems*. Ann Arbor: University of Michigan Press.
- Huang, Xiaoxia, and Tianyi Zhao. 2014. Mean-chance model for portfolio selection based on uncertain measure. *Insurance: Mathematics and Economics* 59: 243–50. [[CrossRef](#)]
- Hwang, Soosung, and Steve E. Satchell. 2010. How loss averse are investors in financial markets. *Journal of Banking & Finance* 34: 2425–38.
- Jin, Zhuo, Zhixin Yang, and Quan Yuan. 2019. A genetic algorithm for investment-consumption optimization with Value-at-Risk constraint and information-processing cost. *Risks* 7: 32. [[CrossRef](#)]
- Josa-Fombellida, Ricardo, Paula López-Casado, and Juan Pablo Rincón-Zapatero. 2018. Portfolio optimization in a defined benefit pension plan where the risky assets are processes with constant elasticity of variance. *Insurance: Mathematics and Economics* 82: 73–86. [[CrossRef](#)]
- Judd, Kenneth L. 1998. *Numerical Methods in Economics*. Cambridge and London: The MIT Press.
- Jung, Hyeonjong, Dong-Hwa Lee, and Do Young Cheong. 2022. Surplus optimization in defined benefit pensions using the regime-switching model: Occupational pension plans in South Korea. *Asia-Pacific Journal of Financial Studies* 51: 709–34. [[CrossRef](#)]
- Leibowitz, Martin L., Stanley Kogelman, and Lawrence N. Bader. 1992. Asset performance and surplus control: A dual-shortfall approach. *Journal of Portfolio Management* 18: 28–37. [[CrossRef](#)]
- Liu, Yong-Jun, and Wei-Guo Zhang. 2013. Fuzzy portfolio optimization model under real constraints. *Insurance: Mathematics and Economics* 53: 704–11. [[CrossRef](#)]
- Markowitz, Harry. 1952. Portfolio selection. *The Journal of Finance* 7: 77–91.
- Maurer, Raimond, Olivia S. Mitchell, and Ralph Rogalla. 2009. Managing contribution and capital market risk in a funded public defined benefit plan: Impact of CVaR cost constraints. *Insurance: Mathematics and Economics* 45: 25–34. [[CrossRef](#)]
- Miao, Jerry C. Y., and Jennifer L. Wang. 2006. Intertemporal stable pension funding. *Asia-Pacific Journal of Risk and Insurance* 1: 50–63. [[CrossRef](#)]
- Ngwira, Bernard, and Russell Gerrard. 2007. Stochastic pension fund control in the presence of Poisson jumps. *Insurance: Mathematics and Economics* 40: 283–92. [[CrossRef](#)]
- Peskin, Michael. W. 1997. Asset allocation and funding policy for corporate-sponsored defined benefit pension plans. *Journal of Portfolio Management* 23: 66–73. [[CrossRef](#)]
- Qian, Edward. 2012. Pension liabilities and risk parity. *Journal of Investing* 21: 93–103. [[CrossRef](#)]
- Scrucca, Luca. 2013. GA: A package for genetic algorithms in R. *Journal of Statistical Software* 53: 1–37. [[CrossRef](#)]
- Sharpe, William. F., and Lawrence G. Tint. 1990. Liabilities—a new approach. *The Journal of Portfolio Management* 16: 5–10. [[CrossRef](#)]
- Siegmann, Arjen. 2007. Optimal investment policies for defined benefit pension funds. *The Journal of Pension Economics and Finance* 6: 1–20. [[CrossRef](#)]

- Soleimani, Hamed, Hamid Reza Golmakani, and Mohammad Hossein Salimi. 2009. Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm. *Expert Systems with Applications* 36: 5058–63. [[CrossRef](#)]
- Tversky, Amos, and Daniel Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5: 297–323. [[CrossRef](#)]
- Waring, M. Barton. 2004. Liability-relative investing II. *The Journal of Portfolio Management* 31: 40–53. [[CrossRef](#)]
- Xie, Shuxiang. 2009. Continuous-time mean-variance portfolio selection with liability and regime switching. *Insurance: Mathematics and Economics* 45: 148–55. [[CrossRef](#)]
- Yang, Xiaolou. 2006. Improving portfolio efficiency: A genetic algorithm approach. *Computational Economics* 28: 1–14. [[CrossRef](#)]
- Zhang, Xin-Li, and Ke-Cun Zhang. 2009. Using genetic algorithm to solve a new multi-period stochastic optimization model. *Journal of Computational and Applied Mathematics* 231: 114–23. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.