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The Impact of Management Fees on the Pricing of Variable Annuity Guarantees

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Abstract: Variable annuities, as a class of retirement income products, allow equity market exposure for a policyholder's retirement fund with optional guarantees to limit the downside risk of the market. Management fees andguarantee insurance fees are charged respectively for the market exposure and for the protection from the downside risk. We investigate the pricing of variable annuity guarantees under optimal withdrawal strategies when management fees are present. We consider from both policyholder's and insurer's perspectives optimal withdrawal strategies and calculate the respective fair insurance fees. We reveal a discrepancy where the fees from the insurer's perspective can be significantly higher due to the management fees serving as a form of market friction. Our results provide a possible explanation of lower guarantee insurance fees observed in the market than those predicted from the insurer's perspective. Numerical experiments are conducted to illustrate the results.

Keywords: pricing; variable annuity guarantees; management fees; dynamic programming

JEL Classification: C61; G22

1. Introduction

Variable annuities (VA) with guarantees of living and death benefits are offered by wealth management and insurance companies worldwide to assist individuals in managing their pre-retirement and post-retirement financial plans. These products take advantage of market growth while providing a protection of the savings against market downturns. Similar guarantees are also available for life insurance policies (Bacinello and Ortu 1996). The VA contract cash flows received by the policyholder are linked to the choice of investment portfolio (e.g., the choice of mutual fund and its strategy) and its performance while traditional annuities provide a pre-defined income stream in exchange for a lump sum payment. Holders of VA policies are required to pay management fees regularly during the term of the contract for the management of their investment portfolios (wealth accounts).

A variety of VA guarantees, also known as VA riders, can be added by policyholders at the cost of additional insurance fees. Common examples of VA guarantees include guaranteed minimum accumulation benefit (GMAB), guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum income benefit (GMIB) and guaranteed minimum death benefit (GMDB), as well as a combination of them, e.g., guaranteed minimum withdrawal and death benefit (GMWDB), among others. These guarantees, generically denoted as GMxB, provide different types of protection against market downturns, shortfall of savings due to longevity risk or assurance of stability of income streams. Precise specifications of these products can vary across categories and issuers. See (Bauer et al. 2008; Kalberer and Ravindran 2009; Ledlie et al. 2008) for an overview of these products.

The Global Financial Crisis during 2007–2008 led to lasting adverse market conditions such as low interest rates and asset returns as well as high volatilities for VA providers. Under these conditions, the VA guarantees become more valuable, and the fulfillment of the corresponding liabilities become more demanding. The post-crisis market conditions have called for effective hedging of risks associated with the VA guarantees (Sun et al. 2016). As a consequence, the need for accurate estimation of hedging costs of VA guarantees has become increasingly important. Such estimations consist of risk-neutral pricing of future cash flows that must be paid by the insurer to the policyholder in order to fulfill the liabilities of the VA guarantees.

There have been a number of contributions in the academic literature considering the pricing of VA guarantees. A range of numerical methods are considered, including standard and regression-based Monte Carlo (Huang and Kwok 2016), partial differential equation (PDE) and direct integration methods (Chen and Forsyth 2008; Dai et al. 2008; Milevsky and Salisbury 2006; Bauer et al. 2008; Luo and Shevchenko 2015a, 2015b; Forsyth and Vetzal 2014; Shevchenko and Luo 2017). A comprehensive overview of numerical methods for the pricing of VA guarantees is provided in Shevchenko and Luo (2016).

In this article, we focus on GMWDB, which provides a guaranteed withdrawal amount per year until the maturity of the contract regardless of the investment performance, as well as a lump-sum of death benefit in case the policyholder dies over the contract period. The guaranteed withdrawal amount is determined such that the initial investment is returned over the life of the contract. The death benefit may assume different forms depending on the details of the contract. When pricing GMWDB, one typically assumes either a pre-determined (static) policyholder behavior in withdrawal and surrender, or an active (dynamic) strategy where the policyholder "optimally" decides the amount of withdrawal at each withdrawal date depending on the information available at that date.

One of the most debated aspects in the pricing of GMWDB with dynamic withdrawal strategies is the policyholders' withdrawal behaviors (Chen and Forsyth 2008; Cramer et al. 2007; Moenig and Bauer 2015; Forsyth and Vetzal 2014). It is often customary to refer to the withdrawal strategy that maximizes the hedging cost of the VA guarantee, that is, the risk-neutral value of the guarantee alone, as the "optimal" strategy. Even though such a strategy underlies the worst case scenario for the VA provider with the highest hedging cost, it may not coincide with the real-world behavior of the policyholder. Nevertheless, the value of the guarantee under this strategy provides an upper bound of hedging cost from the insurer's perspective. The real-world behaviors of policyholders often deviate from this "optimal" strategy, as is noted in Moenig and Bauer (2015). Different models have been proposed to account for the real-world behaviors of policyholders, including the reduced-form exercise rules of Ho et al. (2005), and the subjective risk neutral valuation approach taken by Moenig and Bauer (2015). In particular, it is concluded by Moenig and Bauer (2015) that a subjective risk-neutral valuation methodology that takes different tax structures into consideration is in line with the corresponding findings from empirical observations.

When the management fee of the policyholder's wealth account is zero, and deterministic withdrawal behavior is assumed, Hyndman and Wenger (2014) and Fung et al. (2014) show that risk-neutral pricing of guaranteed withdrawal benefits in both a policyholder's and an insurer's perspectives will result in the same fair insurance fee. Feng and Volkmer (2016) obtains similar results based on an application of an identity of hitting times. Several studies that take management fees into account in the pricing of VA guarantees include Bélanger et al. (2009), Chen et al. (2008) and Kling et al. (2011). In these studies, fair insurance fees are considered from the insurer's perspective with the management fees as given. Feng and Volkmer (2012), Feng and Jing (2016), Feng and Huang (2016) show that it is possible to obtain closed-form solutions for the valuation and risk measures of guaranteed benefits under certain assumptions including deterministic withdrawal behaviours. Feng and Vecer (2017) studies a PDE approach for the calculation of risk-based capital for GMWB. A comonotonic approximation approach for the calculation of risk metrics for VA is considered in (Feng et al. 2017) where a dynamic lapse rate is taken into account. Cui et al. (2017) studies the pricing of VA with VIX-linked fee structure under a Heston-type stochastic volatility model.

Similar to the tax consideration in (Moenig and Bauer 2015), the management fee is a form of market friction that would affect policyholders' rational behaviors. Despite a large range of papers mentioned above on VA guarantee pricing with management fees, the important question of how the management fees as a form of market friction will impact withdrawal behaviors of the policyholder, and hence the hedging cost for the insurer, is yet to be specifically examined in a dynamic withdrawal setting. The main goal of the paper is to address this question.

The paper contributes to the literature in three aspects. First, we consider two pricing approaches based on the policyholder's and the insurer's perspective. In the literature, it is most often the case that only an insurer's perspective is considered, which might result in mis-characterisation of the policyholder's withdrawal strategies. Second, we characterize the impact of management fees on the pricing of GMWDB, and demonstrate that the two afore-mentioned pricing perspectives lead to different fair insurance fees due to the presence of management fees. In particular, the fair insurance fees from the policyholder's perspective is lower than those from the insurer's perspective. This provides a possible justification of lower insurance fees observed in the market. Third, the sensitivity of the fair insurance fees to management fees under different market conditions and contract parameters are investigated and quantified through numerical examples.

The paper is organized as follows. In Section 2, we present the contract details of the GMWDB guarantee together with its pricing formulation under a stochastic optimal control framework. Section 3 derives the total value function of the contract under the risk-neutral pricing approach, followed by the net guarantee value function in Section 4. In Section 5, we introduce the wealth manager's value function that relates the total value and guarantee value functions and the two optimal withdrawal strategies corresponding to these value functions. Section 6 demonstrates our analysis via numerical examples. Section 7 concludes with remarks and discussion.

