*Risks* **2014**, *2*, 132-145; doi:10.3390/risks2020132

OPEN ACCESS

**risks**

**ISSN 2227-9091**

www.mdpi.com/journal/risks

*Article*

# Effectively Tackling Reinsurance Problems by Using Evolutionary and Swarm Intelligence Algorithms

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*Received: 15 January 2014; in revised form: 28 February 2014 / Accepted: 3 March 2014 / Published: 1 April 2014*

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**Abstract:** This paper is focused on solving different hard optimization problems that arise in the field of insurance and, more specifically, in reinsurance problems. In this area, the complexity of the models and assumptions considered in the definition of the reinsurance rules and conditions produces hard black-box optimization problems (problems in which the objective function does not have an algebraic expression, but it is the output of a system (usually a computer program)), which must be solved in order to obtain the optimal output of the reinsurance. The application of traditional optimization approaches is not possible in this kind of mathematical problem, so new computational paradigms must be applied to solve these problems. In this paper, we show the performance of two evolutionary and swarm intelligence techniques (evolutionary programming and particle swarm optimization). We provide an analysis in three black-box optimization problems in reinsurance, where the proposed approaches exhibit an excellent behavior, finding the optimal solution within a fraction of the computational cost used by inspection or enumeration methods.

**Keywords:** reinsurance; optimization problems; evolutionary-based algorithms

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## 1. Introduction

Reinsurance is an important risk management strategy in insurance, consisting of ceding part of the insurer's risk to a reinsurer, in exchange for a reinsurance premium. Reinsurance is an intelligent mechanism to reduce the insurer's risk retention, when it is possible to control the reinsurance premium [1]. Mathematically, let $X$ be a random variable that stands for the loss (claim) initially set by the insurer and $g(\cdot)$ a reinsurance function, $0 \leq g(X) \leq X$, that divides the total risk, $X$, into two parts: $g(X)$ (the ceded loss part, undertaken by the reinsurer) and $X - g(X)$ or the retention part (undertaken by the insurer). In general, the objective of optimal reinsurance design is to find the optimal function, $g(X)$, under different risk measures, reinsurance strategies and/or premium conditions [2–4].

Independently of the reinsurance model and strategy considered, the final optimal solution for a reinsurance problem involves the solution of an optimization problem. In many occasions, the analysis in research articles does not reach to this final stage of the problem, since specific assumptions on the model's variables and parameters need to be done. Instead, the expression of the optimization problem is described, without a final resolution of specific cases [5–8]. In other cases, numerical results are given, with little explanation of the optimization technique used, or just obtaining optimum values by inspection, whenever this is possible [9].

An interesting characteristic of the optimization problems in reinsurance applications is that many of them can be treated as black-box optimization problems (BBOPs), since the final expression of the problem cannot be represented in the form of a simple algebraic expression, but depends on the resolution of hard models (many times, integro-differential equations) with the appropriate specific parameters and simulation conditions. Evolutionary-based search algorithms have been traditionally applied to solve these problems, with excellent results [10,11].

In this paper, we analyze several different optimization problems in reinsurance that can be treated as a BBOP and solve them by using two state-of-the-art evolutionary and swarm intelligence approaches: the evolutionary programming and particle swarm optimization algorithms [12]. Specifically, we provide a discussion based on three optimization problems related to reinsurance contracts, which may affect the solvency of the insurer and reinsurer. We show that these problems can be solved by using evolutionary and particle swarm approaches in an optimal way, within a fraction of the computational cost used by inspection or enumeration methods.

The rest of the paper is structured in the following way: the next subsection describes the main characteristics of BBOPs and the specific details of three BBOPs that arise in insurance. Section 3 describes the evolutionary and swarm intelligence optimization approaches applied in this paper to solve the optimization problems previously described. In Section 4, numerical results are carried out, to show the good performance of evolutionary-based algorithms in the reinsurance problems discussed. Section 5 closes the paper by giving some final conclusions and remarks.

## 2. Black-Box Optimization Problems in Reinsurance

An optimization problem can be defined as a tuple $(\mathcal{S}, f(\mathbf{x}))$, where:

- $\mathcal{S}$ is a search space, formed by feasible elements $\mathbf{x} \in \mathcal{S}$.

