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An Integrated Approach to Pricing Catastrophe Reinsurance

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Abstract: We propose an integrated approach straddling the actuarial science and the mathematical finance approaches to pricing a default-risky catastrophe reinsurance contract. We first apply an incomplete-market version of the no-arbitrage martingale pricing paradigm to price the reinsurance contract as a martingale by a measure change, then we apply risk loading to price in—as in the traditional actuarial practice—market imperfections, the underwriting cycle, and other idiosyncratic factors identified in the practice and empirical literatures. This integrated approach is theoretically appealing for its merit of factoring risk premiums into the probability measure, and yet practical for being applicable to price a contract not traded on financial markets. We numerically study the catastrophe pricing effects and find that the reinsurance contract is more valuable when the catastrophe is more severe and the reinsurer's default risk is lower because of a stronger balance sheet. We also find that the price is more sensitive to the severity of catastrophes than to the arrival frequency; implying (re)insurers should focus more on hedging the severity than the arrival frequency in their risk management programs.

Keywords: mathematical finance; actuarial science; catastrophe arrivals; catastrophe reinsurance; default risk; Monte Carlo simulation

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1. Introduction

Catastrophe (cat hereafter) reinsurance contracts are not traded on financial markets, instead they are one-shot deals made between reinsurers and insurers. In actuarial science, catastrophe reinsurance is conventionally priced according to a two-stage process where in the first stage, catastrophe modelers like RMS (Risk Management Solutions), AIR (AIR Worldwide) and EQE (EQECAT) produce loss distributions through catastrophe modeling and calibration, and in the second stage insurers and reinsurers negotiate prices based on the given loss distributions with additional consideration of non-modeled market factors. This two-stage process is complex as many elements, including input data quality, catastrophe modeling output, and non-modeled market factors, can affect premiums, and some of these elements cannot be quantified but require subjective judgements. It is therefore fair to say that cat reinsurance pricing, as summarized in the following pricing norm, is a mix of art and science:

$$\textbf{Layer premium = Expected layer loss + Risk load factor} \times \textbf{Standard Deviation,}$$

where the expected layer loss and standard deviation are obtained from catastrophe models inclusive of the consideration of the modeled factors of input quality, model errors and output uncertainty, while the risk load factor is a subjective entry that depends on risk preference, reinsurers' capital position, and (re)insurance market conditions, among other considerations.

Reinsurers may also load non-modeled market factors on top of the modeled factors to increase the premium. In the academic literature, Froot (2001) and Froot and O'Connell (2008) have shown that reinsurers' monopolistic power and the high costs of capital resulted from inefficient corporate forms[1] have led to unusually high risk loadings with spreads at around 5–8 times the expected loss. Moreover, these multiples tend to jump even higher in the aftermath of a major cat event due to reinsurers' "reloading" of balance sheets. In practice, reinsurers may also load geographic concentration, data quality in capturing exposure characteristics, a cedant's loss experience, and reinsurance relationship.

The merit of the traditional actuarial pricing approach is that it works in the imperfect global reinsurance market, where many reinsurers also ensemble multiple pricing models to reign in model errors. The drawback is that the pricing norm is ad hoc but not derived from prevailing financial valuation theories.

In mathematical finance, theoreticians have attempted to price cat reinsurance as if it were traded on financial markets by using option pricing theory, non-arbitrage martingale pricing theory, and utility optimization theory. They also take cat losses as exogenously given with a common assumption that loss arrivals follow a compound Poisson process or its variations as in Bowers et al. (1986), in lieu of using the loss output from a cat modeler. For example, Chang et al. (1989) value reinsurance in an option theoretical framework, Dassios and Jang (2003) value cat reinsurance using the Cox process with shot noise intensity under a no-arbitrage martingale probability measure, and Lee and Yu (2007) value default-risky cat reinsurance with catastrophe bonds under a no-arbitrage martingale probability measure. However, the unrealistic perfect-market assumptions, the overly-simplified specification of the loss arrival process, and the lack of data to calibrate the process have made direct implementation of these theories in the global reinsurance market impractical.