2. Formulation of the GMWDB Pricing Problem

We begin with the setup of the framework for the pricing of GMWDB contract and describe the features of this type of guarantees. (In the sequel, we refer to the VA contract with GMWDB rider simply as the GMWDB contract, unless explicitly stated otherwise.) The pricing problem is formulated under a general setting so that the resulting pricing formulation can be applied to different contract specifications. Besides the general setting, we also consider a specific GMWDB contract, which will be subsequently used for illustration purposes in numerical experiments presented in Section 6.

The VA policyholder's retirement fund is usually invested in a managed wealth account that is exposed to financial market risks. A management fee is usually charged for this investment service. In addition, if the GMWDB rider is selected, extra insurance fees will be charged for the protection offered by the guarantee provider (insurer). We assume the wealth account guaranteed by the GMWDB is subject to continuously charged proportional management fees independent of any fees charged for the guarantee insurance. The purpose of these management fees is to compensate for the wealth management services provided, or perhaps merely for the access to the guarantee insurance on the investment. This fee should not be confused with other forms of market frictions, e.g., transaction costs, if any, that must incur when tracking a given equity index. Given the proliferation of index-tracking exchange-traded funds in recent years, with much desired liquidity at a fraction of the costs of the conventional index mutual funds, see, e.g., (Agapova 2011; Kostovetsky 2003; Poterba and Shoven 2002), regarding these management fees as additional costs to policyholders beyond the normal market frictions seems to be a reasonable assumption.

The hedging cost of the guarantee, on the other hand, is paid by proportional insurance fees continuously charged to the wealth account. The fair insurance fee rate, or the fair fee in short, refers to the minimal insurance fee rate required to fund the hedging portfolio, so that the guarantee provider can eliminate the market risk associated with the selling of the guarantees.

We consider the situation where a policyholder purchases the GMWDB rider in order to protect his wealth account that tracks an equity index $S(t)$ at time $t \in [0, T]$, where 0 and T correspond to the inception and expiry dates. The equity index account is modelled under the risk-neutral probability measure $\mathbb{Q}$ following the stochastic differential equation (SDE)

$$dS(t) = S(t)\left(r(t)dt + \sigma(t)dB(t)\right), \quad t \in [0, T], \tag{1}$$

where $r(t)$ is the risk-free short interest rate, $\sigma(t)$ is the volatility of the index, which is time-dependent and can be stochastic, and $B(t)$ is a standard $\mathbb{Q}$-Brownian motion modelling the uncertainty of the index. Here, we follow standard practices in the literature of VA guarantee pricing by modelling under the risk-neutral probability measure $\mathbb{Q}$, which allows the pricing of stochastic cash flows to be given as the risk-neutral expectation of the discounted cash flows. The risk-neutral probability measure $\mathbb{Q}$ exists if the underlying financial market satisfies certain "no-arbitrage" conditions. Adopting risk-neutral pricing here assumes that stochastic cash flows in the future can be replicated by dynamic hedging without transaction fees. For details on risk-neutral pricing and the underlying assumptions, see, e.g., Delbaen and Schachermayer (2006) for an account under very general settings.

The wealth account $W(t), t \in [0, T]$ over the lifetime of the GMWDB contract is invested into the index S, subject to management fees charged by a wealth manager at the rate $\alpha_{\mathrm{m}}(t)$. An additional charge of insurance fees at rate $\alpha_{\mathrm{ins}}(t)$ for the GMWDB rider is collected by the insurer to pay for the hedging cost of the guarantee. We assume that both fees are deterministic, time-dependent and continuously charged to the wealth account. (Sometimes, the insurance fees are charged to the guarantee account mentioned shortly.) Discrete fees may be modelled similarly without any difficulty. The wealth account in turn evolves as

$$dW(t) = W(t)\left((r(t) - \alpha_{\mathrm{tot}}(t))dt + \sigma(t)dB(t)\right), \tag{2}$$

for any $t \in [0, T]$ at which no withdrawal of wealth is made. Here, $\alpha_{\mathrm{tot}}(t) = \alpha_{\mathrm{ins}}(t) + \alpha_{\mathrm{m}}(t)$ is the total fee rate. The GMWDB contract allows the policyholder to withdraw from a guarantee account $A(t), t \in [0, T]$ on a sequence of pre-determined contract event dates, $0 = t_0 < t_1 < \cdots < t_N = T$. The initial guarantee $A(0)$ usually matches the initial wealth $W(0)$. The guarantee account stays constant unless a withdrawal is made on one of the event dates, which changes the guarantee account balance. If the policyholder dies on or before the maturity T, the death benefit will be paid at the next event date immediately following the death of the policyholder. Additional features such as early surrender can be included straightforwardly but will not be considered in this article to avoid unnecessary complexities.

To simplify notations, we denote by $\mathbf{Y}(t)$ the vector of state variables at t, given by

$$\mathbf{Y}(t) = (r(t), \sigma(t), S(t), W(t), A(t)), \quad t \in [0, T], \tag{3}$$

where we assume that all state variables follow Markov processes under the risk-neutral probability measure $\mathbb{Q}$, so that $\mathbf{Y}(t)$ contains all the market and account balances information available at t. For simplicity, we assume the state variables $r(t)$, $\sigma(t)$ and $S(t)$ are continuous, and $W(t)$ and $A(t)$ are left continuous with right limit (LCRL). We include the index value $S(t)$ in $\mathbf{Y}(t)$, which under the current model may seem redundant, due to the scale-invariance of the geometric Brownian motion type model (1). In general, however, $S(t)$ may determine the future dynamics of S in a nonlinear fashion, as is the case under, e.g., the minimal market model described in Platen and Heath (2006).

We define $I(t), t \in [0, T]$ as the life indicator function of an individual policyholder as the following: $I(t) = 1$ if the policyholder was alive on the last event date on or before t; $I(t) = 0$ if the policyholder was alive on the second-to-the-last event date prior to t but died on or before the last event date; $I(t) = -1$ if the policyholder died on or before the second-to-the-last event date prior to t. We assume the policyholder is alive at t_0. The life indicator function $I(t)$ therefore starts at $I(t_0) = 1$, is right continuous with left limit (RCLL), and remains constant between two consecutive event dates. Note that mortality information contained in $I(t)$ (RCLL) comes before any jumps of the LCRL account balances $W(t)$ and $A(t)$ on the event dates, reflecting the situation that any jumps in these account balances may depend on the mortality information. We denote the vector of state variables including $I(t)$ as $\mathbf{X}(t) = (\mathbf{Y}(t)^\top, I(t))^\top$, and we denote by $\mathrm{E}_t^{\mathbb{Q}}[\cdot]$ the risk-neutral expectation conditional on the state variables $\mathbf{X}(t)$ at t, i.e., $\mathrm{E}_t^{\mathbb{Q}}[\cdot] := \mathrm{E}^{\mathbb{Q}}[\cdot|\mathbf{X}(t)]$. Note that the risk-neutral measure $\mathbb{Q}$ is assumed to extend to the mortality risk represented by the life indicator $I(t)$.

On event dates $t_n, n = 1, \ldots, N$, a nominal withdrawal γ_n from the guarantee account is made. The policyholder, if alive, may choose γ_n on $t_n < T$. Otherwise, a liquidation withdrawal of $\max(W(t_n), A(t_n))$ is made. That is,

$$\gamma_n = \Gamma(t_n, \mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=1, n<N\}} + \max\left(W(t_n), A(t_n)\right)\mathbb{1}_{\{I(t_n)=0 \,\text{or}\, n=N\}}, \tag{4}$$

where $\mathbb{1}_{\{\}}$ denotes the indicator function of an event, and $\Gamma(\cdot,\cdot)$ is referred to as the *withdrawal strategy* of the policyholder. The real cash flow received by the policyholder, which may differ from the nominal amount, is denoted by $C_n(\gamma_n, \mathbf{X}(t_n))$. This is given by

$$C_n(\gamma_n, \mathbf{X}(t_n)) = P_n(\gamma_n, \mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=0\}}, \tag{5}$$

where $P_n(\gamma_n, \mathbf{Y}(t_n))$ is the payment received if the policyholder is alive, and $D_n(\mathbf{Y}(t_n))$ denotes the death benefit if the policyholder died during the last period. As a specific example, $P_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$P_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n - \beta \max(\min(\gamma_n, A(t_n)) - g_n, 0). \tag{6}$$