- $f(\mathbf{x})$ is an objective function $\mathcal{S} \rightarrow \mathbb{R}$, to be optimized (maximized or minimized).

The problem consists of obtaining $\mathbf{x}_o$, such that $f(\mathbf{x}_o) > f(\mathbf{x})$, if the problem consists of maximizing the objective function (or $f(\mathbf{x}_o) < f(\mathbf{x})$ for minimizing), with $\{\mathbf{x}_o, \mathbf{x}\} \in \mathcal{S}$.

Optimization problems can be either continuous or discrete (combinatorial optimization), depending on whether the variables involved are continuous or discrete, and can be also characterized by its structure (linear, quadratic optimization, *etc.*) or the degree of constraints and locality (constrained global optimization, *etc.*). An optimization problem is called a black-box when the objective function to be optimized does not have an algebraic expression, but it is the output of a computer program (black-box) [11]. Black-box optimization problems (BBOPs) have several characteristics that make them especially difficult to be solved: first, no derivatives can be calculated on the objective function, which reduces the techniques available to solve these problems. The computation time of the objective function can also be a problem, since it can be prohibitively high (subrogate models are sometimes useful in these cases [11]). In addition, the structure of the problem cannot be exploited in the majority of BBOP cases, and sometimes, BBOPs involve some kind of noise in the objective function or their parameters, which makes the optimization even more complicated [10].

BBOP appears in the reinsurance field and, specifically, in the analysis of the the effect of reinsurance contracts on the solvency of the two agents that participate in the contract (insurer and reinsurer) [1,6,9,13]. One of the main measures used to control solvency is the ruin probability. In fact, we analyze three different optimization BBOPs in reinsurance focused on this measure.

### *2.1. Problem 1: Excess of Loss Reinsurance*

Let us consider as the first example of the optimization problem the optimality problem of minimizing the joint ruin probability of the insurer and reinsurer over a finite-time horizon, *i.e.*, the probability that at least one of them gets ruined before the fixed horizon. The aim of this problem is to find the optimal split of the total premium earned by the insurer between the insurer (cedent) and the reinsurer.

This joint ruin probability depends on the statistical characteristics of the insured risk, the initial reserves of the insurer and reinsurer, the time horizon and the premiums established by both companies. As we are considering here an excess of a loss contract, the parameters of this specific contract (deductible and maximum) will also have an influence on this probability.

The calculus of the joint ruin probability is not easy, nonetheless [14], and the problem can be treated as a BBOP after discretization of the variables involved in it. Let $\psi_{I,R}(c_R)$ be the joint ruin probability of the insurer and reinsurer when it is considered that all the variables that influence this probability are fixed, except the reinsurer premium ($c_R$). Then, the problem takes the following form:

$$\begin{array}{cc} \min\limits_{c_R} & \psi_{I,R}(c_R) \\ 0 \leq c_R \leq c & \end{array} \tag{1}$$

where $c$ is the total premium earned by the insurer (and paid by the policyholder) that will be split into two parts: the premium that is retained by the insurer and the premium that will receive the reinsurer ($c_R$).

### *2.2. Problem 2: Stop-Loss Reinsurance*

In the second optimization problem considered, the function to be minimized is the absolute value of the difference between the probability of survival of the insurer and the probability of survival of the reinsurer given the insurer's survival, over a finite time horizon. The decision variable is, as in the previous case, the reinsurer premium ($c_R$). Now, the reinsurance contract is a stop-loss, and therefore, the way of splitting the risk between the insurer and the reinsurer differs from that of the excess of loss, and the ruin and survival probabilities are different, too. The parameters of the stop-loss (deductible and maximum) influence the probabilities and, thus, the differences to be minimized. The other factors to be taken into account are the same as in the excess of loss contract.

The calculus of this difference is a difficult one, and it is not possible to find an explicit expression, so it can be solved as a BBOP. It is necessary to consider the probability of survival of the insurer with a stop-loss ($\phi_I(c_R)$), which can be obtained by adapting the univariate model explained in [15]. We must also consider the probability of the survival of the reinsurer given the insurer's survival. This conditional probability can be calculated as the quotient between the joint survival probability of the insurer and the reinsurer and the insurer's survival probability. The process to derive the joint survival probability, $\phi_{I,R}(c_R)$, can be found in [14] (Proposition 1), and a discretization of the claim amount distribution is also needed.