In this paper, we propose an integrated approach to pricing a default-risky cat reinsurance contract. First, we apply an incomplete-market version of the no-arbitrage martingale pricing paradigm of Harrison and Kreps (1979) and Harrison and Pliska (1981)[2], which is internally consistent with option pricing theory, to factor in the modeled risk premiums by means of a measure change. We then add risk loading/markup (used interchangeably hereafter) to price in the non-modeled market factors identified in practice and the empirical literature. This integrated approach is theoretically appealing in being able to factor the risk premiums into the probability measure, and yet applicable to pricing a contract traded on the global reinsurance market by taking into consideration non-modeled risk loading. We could have used the equilibrium approach by imposing a preference assumption, however applying utility functions to resolve problems incurred in incomplete markets is often impractical as the results become overly complicated to be fully specified by practitioners (see Carr et al. (2001)). Another problem with the equilibrium approach as documented by many empirical studies is that

1 Traditional reinsurers tend to invest in illiquid and information-intensive financial activities, and so they charge premiums based on correlations with their own pre-existing portfolios and U.S. nationwide cat risks, rather than with any market portfolio, resulting in significantly higher cost of capital.

2 A martingale is a stochastic variable with no drift such that the current expectation of the future value of the variable is always equal to the current value of the variable. The above authors show that the absence of arbitrage is equivalent to the existence of an equivalent martingale pricing measure Q (or risk-neutral/risk-neutralized pricing measure as used by some authors) such that all normalized (with respect to a chosen numeraire) security prices are martingales and as such they can be priced by taking expectations under the measure Q. The beauty of martingale pricing is that it applies to both complete and incomplete markets as "absence of arbitrage" is the only required assumption. When markets are complete with the absence of arbitrage, martingale pricing theory guarantees that the equivalent martingale measure is unique, thus the market price of risk does not explicitly enter into the valuation process. In this context, options can be priced preference-free *as if* agents were risk-neutral as in the Black–Scholes model. When markets are incomplete, however, the absence of arbitrage no longer guarantees a unique martingale measure. In this case, information related to the market prices of risk that embeds the risk-aversion behavior of agents is needed to uniquely identify the equivalent martingale measure (see Geman (2005) for a review). The procedure of embedding the market price of risk to obtain such a unique measure is often called *risk-neutralization*.

for a CRRA (Constant Relative Risk Aversion) utility to work, the risk-aversion parameter must be unreasonably small (e.g., Bollerslev et al. (2011)).

Our integrated approach proceeds as follows:

(1) We first model a default-risky reinsurer by employing Merton (1974) structural approach to endogenize default.

- On the asset side, since the reinsurer holds a large proportion of fixed-income assets in the asset portfolio, we model the asset dynamics taking into account explicitly the impact of stochastic interest rates. We make a measure change from the physical pricing measure to the equivalent martingale pricing measure Q by embedding the market price of interest rate risk.
- On the liability side, since the reinsurer's non-catastrophic liability shocks are idiosyncratic and small, we apply the law of large numbers to assume away this risk premium. We also make a measure change from the physical pricing measure to the equivalent martingale pricing measure Q by embedding the market price of interest rate risk.

(2) We model catastrophe arrivals. We make use of the empirical finding that catastrophe derivatives are zero-beta assets, and thus both the loss number and the amount of losses have zero risk premiums.

(3) We proceed to price a default-risky reinsurance contract under the equivalent martingale pricing measure Q as a martingale.

(4) Finally, we extend the pricing formula to incorporate risk load/markup to account for the observed empirical characteristics of the (re)insurance market. The interpretation of the markup is the same as in the traditional actuarial pricing approach except that (1) our expectation is taken with respect to the risk-neutralized martingale pricing measure, but in the traditional approach the expectation is taken with respect to the physical pricing measure; and (2) our markup only needs to account for market imperfections and other idiosyncratic factors, while the markup in the traditional actuarial approach accounts for both modeled and non-modeled factors.

To load non-modeled market factors, we suggest calibrating the markup using empirical cat reinsurance data as it has become increasingly available because of the growth of the market. Although endogenizing the markup could further enrich the analysis and provide more intuition, it constitutes a major undertaking in an incomplete-market general equilibrium setting encompassing two heterogeneous agents. Besides considering market imperfections such as "taxes, agency costs, issuance costs", risk aversions, and costs of capital associated with bearing the catastrophe risk of the two counterparties, one may also need to consider the insurance underwriting cycle, optimum total catastrophe financing, the tradeoff between capital reserve and hedging, reinsurers' monopolistic/oligopolistic positions in the reinsurance markets, and other idiosyncratic factors identified in the empirical literature. We will leave this task to a future project.