Here, the contracted withdrawal g_n is a pre-determined withdrawal amount specified in the GMWDB contract, and β is the penalty rate applied to the part of the withdrawal from the guarantee account exceeding the contracted withdrawal g_n. Note that the $\min(\gamma_n, A(t_n))$ term in (6) accommodates the situation that, at expiration of the contract, both accounts are liquidated, but only the *guarantee account withdrawals* exceeding the contracted rate g_n will be penalized. Excess balance on the wealth account after the guaranteed withdrawal is not subject to this penalty. An example of the death benefit may simply be taken as the total withdrawal without penalty, i.e.,

$$D_n(\mathbf{Y}(t_n)) = \max\left(W(t_n), A(t_n)\right). \tag{7}$$

Upon withdrawal by the policyholder, the guarantee account is changed by the amount $J_n(\gamma_n, \mathbf{Y}(t_n))$, that is,

$$A(t_n^+) = A(t_n) - J_n(\gamma_n, \mathbf{Y}(t_n)), \tag{8}$$

where $A(t_n^+)$ denotes the guarantee account balance "immediately after" the withdrawal. For example, $J_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$J_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n \mathbb{1}_{\{I(t_n)=1\}} + A(t_n)\mathbb{1}_{\{I(t_n)\leq 0\}}, \tag{9}$$

i.e., the guarantee account balance is reduced by the withdrawal amount if the policyholder is alive and the policy has not expired. Otherwise, the account is liquidated. The guarantee account stays nonnegative, that is, γ_n if chosen by the policyholder must be such that $J_n(\gamma_n, \mathbf{Y}(t_n)) \leq A(t_n)$. The wealth account is reduced by the amount γ_n upon withdrawal and remains nonnegative. That is,

$$W(t_n^+) = \max(W(t_n) - \gamma_n, 0), \tag{10}$$

where $W(t_n^+)$ is the wealth account balance immediately after the withdrawal. It is assumed that $\gamma_0 = 0$, i.e., no withdrawals at the start of the contract. Both the wealth and the guarantee account balance are 0 after contract expiration. That is,

$$W(T^+) = A(T^+) = 0. \tag{11}$$

The total value function at time t is denoted by $V(t, \mathbf{X}(t)), t \in [0, T]$, which corresponds to the risk-neutral expected value of all cash flows to the policyholder on or after time t. The remaining policy value after the final cash flow is thus 0, i.e.,

$$V(T^+, \mathbf{X}(T^+)) = 0. \tag{12}$$

3. Calculating the Total Value Function

We now calculate the policyholder's total value function $V(t, \mathbf{X}(t))$ as the risk-neutral expected value of policyholder's future cash flows at time $t \in [0, T]$. The risk-neutral valuation of the policyholder's future cash flows can be regarded as the value of the remaining term of the VA contract from the policyholder's perspective. As mentioned in the beginning of Section 2, valuation under the risk-neutral pricing approach assumes that the cash flows may be replicated by hedging portfolios without market frictions. This may be carried out by a third-party independent agent, if not directly by the individual policyholder.

Following Section 2, the total value function on an event date t_n can be written as

$$V(t_n, \mathbf{X}(t_n)) = C_n(\gamma_n, \mathbf{X}(t_n)) + V(t_n^+, \mathbf{X}(t_n^+)), \tag{13}$$

which by (5) can be further written as

$$V(t_n, \mathbf{X}(t_n)) = (P_n(\gamma_n, \mathbf{Y}(t_n)) + V_n(t_n^+, \mathbf{X}(t_n^+)))\mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=0\}}, \tag{14}$$

since, if the policyholder died during last period, the death benefit is the only cash flow to receive. Taking the risk-neutral expectation $\mathrm{E}_{t_n^-}^{\mathbb{Q}}[\cdot]$, we obtain the jump condition

$$V(t_n^-, \mathbf{X}(t_n^-)) = \Big((1 - q_n)\Big(P_n(\gamma_n, \mathbf{Y}(t_n)) + V(t_n^+, \mathbf{X}(t_n^+))\Big) + q_n D_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{15}$$

where q_n is the risk-neutral probability that the policyholder died over (t_{n-1}, t_n), given that he is alive on the last withdrawal date t_{n-1}. That is,

$$q_n = \mathbb{Q}[I(t_n) = 0 | I(t_n^-) = 1]. \tag{16}$$

Here, we assume that the mortality risk is independent of the market risk under the risk-neutral probability measure. Under the assumption that the mortality risk is completely diversifiable, the risk-neutral mortality rate may be identified with that under the real-world probability measure and inferred from a historical life table. (Since an individual policyholder cannot hedge the mortality risk through diversification, risk-neutral pricing of the total value function essentially assumes that the policyholder is risk-neutral toward the mortality risk.) Here, the mortality information over $(t_{n-1}, t_n]$ is revealed at t_n, thus at t_n^- such information is not yet available. This assumption is not a model constraint since all decisions are made only on event dates.

The total value at $t \in (t_{n-1}, t_n)$ is given by the expected discounted future total value under the risk-neutral probability measure, given by

$$V(t, \mathbf{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} V(t_n^-, \mathbf{X}(t_n^-))\Big], \tag{17}$$

where $e^{-\int_t^{t_n} r(s)ds}$ is the discount factor. The initial policy value, given by $V(0, \mathbf{X}(0))$, can be calculated backward in time starting from the terminal condtion (12), using (15) and (17), as described in Algorithm 1.

As an illustrative example, we assume $r(t) \equiv r$, $\sigma(t) \equiv \sigma$ and $\alpha_{\text{tot}}(t) \equiv \alpha_{\text{tot}}$ as constants. Since $A(t)$ and $I(t)$ are constants between the withdrawals, the only effective state variable between the withdrawal dates are $W(t)$. Thus, we can simply write $V(t, \mathbf{X}(t)) = V(t, W(t))$ for $t \in (t_{n-1}, t_n)$ without confusion. We now derive the partial differential equation (PDE) satisfied by the value function V through a hedging argument. We consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, takes a long position of the value function V and a short position of $\frac{W(t)\partial_W V(t,W(t))}{S(t)}$ shares of the index S. Here, $\partial_W V(t, W(t)) \equiv \frac{\partial V(t,W)}{\partial W}|_{W=W(t)}$ is the partial derivative of $V(t, W)$ with respect to the second argument, evaluated at $W(t)$. Denoting the total value of this portfolio at $t \in (t_{n-1}, t_n)$ as $\Pi^V(t)$, the value of the delta hedging portfolio is given by

$$\Pi^V(t) = V(t, W(t)) - W(t)\partial_W V(t, W(t)). \tag{18}$$

By Ito's formula and (1), the SDE for Π^V can be obtained as

$$d\Pi^V(t) = \Big(\partial_t V(t, W(t)) - \alpha_{\text{tot}} W(t)\partial_W V(t, W(t)) + \tfrac{1}{2}\sigma^2 W(t)^2 \partial_{WW} V(t, W(t))\Big)\, dt \tag{19}$$

for $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^V is locally riskless, it must grow at the risk-free rate r, that is $d\Pi^V(t) = r\Pi^V(t)dt$. This along with (18) implies that the PDE satisfied by the value function $V(t, W)$ is given by

$$\partial_t V - rV + (r - \alpha_{\text{tot}})W\partial_W V + \frac{1}{2}\sigma^2 W^2 \partial_{WW} V = 0, \tag{20}$$

for $t \in (t_{n-1}, t_n)$ and $n = 1, \ldots, N$. The boundary conditions at t_n are specified by (12) and (15). The valuation formula (17) or the PDE (20) may be solved recursively by following Algorithm 1 to compute the initial policy value $V(0, \mathbf{Y}(0))$. It should be noted that (17) is general, and does not depend on the simplifying assumptions made in the PDE derivation.