The statement of this problem is the following:

$$\min_{\substack{c_R \\ 0 \le c_R \le c}} \quad f(c_R) = \left| \phi_I(c_R) - \frac{\phi_{I,R}(c_R)}{\phi_I(c_R)} \right| \tag{2}$$

where $c$ is the total premium earned by the insurer (and paid by the policyholder) that will be split in two parts: the premium that is retained by the insurer and the premium that will receive the reinsurer ($c_R$).

### *2.3. Problem 3: Threshold Proportional Reinsurance*

The final problem tackled is the hardest one and has been previously tackled in [9]. It consists of minimizing the ultimate ruin probability of the insurer in a threshold proportional reinsurance, *i.e.*, the probability that the insurer's surplus level eventually falls below zero in the case that the insurer cedes a percentage of the insured risk to a reinsurer. Our aim is to find the optimal value of the parameters of this kind of reinsurance that minimize this probability.

The probability of ultimate ruin depends on the statistical characteristics of the insured risk (the distribution of the amount of each claim and the distribution of the number of claims), the initial surplus of the insurer and the premiums established by both companies. We consider that the insurer and the reinsurer use the expected value principle to calculate their premiums, and then, they have to apply positive loading factors. A specific threshold proportional reinsurance can be identified by three parameters: $k_1$, the retention level of the insurer when its reserves are less than the threshold, $k_2$, the retention level of the insurer when its reserves are greater than or equal to the threshold, and $b$, the level of the threshold.

Let $\psi_I(k_1, k_2, b)$ be the ultimate ruin probability of the insurer when all the variables that influence this probability are considered to be fixed except the parameters of the threshold reinsurance. Then, this problem can be expressed as follows:

$$\begin{array}{cc} \min\limits_{k_1, k_2, b} & \psi_I(k_1, k_2, b) \\ \frac{\rho_R - \rho}{\rho_R} < k_1 \leq 1 & \\ \frac{\rho_R - \rho}{\rho_R} < k_2 \leq 1 & \\ b > 0 & \end{array}$$

where $\rho$ and $\rho_R$ are the loading factors of the insurer and the reinsurer, respectively. If we assume certain statistical distributions for the claim amount, explicit expressions for $\psi_I(k_1, k_2, b)$ can then be found.

## 3. Evolutionary-Based Algorithms

Evolutionary-based algorithms [16–19] are robust problem solving techniques based on natural evolution processes. They are population-based techniques, which codify a set of possible solutions to the problem and evolve it through the application of certain evolution rules. Evolutionary-based algorithms have been previously discussed in economic applications [20–23], including insurance problems [24,25]. In this paper, we consider two different types of evolutionary-based approaches, focused on continuous optimization problems: evolutionary programming and particle swarm optimization.

### *3.1. Evolutionary Algorithms: Evolutionary Programming*

Among evolutionary approaches, evolutionary programming (EP) approaches have been successfully applied to continuous optimization problems. This algorithm is characterized by only using mutation and selection operators (no crossover is applied). Several versions of the algorithm have been proposed in the literature: A classical evolutionary programming algorithm, first described in the work by Bäck and Schwefel in [16], and analyzed later by Yao *et al.* in [19,26], can be defined as follows:

1. Generate an initial population of $\mu$ individuals (solutions). Let $t$ be a counter for the number of generations; set it to $t = 1$. Each individual is taken as a pair of real-valued vectors, $(\mathbf{x}_i, \boldsymbol{\sigma}_i)$, $\forall i \in \{1, \cdots, \mu\}$, where $\mathbf{x}_i$'s are objective variables and $\boldsymbol{\sigma}_i$'s are standard deviations for Gaussian mutations.
2. Evaluate the fitness value for each individual ($\mathbf{x}_i$, $\boldsymbol{\sigma}_i$) (using the problem's objective function).
3. Each parent ($\mathbf{x}_i$, $\boldsymbol{\sigma}_i$), $\{i = 1, \cdots, \mu\}$, then creates a single offspring ($\mathbf{x}'_i$, $\boldsymbol{\sigma}'_i$) as follows ($j$ denotes components of the $i$-th vector):