2. Modeling Reinsurer Default with Asset, Interest Rate, Liability, and Catastrophe Loss

Dynamics

Since the reinsurer's asset-liability structure and catastrophe loss specification are important factors in modeling the reinsurer's default, we begin by employing Merton (1974) structural approach to model the balance sheet of the reinsurer in a continuous-time no-arbitrage martingale framework. The Merton approach has the advantage of linking the valuation of financial claims to the firm's assets and capital structure. Thus unlike the more recent approach to incorporate jumps and to integrate with the capital asset pricing model (CAPM) in order to analyze corporate bond spreads (e.g., Collin-Dufresne et al. (2012) and Coval et al. (2009)), our structural model builds upon the work of Duan et al. (1995), Duan and Yu (2005), and Lee and Yu (2007) to properly allow for the asset, liability, interest rate, and cat loss dynamics.

We employ a one-factor interest rate model assuming the asset price and the instantaneous interest rate processes are governed by the following correlated general Wiener process with drifts:

$$\frac{dV_t}{V_t} = \mu(V_t, t)dt + \sigma(V_t, t)dZ_{V,t}, \tag{1}$$

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dZ_{r,t}, \tag{2}$$

where $Z_{v,t}$ and $Z_{r,t}$ are correlated Wiener processes.

Typically, reinsurers hold a large proportion of fixed-income assets in the asset portfolio and issue catastrophe bonds to lay off catastrophe risk (see Lee and Yu (2007)). To explicitly examine the interest rate exposure, a further decomposition of the asset process is required to provide a direct interpretation of interest rate risk. Projecting $dZ_{V,t}$ onto $dZ_{r,t}$ yields

$$dZ_{V,t} = \varphi dZ_{r,t} + \sqrt{1-\varphi^2}dW_{V,t}, \tag{3}$$

where $\varphi = \frac{Cov(dZ_{V,t}, dZ_{r,t})}{dt}$. As a result of this projection, $W_{V,t}$, denoting the credit risk on the assets of the reinsurance company, is orthogonal to Z_t by construction. As explained in footnote 7 of Duan et al. (1995), the term credit risk here is used to designate all risks other than the interest rate risk. To the extent that defaults on fixed income assets are related to interest rates, the credit risk is considered as part of the interest rate risk, thus $W_{V,t}$ may not strictly correspond to the conventional definition of credit risk.

Theorem 1. *In the one-factor interest rate model, Equation (1) can be risk-neutralized to the following risk-neutralized asset price process with a measure change,* $dZ_t^* = dZ_t + \frac{\lambda_r \sqrt{r_t}}{v}dt$ *and* $dW_{V,t}^* = dW_{V,t}$, *from the physical measure to the equivalent martingale measure Q:*

$$\frac{dV_t}{V_t} = r_t dt + \phi_V v\sqrt{r_t}dZ_t^* + \sigma_V dW^*{}_{V,t}. \tag{4}$$

Proof.

As shown in Appendix A of Lee and Yu (2007), substituting Equation (3) into Equation (1) yields

$$\frac{dV_t}{V_t} = \mu(V_t, t)dt + \sigma(V_t, t)\varphi dZ_{r,t} + \sigma(V_t, t)\sqrt{1-\varphi^2}dW_{V,t}, \tag{5}$$

and further substituting Equation (2) into Equation (5) yields

$$\frac{dV_t}{V_t} = \mu_V dt + \phi_V dr_t + \sigma_V dW_{V,t}, \tag{6}$$

where $\mu_V = [\mu(V_t, t) - \frac{\sigma(V_t, t)\varphi\mu(r_t, t)}{\sigma(r_t, t)}]$, $\phi_V = \sigma(V_t, t)\varphi/\sigma(r_t, t)$ is the instantaneous interest rate elasticity of the assets of the reinsurance company, and $\sigma_V = \sigma(V_t, t)\sqrt{1-\varphi^2}$ is the volatility of the credit risk.

Next, we assume the general instantaneous interest rate process, Equation (2), is specified by the squared-root process of Cox et al. (1985 and CIR hereafter) to avoid negative interest rates:

$$dr_t = \kappa(m - r_t)dt + v\sqrt{r_t}dZ_t, \tag{7}$$

where r_t denotes the instantaneous interest rate at time t; κ is the mean-reverting force measurement; m is the long-run mean of the interest rate; and v is the volatility parameter for the interest rate.