Algorithm 1 Recursive computation of $V(0, \mathbf{X}(0))$

```
choose a withdrawal strategy Γ
initialize V(T^+, X(T^+)) = 0
set n = N
while n > 0 do
    compute the withdrawal amount γ_n by by (4)
    compute V(t_n^-, X(t_n^-)) by applying jump condition (15) with appropriate cash flows
    compute V(t_{n-1}^+, X(t_{n-1}^+)) by solving (17) or (20) with terminal condition V(t_n^-, X(t_n^-))
    n = n − 1
end while
return V(0, X(0)) = V(0^+, X(0^+))
```

4. Calculating the Net Guarantee Value Function

The GMWDB contract may be considered from the insurer's perspective by examining the insurer's liabilities to the guarantee, given by the risk-neutral value of the cash flows that must be paid by the insurer in order to fulfill the guarantee contract. We define the *net guarantee value function* $G(t, \mathbf{X}(t)), t \in [0, T]$ as the time-t risk-neutral value of all future payments on or after t made to the policyholder by the insurer, less that of all insurance fees received over the same period.

The insurance fees, charged at the rate $\alpha_{\text{ins}}(t)$, $t \in [0, T]$, is called fair if the total fees exactly compensate for the insurer's total liability, such that the net guarantee value is zero at time $t = 0$. That is,

$$G(0, \mathbf{X}(0)) = 0, \tag{21}$$

and if $\alpha_{\text{ins}}(t) \equiv \alpha_{\text{ins}}$ is a constant, its value can be found by solving (21). The fair insurance fees represent the hedging cost for the insurer to deliver the GMWDB rider to the policyholder. We emphasize here that the net guarantee value may not be equal to the added value of the GMWDB rider to the policyholder's wealth account, as we will show in Section 5.

On an event date t_n, the actual cash flow received by the policyholder is given by (5). This cash flow is first paid out of the policyholder's real withdrawal from the wealth account, which is equal to $\min(W(t_n), \gamma_n)$, the smaller of the nominal withdrawal and the available wealth. If the wealth account has an insufficient balance, the rest must be paid by the insurer. If the real withdrawal exceeds the cash flow entitled to the policyholder, the insurer keeps the surplus. The payment made by the insurer at t_n is thus given by

$$c_n(\gamma_n, \mathbf{X}(t_n)) = C_n(\gamma_n, \mathbf{X}(t_n)) - \min(W(t_n), \gamma_n). \tag{22}$$

To compute $G(t, \mathbf{X}(t))$ for all $t \in [0, T]$ we first note that, at maturity T, the terminal condition on G is given by

$$G(T^+, \mathbf{X}(T^+)) = 0, \tag{23}$$

i.e., no further liability or insurance fee income after maturity. Analogous to (13) and (15), the jump condition of G on event date t_n is given by

$$G(t_n, \mathbf{X}(t_n)) = c_n(\gamma_n, \mathbf{X}(t_n)) + G(t_n^+, \mathbf{X}(t_n^+)) \tag{24}$$

and

$$G(t_n^-, \mathbf{X}(t_n^-)) = \Big((1-q_n)\Big(p_n(\gamma_n, \mathbf{Y}(t_n)) + G(t_n^+, \mathbf{Y}(t_n^+))\Big) + q_n d_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{25}$$

where the insurance payments under $I(t_n) = 1$ and $I(t_n) = 0$ are given by $p_n(\gamma_n, \mathbf{Y}(t_n)) = P_n(\gamma_n, \mathbf{Y}(t_n)) - \min(W(t_n), \gamma_n)$ and $d_n(\mathbf{Y}(t_n)) = D_n(\mathbf{Y}(t_n)) - \min(W(t_n), \gamma_n)$, respectively. See (4), (5) and (22).

At $t \in (t_{n-1}, t_n)$, the net guarantee value function $G(t, \mathbf{X}(t))$ is given by the risk-neutral value of the net guarantee value just before the next withdrawal date t_n^- less any insurance fee incomes over the period (t, t_n), discounted at the risk-free rate. Specifically, we have

$$G(t, \mathbf{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} G(t_n^-, \mathbf{X}(t_n^-))\Big] - \mathrm{E}_t^{\mathbb{Q}}\Big[\int_t^{t_n} e^{-\int_t^s r(u)du} \alpha_{\text{ins}}(s) W(s) ds\Big]. \tag{26}$$

Note that the net guarantee value at t is reduced by expecting to receive insurance fees over time. Since this reduction decreases with time, the net guarantee value *increases* with time over (t_{n-1}, t_n).

To give an example, we again assume constant $r(t) \equiv r$, $\sigma(t) \equiv \sigma$, $\alpha_{\text{ins}}(t) \equiv \alpha_{\text{ins}}$, $\alpha_{\text{tot}}(t) \equiv \alpha_{\text{tot}}$. Under these simplifying assumptions, we have $G(t, \mathbf{X}(t)) = G(t, W(t))$ for $t \in (t_{n-1}, t_n)$. To derive the PDE satisfied by $G(t, W)$, consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, consists of a long position in the net guarantee value function G and a short position of $\frac{W(t)\partial_W G(t, W(t))}{S(t)}$ shares of the index S. The value of the delta hedging portfolio, denoted as $\Pi^G(t)$, is given by

$$\Pi^G(t) = G(t, W(t)) - W(t)\partial_W G(t, W(t)). \tag{27}$$

By Ito's formula and (1), we obtain the SDE for Π^G as

$$d\Pi^G(t) = \Big(\partial_t G(t, W(t)) - \alpha_{\text{tot}} W(t)\partial_W G(t, W(t)) + \tfrac{1}{2}\sigma^2 W(t)^2 \partial_{WW} G(t, W(t))\Big)\, dt, \tag{28}$$

where $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^G is locally riskless and must grow at the risk-free rate r, as well as increase with the insurance fee income at rate $\alpha_{\text{ins}} W(t)$ (see remarks after (26)), we must also have $d\Pi^G(t) = \big(r\Pi^G(t) + \alpha_{\text{ins}}W(t)\big)\,dt$. This along with (27) implies that the PDE satisfied by $G(t, W)$ is given by

$$\partial_t G - \alpha_{\text{ins}} W - rG + (r - \alpha_{\text{tot}}) W \partial_W G + \frac{1}{2}\sigma^2 W^2 \partial_{WW} G = 0, \tag{29}$$

for $t \in (t_{n-1}, t_n)$. The initial net guarantee value can thus be computed by recursively solving (26) or (29) from terminal and jump conditions (23) and (25), as described in Algorithm 2.

Algorithm 2 Recursive computation of $G(0, \mathbf{X}(0))$

1: choose a withdrawal strategy Γ
2: initialize $G(T^+, \mathbf{X}(T^+)) = 0$
3: set $n = N$
4: **while** $n > 0$ **do**
5: compute the withdrawal amount γ_n by (4)
6: compute $G(t_{n-1}^-, \mathbf{X}(t_{n-1}^-))$ by applying jump condition (25) with appropriate cash flows
7: compute $G(t_{n-1}^+, \mathbf{X}(t_{n-1}^+))$ by solving (26) or (29) with terminal condition $G(t_n^-, \mathbf{X}(t_n^-))$
8: $n = n - 1$
9: **end while**
10: return $G(0, \mathbf{X}(0)) = G(0^+, \mathbf{X}(0^+))$

5. The Wealth Manager's Value Function and Optimal Withdrawals

In the previous sections, the withdrawal strategy Γ has been assumed to be given. The withdrawal strategy serves as a control sequence affecting the total value function and the (net) guarantee value function. These withdrawals may thus be chosen to maximize either of these functions, leading to two distinct withdrawal strategies. In this section, we formulate these two strategies and discuss their relations. In particular, we identify the wealth manager's value function that connects the two perspectives.