$$\mathbf{x}'_i(j) = \mathbf{x}_i(j) + \sigma_i(j) \cdot N_j(0, 1) \tag{3}$$

$$\sigma'_i(j) = \sigma_i(j) \cdot \exp(\tau' \cdot N(0, 1) + \tau \cdot N_j(0, 1)) \tag{4}$$

where $N(0, 1)$ denotes a normally distributed one-dimensional random number with mean zero and standard deviation one, and $N(0, 1)$ and $N_j(0, 1)$ are random numbers of mean zero and standard deviation one, generated anew for each value of $i$ or $j$, respectively. The parameters,

$\tau$ and $\tau'$, are commonly set to $(\sqrt{2\sqrt{n}})^{-1}$ and $(\sqrt{2n})^{-1}$, respectively [19], where $n$ is the length of the individuals.

4. If $x_i(j) > lim\_sup$, then $x_i(j) = lim\_sup$, and if $x_i(j) < lim\_inf$, then $x_i(j) = lim\_inf$.
5. Calculate the fitness values associated with each offspring $(\mathbf{x}'_i, \boldsymbol{\sigma}'_i)$, $\forall i \in \{1, \cdots, \mu\}$.
6. Conduct pairwise comparison over the union of parents and offspring: for each individual, $p$ opponents are chosen uniformly at random from all the parents and offspring. For each comparison, if the individual's fitness is better than the opponent's, it receives a "win".
7. Select the $\mu$ individuals out of the union of parents and offspring that have the most "wins" to be parents of the next generation.
8. Stop if the halting criterion is satisfied, and if not, set $t = t + 1$ and go to Step 3.

### *3.2. Particle Swarm Optimization*

Particle swarm optimization (PSO) is another population-based stochastic optimization technique developed by Eberhart and Kennedy [27], inspired by the social behavior of bird flocking and fish schooling. PSO belongs to the family of algorithms known as swarm intelligence approaches, and it has also been mainly applied to solve continuous optimization problems. A PSO system is initialized with a population of random solutions and searches for the optimal one by updating the population over several generations. PSO has no evolution operators, such as crossover and mutation, as genetic algorithms do, but potential solutions instead, called *particles*, which fly through the problem search space to look for promising regions on the basis of their own experiences and those of the whole group. Thus, social information is shared, and also, individuals profit from the discoveries and previous experiences of other particles in the search. The PSO is considered a global search algorithm.

Mathematically, given a swarm of $N$ particles, each particle $i \in \{1, 2, \cdots, N\}$ is associated with a position vector $\mathbf{x}_i = (x_1^i, x_2^i, \cdots, x_K^i)$, $K$ being the number of parameters to be optimized in the problem. Let $\mathbf{p}_i$ be the best previous position that particle $i$ has ever found, *i.e.*, $\mathbf{p}_i = (p_1^i, p_2^i, \cdots, p_K^i)$, and $\mathbf{g}$ be the group's best position ever found by the algorithm, *i.e.*, $\mathbf{g} = (g_1, g_2, \cdots, g_K)$. At each iteration step, $k + 1$, the position vector of the $i$-th particle is updated by adding an increment vector, $\Delta\mathbf{x}_i(k + 1)$, called velocity $\mathbf{v}_i(k + 1)$, as follows:

$$v_d^i(k+1) = \omega v_d^i(k) + c_1 r_1 (p_d^i - x_d^i(k)) + c_2 r_2 (g_d - x_d^i(k)) \tag{5}$$

$$x_d^i(k+1) = x_d^i(k) + v_d^i(k+1) \tag{6}$$

where $\omega$ is the inertia weight, $c_1$ and $c_2$ are two positive constants and $r_1$ and $r_2$ are two random parameters, which are found uniformly within the interval $[0, 1]$. In addition the following constraint holds:

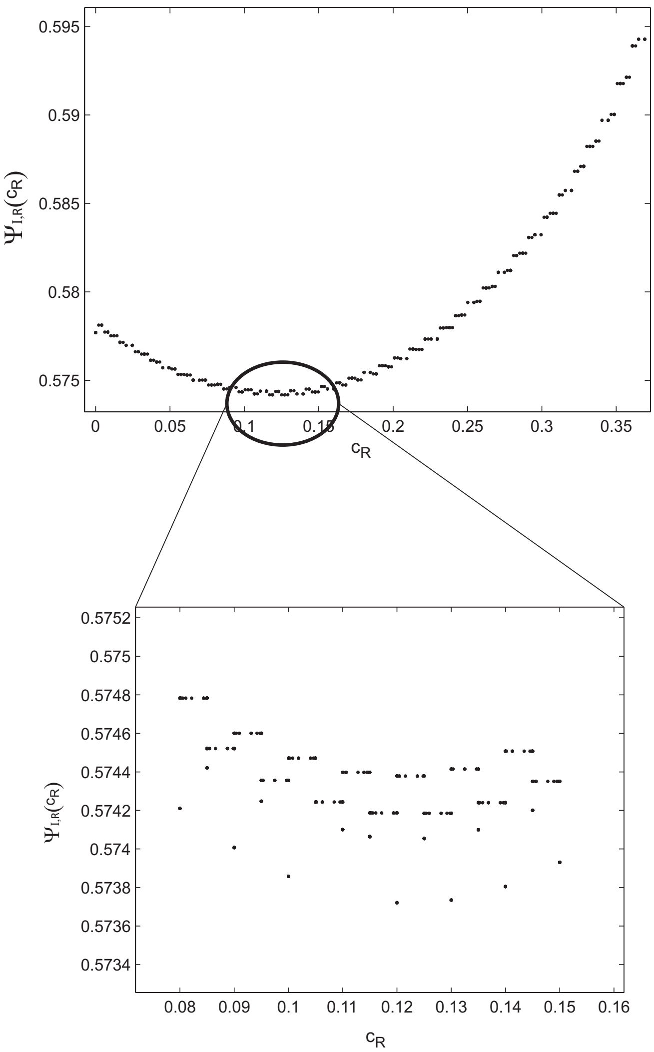
$$v_d^i(k+1) = \frac{v_d^i(k+1) \cdot V_d^{max}}{|v_d^i(k+1)|}, if |v_d^i(k+1)| > v_d^{max} \tag{7}$$

where $v_d^{max}$ is a parameter that limits the velocity of the particle in the $d$-th coordinate direction. This iterative process will continue until a stop criterion is fulfilled, this forming the basic iterative process of a standard PSO algorithm [27].

## 4. Numerical Results

In this section, we show the results obtained applying the considered evolutionary-based approaches to the three BBOPs in reinsurance. The first two problems are described by just one variable. They can be solved by an inspection algorithm that covers enough of a range of $c_R$ values. The third problem is more complicated, since it involves three variables ($k_1$, $k_2$ and $b$). Note that in the case of the EP, the encoding of the individuals include the variance of each variable, as shown in Section 3.1. A more advanced inspection algorithm can be used to solve it, DIRECT [28]. The analysis and discussion carried out consists in evaluating the quality of the solutions given by the evolutionary algorithms proposed and also the computation time to reach this optimal solution. The parameters of the algorithms compared in this section are the following: for the EP, $\mu = 20$ individuals, $p = 85\%$ (tournament selection), during 50 generations. Regarding the parameters of the PSO, $\omega = 0.9$ and $c_1 = c_2 = 2$ provided the best results in the problems considered. The number of particles and generations were set to 20 and 50, respectively. We have set these parameters in such a way that the number of function evaluations for both algorithms are the same.

**Figure 1.** The objective function and zoom of the excess of the loss reinsurance optimization problem.



### *4.1. Results in Problem 1*

For illustration in this problem, let us assume that the number of claims in Problem 1 can be modeled as a Poisson random variable with a parameter of one, and the claim amount is exponentially distributed with a mean of one monetary unit (m.u.) [29]. The initial reserves of the insurer are $0.1$ m.u., and the initial reserves of the reinsurer are $0.25$ m.u. We consider a time horizon of two years, and the premium established by the insurer (and paid by the policyholder) is $1.05$ m.u. We consider, in addition, an excess of loss contract with a deductible of $0.8$ m.u. and without a maximum. The span of discretization used is $0.01$. Figure 1 shows the function and a zoom in the zone of interest (function optimum).