Substituting Equation (7) into Equation (6), the reinsurance company's asset dynamics can be described as:

$$\frac{dV_t}{V_t} = (\mu_V + \phi_V \kappa m - \phi_V \kappa r_t)dt + \phi_V v \sqrt{r_t} dZ_t + \sigma_V dW_{V,t}. \quad (8)$$

The dynamics for the interest rate process under the risk-neutralized pricing measure, denoted by Q, can be written as

$$dr_t = \kappa^*(m^* - r_t)dt + v\sqrt{r_t}\, dZ_t^*, \quad (9)$$

where $\kappa^* = \kappa + \lambda_r$; $m^* = \frac{\kappa m}{\kappa + \lambda_r}$; $dZ_t^* = dZ_t + \frac{\lambda_r \sqrt{r_t}}{v} dt$; λ_r, the market price of interest rate risk, is constant under the assumptions of Cox et al. (1985); and Z_t^* is a Wiener process under Q.

As $W_{V,t}$ is orthogonal to Z_t by construction and is considered as part of the interest rate risk, we can risk-neutralize Equation (8) by making use of the market price of the interest rate risk alone so that $\mu_V + \phi_V \kappa m - \phi_V \kappa r_t = r_t + \phi_V r_t$, and a measure change from the physical measure to the equivalent martingale measure Q that $dZ_t^* = dZ_t + \frac{\lambda_r \sqrt{r_t}}{v} dt$ and $d = dW_{V,t}$. Upon rearrangement we derive the risk-neutralized asset price process. □

On the liability side, the reinsurer faces the liability of providing reinsurance coverage to other lines of business, plus the liability of providing reinsurance coverage for catastrophes. Since the former liability represents the present value of future claims related to non-catastrophic policies, its value, denoted as L_t, can be modeled like Equation (6) as follows:

$$dL_t = \mu_L L_t dt + \phi_L L_t dr_t + \sigma_L L_t dW_{L,t}, \quad (10)$$

where $\mu_L = [\mu(L_t, t) - \frac{\sigma(L_t,t)\varphi\mu(r_t,t)}{\sigma(r_t,t)}]$ and $\phi_L = \sigma(L_t, t)\varphi/\sigma(r_t, t)$ denote respectively the instantaneous expected return for the non-catastrophic policies and the interest rate elasticity of the liability of the reinsurance company; $\varphi = \frac{Cov(dZ_{L,t}, dZ_{r,t})}{dt}$; and $\sigma_V = \sigma(L_t, t)\sqrt{1 - \varphi^2}$ is the volatility of the credit risk. This continuous diffusion process reflects the effects of interest rate changes and other small day-to-day shocks. Since the small day-to-day shocks, denoted as $W_{L,t}$, pertain to idiosyncratic shocks to the capital market, we assume a zero risk premium for this risk.

Lemma 1. *The liability process, Equation (10), can be risk-neutralized under Q using the market price of interest rate risk as:*

$$dL_t = r_t L_t dt + \phi_L v \sqrt{r_t} L_t dZ_t^* + \sigma_L L_t dW_{L,t}^*. \quad (11)$$

Proof.

Substituting Equation (7) into Equation (10), the reinsurance company's liability dynamics can be described as:

$$\frac{dL_t}{L_t} = (\mu_L + \phi_L \kappa m - \phi_L \kappa r_t)dt + \phi_L v \sqrt{r_t} dZ_t + \sigma_L dW_{L,t}. \quad (12)$$

Next the dynamics for the interest rate process under the risk-neutralized pricing measure Q is given in Equation (9) as

$$dr_t = \kappa^*(m^* - r_t)dt + v\sqrt{r_t}\, dZ_t^*,$$

where $\kappa^* = \kappa + \lambda_r$; $m^* = \frac{\kappa m}{\kappa + \lambda_r}$; $dZ_t^* = dZ_t + \frac{\lambda_r \sqrt{r_t}}{v} dt$; λ_r, the market price of interest rate risk, is constant, and Z_t^* is a Wiener process under Q.