5.1. The Wealth Manager's Value Function

We establish the relationship between the total value V and the net guarantee value G by defining the process

$$M(t, \mathbf{X}(t)) := G(t, \mathbf{X}(t)) + W(t) - V(t, \mathbf{X}(t)), \tag{30}$$

for $t \in [0, T]$. From (11) and (12), we obtain

$$M(T^+, \mathbf{X}(T^+)) = 0 \tag{31}$$

as the terminal condition for M. The jump condition for M can be obtained from (10), (13), (22) and (24) as

$$M(t_n, X(t_n)) = M(t_n^+, \mathbf{X}(t_n^+))), \tag{32}$$

and furthermore

$$M(t_n^-, X(t_n^-)) = (1 - q_{n-1}) M(t_n^+, \mathbf{X}(t_n^+))) \mathbb{1}_{\{I(t_n^-)=1\}} \tag{33}$$

due to the possible death occurence of the policyholder over the last period. Note that the death information is not revealed until t_n.

From (17) and (26), we find the recursive relation for M as

$$\begin{aligned} M(t, \mathbf{X}(t)) = {} & \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} M(t_n^-, \mathbf{X}(t_n^-))\Big] \\ & + W(t) - \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} W(t_n)\Big] \\ & - \mathrm{E}_t^{\mathbb{Q}}\Big[\int_t^{t_n} e^{-\int_t^{s} r(u)du} \alpha_{\text{ins}}(s) W(s) ds\Big]. \end{aligned} \tag{34}$$

Note that the second and third lines in (34) can be identified with the time-t risk-neutral value of management fees over (t, t_n). To see this, we first note that the difference of the two terms in the second line is the time-t risk-neutral value of the total fees charged on the wealth account over (t, t_n), and the expectation in the third line is the time-t risk-neutral value of the insurance fees over the same period.

In light of (31), (33) and (34), the process $M(t, \mathbf{X}(t)), t \in [0, T]$ defined by (30) is precisely the time-t risk-neutral value of future management fees, or *wealth manager's value function*. From (30), the policy value may be written as

$$V(t, \mathbf{X}(t)) = W(t) + G(t, \mathbf{X}(t)) - M(t, \mathbf{X}(t)), \tag{35}$$

i.e., the sum of the wealth and the net value of the guarantee, less the the wealth manager's value. At $t = 0$, this gives

$$V(0, \mathbf{X}(0)) = W(0) + G(0, \mathbf{X}(0)) - M(0, \mathbf{X}(0)). \tag{36}$$

Therefore, a withdrawal strategy that maximizes the net guarantee value $G(0, \mathbf{X}(0))$ in general is sub-optimal in maximizing the total value $V(0, \mathbf{X}(0))$, since the wealth manager's value $M(0, \mathbf{X}(0))$ depends on the withdrawals. The fair fee condition (21) becomes

$$V(0, \mathbf{X}(0)) = W(0) - M(0, \mathbf{X}(0)), \tag{37}$$

in contrast to the $V(0) = W(0)$ condition often seen in the literature, when no management fees are charged.

5.2. Formulation of Two Optimization Problems

We first formulate the total value maximization problem, i.e., maximizing the initial total value $V(0, \mathbf{X}(0))$ by optimally choosing the sequence γ_n under $I(t_n) = 1$ for $n = 1, \ldots, N-1$. Following the principle of dynamic programming, this is accomplished by choosing the withdrawal γ_n as

$$\gamma_n = \Gamma^V(t_n, \mathbf{Y}(t_n)) = \arg\max_{\gamma \in \mathcal{A}} \{ P_n(\gamma, \mathbf{Y}(t_n)) + V\left(t_n^+, \mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)\right) \} \tag{38}$$

in the admissible set $\mathcal{A} = \{\gamma : \gamma \geq 0, A(t_n^+ | \mathbf{X}(t_n), \gamma) \geq 0\}$. Here, we used $\mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)$ and $A(t_n^+ | \mathbf{X}(t_n), \gamma)$ to denote the state variables $\mathbf{X}$ and guarantee account balance A after withdrawal γ is made, given the value of the state variables $\mathbf{X}$ before the withdrawal. At any withdrawal time t_n, the policyholder chooses the withdrawal $\gamma \in \mathcal{A}$ to maximize the sum of the payment $P_n(\gamma, \mathbf{Y}(t_n))$ received and the present value of the remaining term of the policy $V(t_n^+, \mathbf{X}(t_n^+))$. The strategy Γ^V given by (38) is called the *total value maximization strategy*.

On the other hand, the optimization problem from the insurer's perspective considers the most unfavourable situation for the insurer. That is, by making suitable choices of γ_n's, the policyholder attempts to maximize the initial net guarantee value function $G(0, \mathbf{X}(0))$. By maximizing the net guarantee value, the fair fee rate under this strategy is sufficient to cover the hedging cost of the GMWDB rider regardless of the withdrawal strategy of the policyholder (assuming the insurer can perfectly hedge the market risk). The withdrawal γ_n under $I(t_n) = 1$ for this strategy is given by

$$\gamma_n = \Gamma^G(t_n, \mathbf{Y}(t_n)) = \arg\max_{\gamma \in \mathcal{A}} \{ p_n(\gamma, \mathbf{Y}(t_n)) + L\left(t_n^+, \mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)\right) \}, \tag{39}$$

i.e., the sum of the payment made by the insurer and the net guarantee value of the remaining term of the contract is maximized. The strategy Γ^G given by (39) is referred to as the *guarantee value maximization strategy*. Due to (36), this withdrawal strategy differs from (38) when there are management fees. Section 6 illustrates this discrepancy through a numerical example.

6. Numerical Examples

To illustrate the analysis presented so far, we carry out in this section several numerical experiments. We demonstrate how the presence of management fees leads to different fair fees for the two withdrawal strategies studied in previous sections under different market conditions and contract parameters. For illustration purposes, we assume a simple GMWDB contract as specified by (6), (9) as well as constant r, σ, α_{m} and α_{ins} so that the PDEs (20) and (29) hold, and set all mortality rates to zero.

6.1. Withdrawal Strategies When Management Fees Are Present

To compare the two withdrawal strategies (38) and (39), we consider an example with the market conditions $r = 1\%$, $\sigma = 30\%$, and a GMWDB contract of 20-year maturity, a management fee rate $\alpha_{\mathrm{m}} = 2\%$ and an insurance fee rate $\alpha_{\mathrm{ins}} = 0.4\%$. We assume that the wealth and the guarantee accounts start at $W(0) = A(0) = 1$. The annual contracted withdrawals are rated at $1/20$ per annum. Withdrawals are made at the end of each year of the contract term. The penalty rate for over withdrawal is rated at $\beta = 10\%$.

By following Algorithm 1 and applying the two strategies respectively, where the net guarantee value needed by the Γ^G strategy was computed following Algorithm 2, we compute the total value $V(t, \mathbf{X}(t))$ and the withdrawals as a function of the state variables $W(t)$, $A(t)$ under both strategies. The PDEs (20) and (29) are solved using a Crank–Nicholson finite difference method (Crank and Nicolson 1947; Hirsa 2012) with appropriate terminal and jump conditions. The results are illustrated in Figures 1 and 2, where Figure 1 shows the total value $V(t, \mathbf{X}(t))$ as a function of wealth $W(t)$, for a guarantee account balance $A(t) = 0.5$ and 1, at the end of 5th, 10th, 15th and 19th years under both strategies, and Figure 2 shows the corresponding withdrawal as a function of wealth for the same guarantee account balances and years into the contract.

It is seen from Figure 1 that, although the difference is relatively small and decreases with time, the total value $V(t, \mathbf{X}(t); \Gamma^V)$ under the strategy Γ^V dominates the total value $V(t, \mathbf{X}(t); \Gamma^G)$ under Γ^G in all situations shown. This is to be expected from the definition of the Γ^V strategy of maximizing V. What is perhaps a bit surprising is that $V(t, \mathbf{X}(t); \Gamma^G)$ may be slightly decreasing with wealth $W(t)$ within a small range of lower wealth levels during the earlier stages, as shown in, e.g., the upper right panel of Figure 1. This reveals the suboptimality of the Γ^G strategy from the policyholder's perspective. The total value $V(t, \mathbf{X}(t); \Gamma^V)$ under the optimal strategy Γ^V is strictly increasing with $W(t)$. (It should be noted that the net guarantee value $G(t, \mathbf{X}(t))$ under the Γ^G strategy, which is not shown here, does maintain strict monotonicity with wealth.).