Table 1 shows the results obtained in this problem with the EP and PSO approaches. Note that both algorithms are able to converge fast (within very few seconds) to almost the same solution (global optimum of the function). In Table 1, we observe that from the optimal split of the total premium earned by the insurer of $1.05$ m.u., a total of $0.12$ m.u. go for the reinsurer and $(1.05–0.12)$ m.u. for the insurer. This split gives a minimum joint ruin probability of $0.5737$ (approximately). Usually, the reinsurer's premium in excess of the loss contract is calculated looking at the cost that is assumed by the reinsurer. Hence, we suggest a new method for calculating reinsurance premiums that takes into account the whole business.

**Table 1.** Results in Problem 1 (an excess of the loss reinsurance model) using the evolutionary programming (EP) and particle swarm optimization (PSO) algorithms.

| **Algorithm** | $\psi_{I,R}(c_R)$ | $c_R$ | **Computation time (s)** |
|---|---|---|---|
| EP | 0.57372111552 | 0.1200013 | 3.5 |
| PSO | 0.5737211153 | 0.1200014 | 2.2 |

### *4.2. Results in Problem 2*

In this case, let us assume that the number of claims in Problem 2 is a Poisson random variable with a parameter of one and that the claim amount is exponentially distributed with a mean of $1$ m.u. The initial reserves of the insurer and the reinsurer are $0$ m.u. The considered horizon is two years, and the premium established by the insurer (and paid by the policyholder) is $1.05$ m.u. We consider a stop-loss contract with a deductible of $0.8$ m.u. and with a maximum of $1.5$ m.u. The span of discretization used is $0.01$. Figure 2 shows the function and a zoom in the zone of interest (function optimum), in this case.

Table 2 shows the results obtained in this problem with the EP and PSO approaches. Note that in this case, both algorithms converge to the same solution within a small computation time. If the objective is to minimize the absolute value of the difference between the probability of survival of the insurer and the probability of survival of the reinsurer given the insurer's survival over a horizon of two years, the reinsurer must receive $0.14$ m.u. as its premium and the insurer must retain $(1.05–0.14)$ m.u. The minimum absolute value obtained is almost zero. Thus, with this split, the probability of survival of the insurer is almost equal to the probability of survival of the reinsurer given the insurer's survival.