As $W_{L,t}$ is orthogonal to Z_t by construction and is considered as part of the interest rate risk, we can risk-neutralize Equation (12) by making use of the market price of interest rate risk alone so that $\mu_L + \phi_L \kappa m - \phi_L \kappa r_t = r_t + \lambda_r \phi_L r_t$, and a measure change from the physical measure to the equivalent martingale measure Q that $dZ_t^* = dZ_t + \frac{\lambda_r \sqrt{r_t}}{v} dt$ and $dW_{L,t}^* = dW_{L,t}$. Upon rearrangement we derive the risk-neutralized liability price process. □

To model the catastrophic component of the liability, we first model the aggregate catastrophe loss as a compound Poisson process as in Bowers et al. (1986). The cumulative catastrophe loss at time t, denoted as C_t, is described as follows:

$$C_t = \sum_{j=1}^{N(t)} c_j, \tag{13}$$

where the loss number process $\{N(t),\ t \geq 0\}$ is assumed to be driven by a Poisson process with intensity λ. Term c_j denotes the amount of loss caused by the jth catastrophe covered by the reinsurance contract during the specific period, where $j = 1, 2, \ldots, N(t)$, and is assumed to be mutually independent, identical, and lognormally-distributed, and independent of the loss number process; its logarithmic means and standard deviations are denoted as μ_c and σ_c, respectively. We assume the loss process is exogenously calibrated by a catastrophe modeler, e.g., AIR Worldwide for hurricanes and windstorms and RMS for earthquakes. We also make the common assumption that catastrophe derivatives are zero-beta assets, and thus both the loss number process $\{N(t)\}$ and the amount of losses (c_t have a zero-risk premium[3]. The loss process, Equation (13), thus retains its original distributional characteristics after changing from the physical probability measure to the risk-neutralized measure.

3. Pricing a Cat Reinsurance Contract and the Monte Carlo Simulation Results

The reinsurer's future payoff to the insurer can be specified as:

$$PO_{R,T} = \begin{cases} M - A & \text{if } C_T \geq M \text{ and } V_T \ \geq L_T + M - A, \\ C_T - A & \text{if } M > C_T \geqq A \text{ and } V_T \ \geq L_T + C_T - A, \\ \left(\frac{V_T}{L_T+M-A}\right)(M-A) & \text{if } C_T \geq M \text{ and } V_T < L_T + M - A, \\ \left(\frac{V_T}{L_T+C_T-A}\right)(C_T-A) & \text{if } M > C_T \geqq A \text{ and } V_T \ < L_T + C_T - A, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where V_T is the value of the reinsurer's assets at T; L_T is the value of the reinsurer's liabilities at T; C_T is the catastrophe loss covered by the reinsurance contract; and M and A are respectively the cap and attachment points arranged in the reinsurance contract. For example, when C_T is larger than the reinsurance cap M and the reinsurer's total assets are larger than total liability inclusive of the reinsurance obligation $M - A$, the payoff is $M - A$. When the reinsurer's total assets are smaller than total liability inclusive of the reinsurance obligation $M - A$, the reinsurer will default, and the payoff to the insurer then is only $(\frac{V_T}{L_T+M-A})(M-A)$. The present value of this future payoff determines the value of the reinsurance contract, computed as a martingale under the risk-neutralized pricing measure Q as:

$$PV_{R,0} = E_0^Q[e^{-\int_0^T r_s\, ds} \times PO_{R,T}], \tag{15}$$

where E_0^Q denotes the expectation taken on the issuing date under the risk-neutralized pricing measure Q. In Appendix A, we demonstrate how to compute $PV_{R,0}$ via Monte Carlo simulation.

To incorporate the observed empirical characteristics of the (re)insurance market, we let the sell-side reinsurance pricing be $(1+u)PV_{R,0}$, where u denotes the risk loading/price markup that is higher in a hard market, but lower in a soft market.

To demonstrate the implementation using Monte Carlo simulation, we consider a base case in which all of the parameter values in the asset, liability, interest rate, and catastrophe loss dynamics are

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[3] Gürtler et al. (2012) show that this premium can be significant when a mega catastrophe strikes, but we assume this scenario away in this paper.

consistent with previous literature[4], as summarized in Table 1 below. We also assume the reinsurance market is still hard with a markup of 40%[5]. Simulations were run on a monthly basis with 20,000 paths.

Table 1. Parameter Definitions and Base Values.