On the other hand, from Figure 2, the withdrawals as a function of wealth under both strategies differ significantly, and show clear patterns. In particular, both strategies usually make withdrawals only at a few critical levels, including 0, the contracted withdrawal amount, and a higher level. These patterns are reminiscent of the bang–bang control as analyzed in Azimzadeh and Forsyth (2015) for a guaranteed lifetime withdrawal benefit (GLWB) contract. It is unclear to the authors whether the optimal withdrawals are always in the form of bang–bang controls under the current settings. Interested readers are referred to Azimzadeh and Forsyth (2015) for discussions on the existence of optimal bang–bang controls for a number of GMxB contracts. In addition, it is interesting to observe some similarities and differences between the two strategies: both strategies withdraw more at low and high wealth levels. The Γ^V strategy tends to withdrawal with less patience than Γ^G especially when the wealth level is high. This is because, at a higher wealth level, the policyholder expects to be charged high management fees, which encourages early withdrawals under the Γ^V strategy. The management fees do not directly affect the net value of the guarantee as seen by the insurer, and thus do not significantly affect the decisions made by the Γ^G strategy. The withdrawals tend to be lower, even zero, when the wealth level is close to the guarantee account level, due to the maximal guarantee premium when the guarantee is "at the money". As maturity gets closer, both strategies tend to reduce the guarantee account due to reduced guarantee premium for shorter time to maturity.

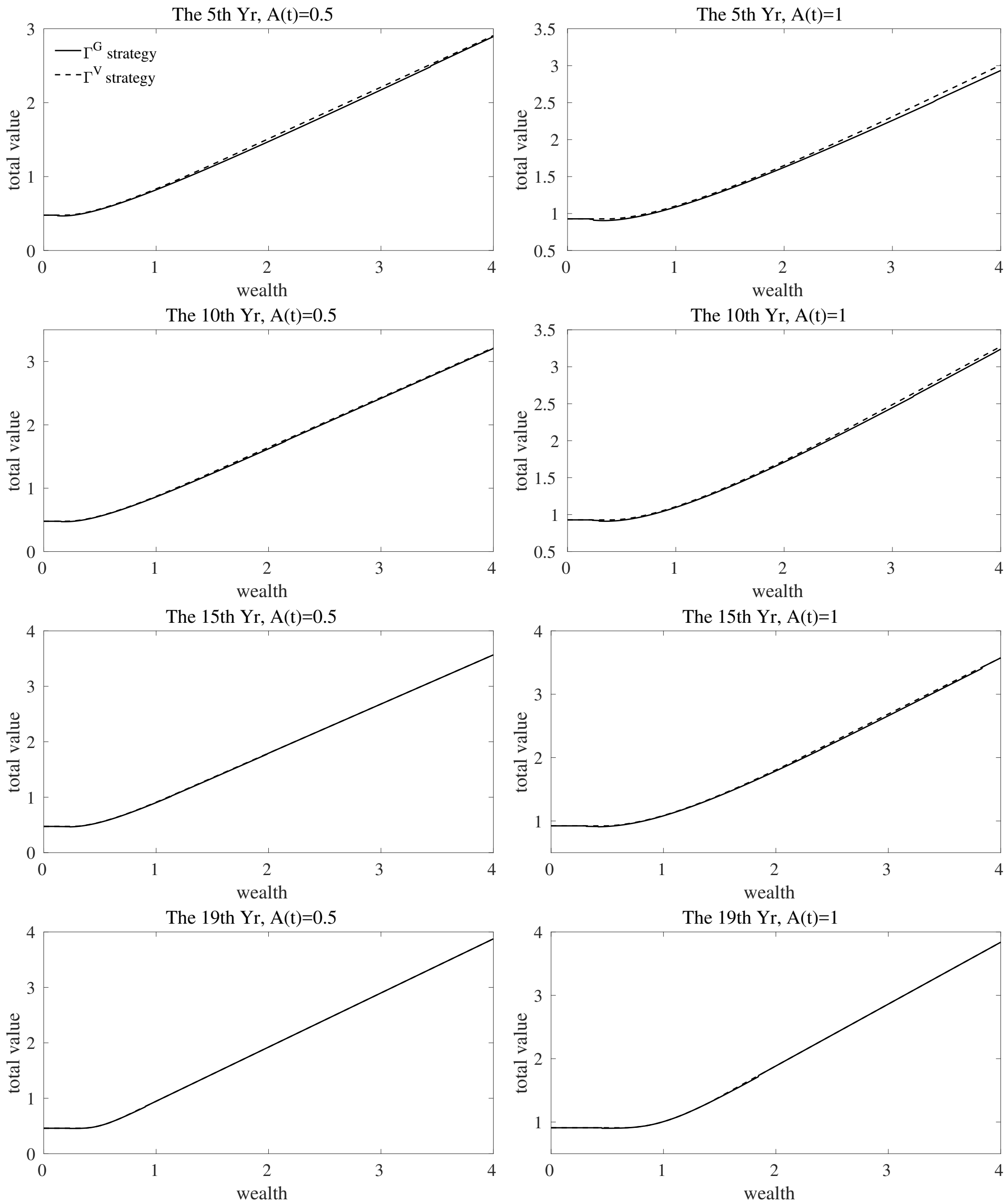

Figure 1. Total value as a function of wealth, respectively under the two strategies for guarantee account levels 0.5 and 1 at the end of 5th, 10th, 15th and 19th years of a contract with a maturity of 20 years. The market conditions are $r = 1\%$ and $\sigma = 30\%$.

It can be seen from this example that, although the total values under both strategies do not differ too much, when viewed as a function of the state variables at times prior to maturity, the withdrawal strategies can differ significantly, leading to potentially large discrepancies in the implied fair insurance fees. This is confirmed in the numerical experiments in Section 6.2.