**Figure 2.** The objective function and zoom of the stop-loss reinsurance optimization problem.

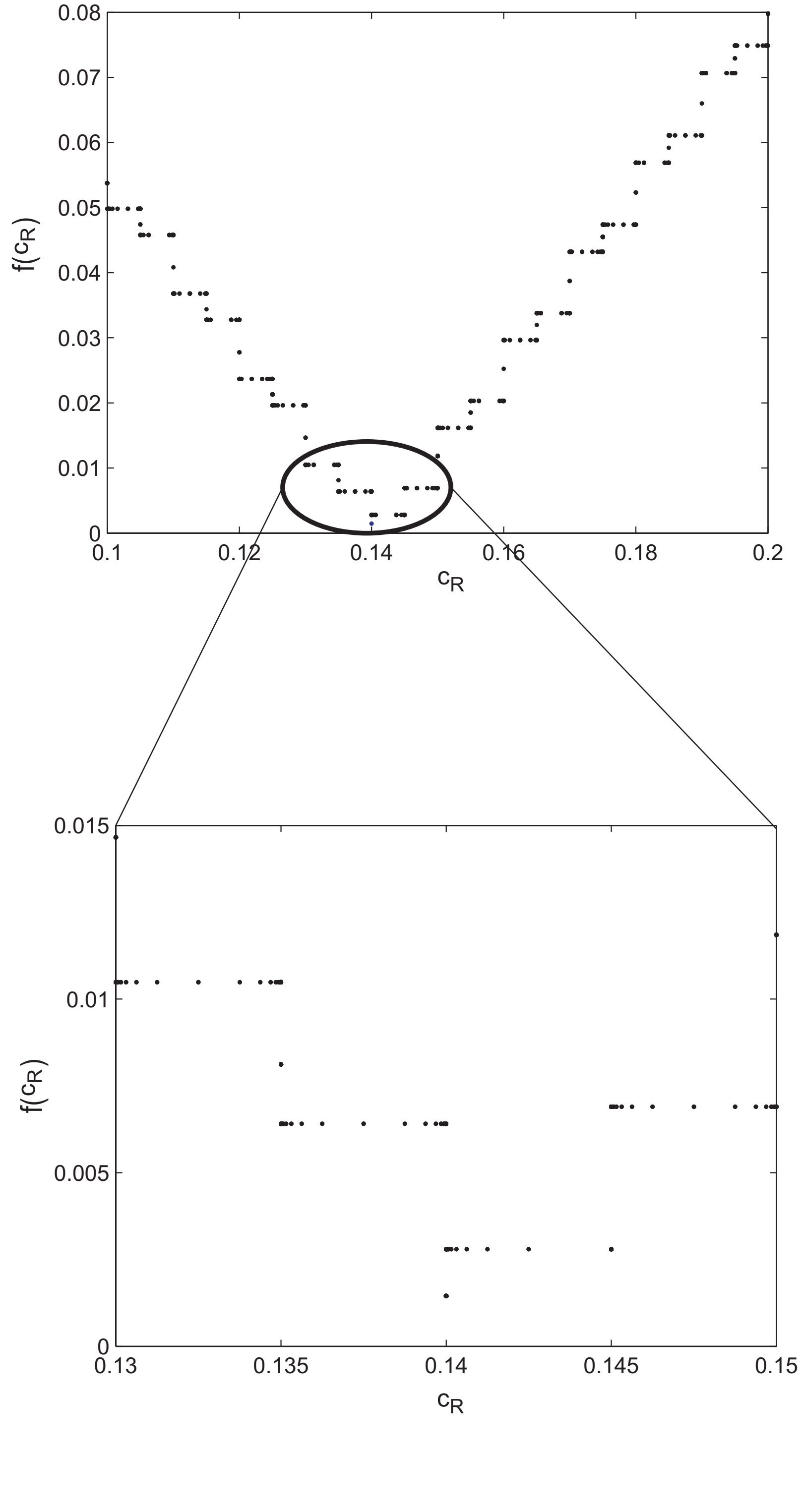


**Table 2.** Results in Problem 2 (the stop-loss reinsurance model) using the EP and PSO algorithms.

| Algorithm | $f(c_R)$ | $c_R$ | Computation time (s) |
|---|---|---|---|
| EP | 0.0014486748 | 0.14 | 2.9 |
| PSO | 0.0014486748 | 0.14 | 2.4 |

### *4.3. Results in Problem 3*

This is the hardest problem we tackle in this paper and may be solved with different assumptions. We consider two cases: first, an exponential distribution with a parameter of one; and second, an Erlang(2,2) distribution (in both cases, the mean claim amount is one m.u.). We also assume that the number of claims follows a Poisson distribution with a parameter of one; the initial surplus of the insurer is $4$ m.u. and the loading factors of the insurer and the reinsurer are $0.15$ and $0.25$, respectively. The optimization problem is then:

$$\begin{array}{cl} \min\limits_{k_1, k_2, b} & \psi_I(k_1, k_2, b) \\ 0.4 < k_1 \leq 1 & \\ 0.4 < k_2 \leq 1 & \\ b > 0 & \end{array}$$

If the claim amount is exponentially distributed,

$$\psi_I(k_1, k_2, b) = \begin{cases} 1 - 1.15A + Ae^{-\frac{0.521739}{k_1}}, & 4 < b \\ \left(1 - 1.15A + Ae^{-\frac{0.130435}{k_1}b}\right) e^{\frac{0.2}{k_2}(b-4)}, & 0 < b \leq 4 \end{cases}$$

where:

$$A = \frac{h}{1.15h + 0.1725(k_1 - k_2)e^{-\frac{b}{k_2}} + (0.15k_2 - h)\, e^{-\frac{0.130435}{k_1}b}}$$

with $h = 0.25(1.15k_1 - 0.15k_2)$.

If the claim amount follows an Erlang(2,2) distribution, the explicit expression of the ultimate ruin probability is more complex than in the exponential case. It can be found in [9].