Asset Parameters		
V	Reinsurer's assets	$V/L = 1.3$
μ_V	Drift due to credit risk	Irreverent
ϕ_V	Interest rate elasticity of asset	-3
σ_v	Volatility of credit risk	5%
$W_{V,t}$	Wiener process for credit shocks	
Liability Parameters		
L	Reinsurer's liabilities	100
μ_L	Drift due to idiosyncratic risk	0
ϕ_L	Interest rate elasticity of liability	-3
σ_L	Volatility of idiosyncratic risk	2%
$W_{L,t}$	Wiener process for idiosyncratic shocks	
Interest Rate Parameters		
r	Initial instantaneous interest rate	2%
κ	Magnitude of mean-reverting force	0.2
m	Long-run mean of interest rate	5%
ν	Volatility of interest rate	10%
λ_r	Market price of interest rate risk	-0.01
Z	Wiener process for interest rate shocks	
Catastrophe loss Parameters for $\mathbf{C_t}$		
$N(t)$	Poisson process for the arrival of catastrophes	
λ	Catastrophe arrival intensity	0.5
μ_C	Mean of the logarithm of the losses per arrival	2
σ_C	Standard deviation of the logarithm of the losses per arrival	0.5
Other Parameters		
A	Attachment level of a reinsurance contract	10~30
M	Cap level of loss paid by a reinsurance contract	60~90
T	Maturity	3 years
u	Reinsurance markup	0.4

Table 2 below summarizes the corresponding reinsurance values across all coverage layers to form the reinsurer's sell-side pricing schedule. As expected, as the attachment point increases, the reinsurance value decreases, while as the cap/detachment point increases, the reinsurance value increases. We also observe that lower layer coverage is more expensive than higher layer coverage, e.g., the price for layer (20, 60) is 2.73468 while the price for layer (25, 65) is 1.59048, reflecting that the lower the layer the higher the probability of penetration.

4 See Lee and Yu (2007) and Duan and Simonato (1999).

5 The size of the markup is an empirical issue, and here we assume a plausible scenario to demonstrate pricing. Zanjani (2002) states that the marginal capital requirement for catastrophe reinsurance is about five times the premium with a price impact of about 30%.

Table 2. Reinsurance Pricing across All Coverage Layers.

Coverage	M = 60	M = 65	M = 70	M = 75	M = 80	M = 85	M = 90
A = 10	7.26957	7.28648	7.29306	7.30003	7.30261	7.30284	7.30634
A = 15	4.45191	4.46950	4.47824	4.48331	4.48579	4.48792	4.99274
A = 20	2.73468	2.75143	2.76143	2.76739	2.16897	2.17120	2.17444
A = 25	1.57234	1.59048	1.59951	1.60448	1.60706	1.60929	1.61162
A = 30	0.93787	0.98590	1.05583	1.06080	1.06338	1.06561	1.06740

Next we examine the catastrophe effects by extending the base value scenario in Table 1 to consider three levels of catastrophe arrival intensity λ = (0.5, 1, 2); three levels of loss volatility: σ_c = (0.5, 1, 2); and three leverage scenarios: V/L = (1.1, 1.3, 1.5), with decreasing leverage. Table 3 below shows that, as expected, as the leverage decreases the default risk decreases such that the reinsurance contract becomes more valuable. It also shows that increases in both catastrophe arrival intensity and loss volatility raise the reinsurance value as hedging becomes more desirable. As catastrophe arrival intensifies and the loss becomes more unpredictable, insurers should increasingly purchase reinsurances as the expected losses accentuate. We further observe that the loss volatility effect is more acute than the arrival intensity effect. For example, when V/L is 1.1, doubling the arrival intensity from 0.5 to 1 increases the reinsurance value from 7.30634 to 13.24426, but doubling the loss volatility from 0.5 to 1 increases the reinsurance value from 7.30634 to 20.75633. This finding suggests that insurers should be more concerned with hedging the severity of a catastrophe than with the intensity of the arrival. Finally, as expected, the reinsurance contract is more valuable when the catastrophe is more severe and the default risk is lower.

Table 3. Impacts of Catastrophe Intensity and Loss Volatility on Pricing.

(λ, σ_C)	Reinsurance Price $(1+u)PV_{R,0}$ for Coverage Payer (90, 10)		
	V/L = 1.1	V/L = 1.3	V/L = 1.5
(0.5, 0.5)	7.30634	7.40443	7.58688
(1, 0.5)	13.24426	13.78995	14.12347
(2, 0.5)	19.56788	19.98765	20.31452
(0.5, 1)	20.75633	21.24536	21.76542
(1, 1)	24.18776	24.35473	24.87682
(2, 1)	27.78653	28.89672	29.45328
(0.5, 2)	40.28763	41.69782	43.13476
(1, 2)	43.21675	43.87862	44.34724
(2, 2)	48.8976	49.90163	51.27658