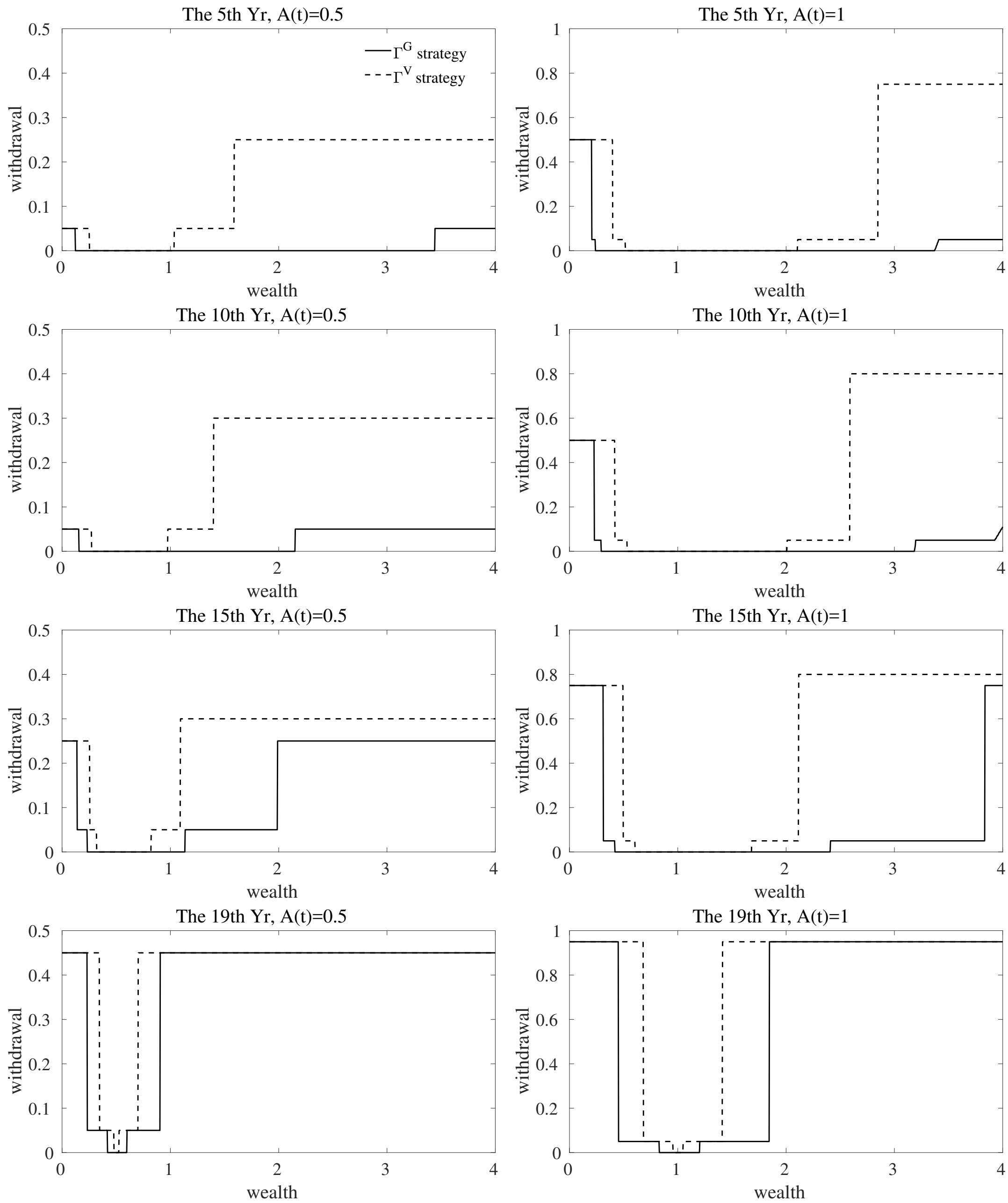

Figure 2. Optimal withdrawal as a function of wealth, respectively under the two strategies for guarantee account levels 0.5 and 1 at the end of 5th, 10th, 15th and 19th years of a contract with a maturity of 20 years. The market conditions are $r = 1\%$ and $\sigma = 30\%$.

6.2. The Impact of Management Fees on the Fair Insurance Fees

To assess the impact of management fees on the fair insurance fees, We consider a number of different market conditions and contract terms of the same GMWDB contract as in the previous example, and calculate the fair fees defined by (21) under the two withdrawal strategies. As in the preceding example, the wealth and the guarantee accounts are assumed to start at $W(0) = A(0) = 1$. The maturities of the contracts range from 5 to 20 years, with constant annual contracted withdrawals summing up to the initial guarantee account level. The management fee rate ranges from 0% up to 2%, and the penalty rate β takes a value of 10% or 20%. We consider several investment environments with the risk-free rate r at levels 1% and 5%, and the volatility of the index σ at 10% and 30%, to represent distinctive market conditions such as low/high growth and low/high volatility scenarios. The fair fees and corresponding total policy values are shown in Figures 3 and 4 for two market conditions: a low interest rate market with high volatility ($r = 1\%, \sigma = 30\%$) and a high interest rate market with low volatility ($r = 5\%, \sigma = 10\%$), respectively. Fair fee rates obtained for all market conditions and contract parameters and the corresponding policy values can be found in Tables 1–4.

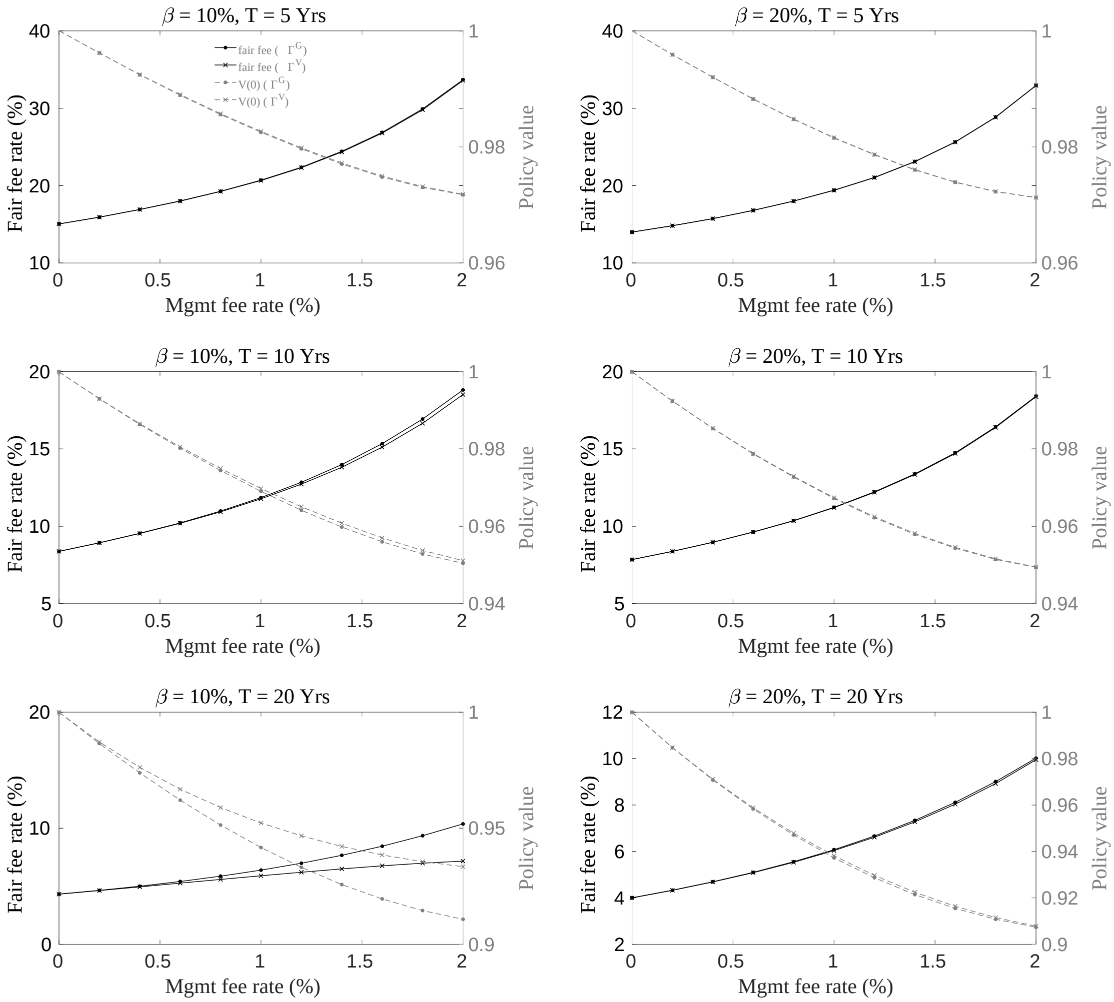

Figure 3. Fair insurance fee rates and policy values as a function of management fee rates α_{m} for risk-free rate $r = 1\%$ and volatility $\sigma = 30\%$, for penalty rates $\beta = 10\%, 20\%$ and maturities $T = 5, 10, 20$ years. The left axis and dark plots refer to the fair fees in percentage; the right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

We first observe from these numerical results that the fair fee rate implied by the guarantee value maximization strategy is always higher, and the corresponding policyholder's total value always lower, than those implied by the total value maximization strategy, unless management fees are absent, in which case these quantities are equal. These are to be expected from the definitions of the two strategies. We also note that, under the market condition of low interest rate with high volatility, a much higher insurance fee rate is required than under the market condition of high interest rate with low volatility. This is because, under adverse market conditions, the guarantee is more valuable; moreover, as the guarantee becomes more valuable, the fair insurance fee increases, leading to less wealth, which in turn makes the guarantee even more valuable. This positive feedback may add significantly to the fair fee rate when market conditions become worse. In addition, this high sensitivity also explains the large discrepancies between the fair fee rates under the two strategies as seen in the lower left panel in Figure 3, despite a relatively modest difference in the total values as shown in

Figure 1 under similar conditions. On the other hand, a higher penalty rate results in a lower insurance fee since it discourages the policyholder from making more desirable withdrawals that exceed the contracted values, and contributes to the insurer's penalty incomes.