The EP and PSO algorithms have been applied to this problem, increasing the maximum generations allowed to 100. Table 3 shows the results obtained with the EP and PSO algorithms in this problem, including both cases of claim amount distributions considered. In this case, both algorithms give also similar results, close to the global optimum of the function. The computation time remains within 10 s.

**Table 3.** Results in Problem 3 (the threshold proportional reinsurance model) using the EP and PSO algorithms.

| Algorithm | $\psi_I(k_1, k_2, b)$ | $k_1$ | $k_2$ | $b$ | Computation time (s) |
|---|---|---|---|---|---|
| Exponential | | | | | |
| EP | 0.4980669653 | 1.0 | 0.7596477801 | 3.2688654179 | 9.6 |
| PSO | 0.4980669664 | 1.0 | 0.7596477914 | 3.2688441178 | 8.5 |
| Erlang(2,2) | | | | | |
| EP | 0.415635 | 1.0 | 0.761572 | 1.9871 | 9.5 |
| PSO | 0.415641 | 1.0 | 0.761564 | 1.9866 | 8.5 |

If the claim amount is exponentially distributed (with a mean of one), the optimal strategy for the insurer is to choose a threshold level of $3.2688$, not to reinsure ($k_1 = 1$) when the reserves are below this level and to reinsure with a retention level $k_2 = 0.7596477$ when the reserves are above the threshold. When the claim amount follows an Erlang(2,2) distribution, the minimum ruin probability is attained when $b = 1.98$, $k_1 = 1$ and $k_2 = 0.7615$. In this example, the exponential distribution and the Erlang(2,2) have the same mean. Then, it can be concluded that the distribution of the claim amount influences the optimal strategy.

### *4.4. Discussion*

Table 4 shows the exact solutions to the three problems considered using inspection algorithms (uniform inspection of $c_R$ for the two first problems and using the DIRECT algorithm [28] in the case of the third one). It is easy to see that the solutions obtained by the evolutionary-based algorithms are very close to these exact solutions, and the computation time is a small fraction of the one employed by the inspection methods. In larger problems involving more variables, the application of exact methods is very often not possible. In these cases, the application of meta-heuristics, such as evolutionary algorithms, is an excellent option to obtain good quality solutions with bounded computation times.

**Table 4.** Exact solutions (inspection-based algorithms).

| Problem # | Optimal solution | Computation time (s) |
|---|---|---|
| Problem 1 | $c_R = 0.1200000$; $\psi_{I,R}(c_R) = 0.573721115524965$ | 2100 |
| Problem 2 | $c_R = 0.1400000$; $f(c_R) = 0.001448674863134$ | 2840 |
| Problem 3 (Exponential) | $k_1 = 1.0000000000$; $k_2 = 0.759647780145614$; $b = 0.759647780145614$; $\psi_I(k_1, k_2, b) = 0.498066965355012$ | 3700 |
| Problem 3 (Erlang) | $k_1 = 1.000000$; $k_2 = 0.761572$; $b = 1.9871$; $\psi_I(k_1, k_2, b) = 0.415635$ | 3850 |

## 5. Concluding Remarks

In this paper, we have done an analysis of the application of evolutionary-based optimization techniques for black-box optimization problems (BBOPs) in reinsurance. Three BBOPs have been tackled with an evolutionary programming approach and a particle swarm optimization algorithm. The BBOPs considered are continuous optimization problems, related to reinsurance contracts, which may have an important impact on the solvency of the insurer and reinsurer. Two of them are one variable problems (an excess of loss and stop-loss reinsurance problems), and the third one is a three-variable problem related to threshold proportional reinsurance. The importance of these problems is that they allow a detailed analysis of the evolutionary-based approaches, since the solutions for the problems are known (which can be obtained exactly by a full inspection algorithm in the two first problems and bounded in the third). Thus, we have shown how the evolutionary algorithms are able to find extremely good solution to the problems within a fraction of the computation time used by inspection algorithms. Moreover, in harder BBOPs, where inspection search algorithms are not applicable (a higher number of variables or high constraints), evolutionary approaches are an excellent option to find good solutions in short computation times.

**Acknowledgments**

This work has been partially supported by the Spanish Ministry of Science and Innovation, under projects number ECO2010-22065-C03-02 and ECO2010-22065-C03-03.

**Conflicts of Interest**

The authors declare no conflict of interest.

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