4. Concluding Remarks

We have proposed an integrated approach to price a default-risky cat reinsurance contract by straddling the traditional actuarial science and the mathematical finance approaches. We first embedded modeled risk premiums by applying an incomplete-market version of the no-arbitrage martingale pricing paradigm in mathematical finance to price the contract as a martingale by a measure change, we then added on risk loading as commonly practiced by actuaries to price in non-modeled market factors such as market imperfections, the underwriting cycle, and other idiosyncratic factors identified in the empirical reinsurance pricing literature. This integrated approach is theoretically appealing in being able to factor model risk premiums into the probability measure, and yet applicable to a contract not traded on financial markets. We demonstrated the catastrophe arrival effects using simulation and found that the price is more sensitive to catastrophe severity than to arrival frequency; implying (re)insurers should focus more on hedging the severity than the arrival frequency in their risk management programs. Finally, as expected, the reinsurance contract is more valuable when the catastrophe is more severe and the reinsurer's default risk is lower with a stronger balance sheet.

Author Contributions: Carolyn W. Chang and Jack S. K. Chang jointly work on the whole paper from conceiving the idea to the implementation of the simulation.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Procedures of the Monte Carlo Simulation Method

In this section, we numerically assess the reinsurance value $PV_{R,0}$ using Equation (15). Because the premium depends on the values of the reinsurer's assets and liabilities, we have to specify the stochastic processes for these variables (Equations (9) and (11)). Applying Ito's lemma to the logarithm of the value of a reinsurer's assets, Equation (9) becomes the following system:

$$d\ln(V_t) = (r_t - \frac{1}{2}\varphi_V v^2 r_t - \frac{1}{2}\sigma_V{}^2)dt + \varphi_V v\sqrt{r_t}dZ_t{}^* + \sigma_V dW_{V,t}{}^*. \quad \text{(A1)}$$

We can solve the above equation. Its solution, for any $0 \le q < 1$, is:

$$V_{t+q} = V_t \exp(\sigma_V(W_{V,t+q}{}^* - W_{V,t}{}^*)) * \exp[(1 - \frac{1}{2}\varphi_V{}^2 v^2)\int_t^{t+q} r_s ds + \varphi_V v \int_t^{t+q} \sqrt{r_s} dZ_s{}^*] \quad \text{(A2)}$$

Similarly, by Ito's lemma, Equation (11) gives rise to:

$$d\ln(L_t) = (\mathrm{r}_t - \frac{1}{2}\varphi_L v^2 r_t - \frac{1}{2}\sigma_L{}^2)dt + \varphi_L v\sqrt{r_t}dZ_t{}^* + \sigma_L dW_{L,t}{}^*. \quad \text{(A3)}$$

Its solution is:

$$L_{t+q} = L_t \exp(\sigma_L(W_{L,t+q}{}^* - W_{L,t}{}^*)) * \exp[(1 - \frac{1}{2}\varphi_L{}^2 v^2)\int_t^{t+q} r_s ds + \varphi_L v \int_t^{t+q} \sqrt{r_s} dZ_s{}^*] \quad \text{(A4)}$$

We set $q = \frac{1}{52}$, say, on a weekly basis, and simulate the risk-neutralized interest rate process per Equation (8) in order to approximate the whole sample path for the term of reinsurance coverage. This in turn allows us to compute two quantities of interest: $\int_t^{t+1} r_s ds$ and $\int_t^{t+1} \sqrt{r_s} dZ_s{}^*$. Second, we simulate $(W_{V,t+q}{}^* - W_{V,t}{}^*)$ and $(W_{L,t+q}{}^* - W_{L,t}{}^*)$ using the fact that they are independent of the path of r_t, and the coefficient of correlation between them is chosen to be zero to reflect that the credit risk shock and the idiosyncratic day-to-day liability shock are likely to be uncorrelated. Combining $(W_{V,t+q}{}^* - W_{V,t}{}^*)$ with $\int_t^{t+1} r_s ds$ and $\int_t^{t+1} \sqrt{r_s} dZ_s{}^*$ yields a simulated value of V_{t+1} as described in Equation (A2). Similarly, combining $(W_{L,t+q}{}^* - W_{L,t}{}^*)$ with $\int_t^{t+1} r_s ds$ and $\int_t^{t+1} \sqrt{r_s} dZ_s{}^*$ yields a simulated value of L_{t+1} as described in Equation (A4). Third, we generate $(N_{t+1} - N_t)$, $(c_{t+1} - c_t)$ and $(c_{index,t+1} - c_{index,t})$. Since $(N_{t+1} - N_t)$ has a Poisson distribution with intensity parameter λ, it can be simulated easily. For a given value of $(N_{t+1} - N_t)$, we then simulate $\sum_{j=N_t}^{N_{t+1}} \ln c_j$ and $\sum_{j=N_t}^{N_{t+1}} \ln c_{index,j}$, knowing that $\ln c_j$ and $\ln c_{index,j}$ are normal random variables and the coefficient of correlation between them is ρ_c. After simulating these processes, $PV_{R,0}$ can be easily calculated via averaging over the contingent payoffs corresponding to the simulated values.

We implemented the simulation using MATLAB and below are two snapshots of the code, where in Figure A1 we display the asset, liability, and interest rate processes, and in Figure A2 the Poisson catastrophe arrival process.

```
Editor – /Users/changd/Documents/Miscellaneous/CatOpMix140928.m
0928.m   CatOpMix140928.m   poissrndd.m   +
/Users/changd/Documents/Miscellaneous/0928.m 131209(A,M,F,K,phiv,phil,V0,Wu,Wd,Crow,Clamda,Csigma)

%clearvars -except rMaxValue rK rF
%tic
close all
%V0=130;%´Áªì Asset
L0=100;%´Áªì Liability
r=0.02;%Interest rate 5% = 0.05
T=3;%3 year=3
dt=1/12;%Monthy
Period=T/dt;%´Á¼Æ
k=0.2;%ªø´Á§¡¼Æ¦^Âk³t²v
m=0.05;%ªø´Á§¡¼Æ
%phiv=-3*(100/V0);%Interest rate elasticity of asset
%phil=-3;%Interest rate elasticity of liability
v=0.10;%Volatility parameter for the Interest rate
St=20000;%simulation times


lamr=0;%Interest rate intensity
Sigmav=0.05;%Volatility of credit risk
Sigmal=0.02;%Volatility of credit risk
k_s=k+lamr;%k-star
m_s=k*m/(k+lamr);%m-star


%Clamda=0.5;%Catastrophe interesity
Cmu=2;% mean of logarithm of CAT losses for the insurer
%Csigma=0.5;% mean of logarithm of CAT losses for the insurer
%Crow=0.8;%Corration of INDEX CATLOSS and com CATLOSS : 0.3 or 0.5 or 0.8
%Wu=0.4;%Åv¼Æ0.4=40%
%Wd=0.05;%Åv¼Æ0.05
R=80;
```

Figure A1. MATLAB code for the asset, liability, and interest rate processes.

Editor – /Users/changd/Documents/Miscellaneous/poissrndd.m

0928.m | CatOpMix140928.m | poissrndd.m

```
function r = poissrndd(lambda,m,n,state,method)

%POISSRND Random matrices from Poisson distribution.

%   R = POISSRND(LAMBDA) returns a matrix of random numbers chosen

%   from the Poisson distribution with parameter LAMBDA.

%

%   The size of R is the size of LAMBDA. Alternatively,

%   R = POISSRND(LAMBDA,M,N) returns an M by N matrix.

%

%   Use R = POISSRND(LAMBDA,M,N,STATE) to initialise the random

%   number generator with STATE. Use R = POISSRND(LAMBDA,[],[],STATE)

%   when M and N are not given and POISSRND([],[],[],STATE) when

%   only the random number generator needs to be initialised.

%   POISSRND([],[],[],0) resets the generator to its initial state.

%   POISSRND([],[],[],J), for integer J, resets the generator to its J-th state.

%   POISSRND([],[],[],sum(100*clock)) resets it to a different state each time.

%   Note: POISSRND actually initialises RAND and RANDN.

%

%   Use R = POISSRND(LAMBDA,M,N,STATE,METHOD) to determine the method.

%   METHOD='normal' (default) uses a waiting time method.

%   METHOD='fast' uses a waiting time method at lambda<1000 and

%           normal distribution with mean and variance equal to lambda

%           at labda>=1000.

%   METHOD=value equal to 'fast' but with the border set at value

%       instead of 1000.

%   Use R = POISSRND(LAMBDA,[],[],[],METHOD) when M, N and STATE are

%   not given.

%

%   For more information see the M-file.



%  Addition 1 (12-3-1999):
```

Figure A2. MATLAB code for the Poisson catastrophe arrival process.

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