Furthermore, the results show that, under most market conditions or contract specifications, the fair insurance fee rate obtained is highly sensitive to the management fee rate regardless of the withdrawal strategies, as seen from Figures 3 and 4. In particular, the fair fee rate implied by the guarantee value maximization strategy always increases with the management fee rate, since the management fees cause the wealth account to decrease, leading to higher liability for the insurer to fulfill. On the other hand, the fair fee rate implied by the total value maximization strategy first increases then decreases with the management fee rate, since, at high management fee rates, a rational policyholder tends to withdraw earlier to avoid the management fees, which in turn reduces the guarantee value from its optimum, and generates more penalty incomes for the insurer.

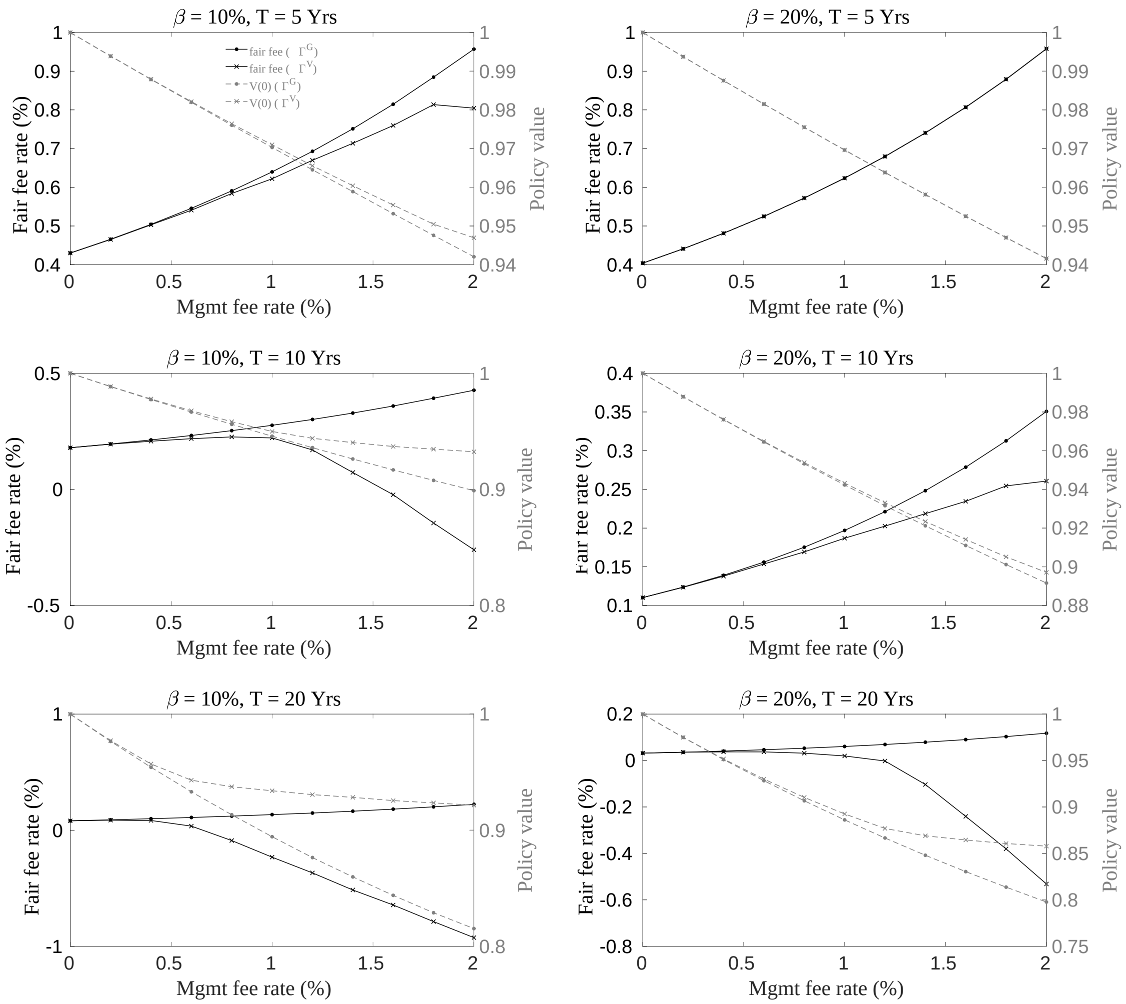

Figure 4. Fair insurance fee rates and policy values as a function of management fee rates $\alpha_{\rm m}$ for risk-free rate $r = 5\%$ and volatility $\sigma = 10\%$, for penalty rates $\beta = 10\%, 20\%$ and maturities $T = 5, 10, 20$ years. The left axis and dark plots refer to the fair fees in percentage; the right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

It is seen by examining Figures 3 and 4 that the fair fee rates implied by the two strategies differ more significantly under the following conditions:

- longer maturity T,
- lower penalty rate β,

- higher interest rate r, and
- higher management fee rate α_{m}.

Moreover, careful examination of results listed in Tables 1 and 2 reveals that the index volatility σ does not seem to contribute significantly to this discrepancy. These observations are intuitively reasonable: the contributors listed above all imply that the wealth manager's value $M(0)$ will be higher. There are more incentives to withdraw early to achieve higher policy value in the form of reduced management fees. The corresponding differences between the total values follow similar patterns. Of particular interest is that, in some cases, as shown in Figure 4, the fair fee rate implied by maximizing total value can become negative. This implies that the policyholder would want to withdraw more and early due to high management fees to such an extent that the penalties incurred exceed the total value of the GMWDB rider. On the other hand, the fair fee rate implied by maximizing the guarantee value is always positive.

Table 1. Fair fee rate α_{ins} (%) based on the guarantee value maximization strategy Γ^{G}.

Parameters				$\boldsymbol{\alpha_{\mathrm{m}}}$										
r(%)	σ(%)	β(%)	T	**0%**	**0.2%**	**0.4%**	**0.6%**	**0.8%**	**1%**	**1.2%**	**1.4%**	**1.6%**	**1.8%**	**2%**
1	10	10	5	3.08	3.48	3.96	4.59	5.41	6.65	8.66	12.18	17.67	23.53	29.92
			10	1.66	1.92	2.25	2.66	3.21	3.97	5.08	6.77	9.26	12.17	15.31
			20	0.82	0.97	1.17	1.43	1.77	2.22	2.83	3.68	4.81	6.19	7.71
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.39	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.13	3.83	4.88	6.60	9.20	12.17	15.31
			20	0.81	0.96	1.16	1.40	1.71	2.13	2.72	3.57	4.74	6.17	7.71
	30	10	5	15.05	15.92	16.92	18.02	19.27	20.70	22.37	24.43	26.88	29.91	33.69
			10	8.38	8.93	9.55	10.22	10.98	11.85	12.84	13.98	15.34	16.93	18.81
			20	4.32	4.64	5.00	5.41	5.87	6.39	6.98	7.66	8.45	9.35	10.37
		20	5	13.99	14.82	15.73	16.80	18.01	19.40	21.06	23.11	25.64	28.86	32.95
			10	7.85	8.38	8.97	9.63	10.36	11.22	12.22	13.38	14.74	16.43	18.41
			20	4.00	4.33	4.69	5.10	5.55	6.07	6.66	7.34	8.11	9.01	10.02
5	10	10	5	0.43	0.47	0.50	0.55	0.59	0.64	0.69	0.75	0.81	0.88	0.96
			10	0.18	0.20	0.21	0.23	0.25	0.28	0.30	0.33	0.36	0.39	0.43
			20	0.08	0.09	0.10	0.11	0.12	0.13	0.15	0.16	0.18	0.20	0.22
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.16	0.18	0.20	0.22	0.25	0.28	0.31	0.35
			20	0.03	0.04	0.04	0.05	0.05	0.06	0.07	0.08	0.09	0.10	0.12
	30	10	5	5.33	5.48	5.65	5.81	5.99	6.17	6.35	6.55	6.75	6.96	7.17
			10	2.91	3.02	3.12	3.23	3.35	3.47	3.60	3.73	3.86	4.01	4.16
			20	1.58	1.65	1.74	1.82	1.91	2.00	2.10	2.21	2.32	2.43	2.55
		20	5	4.97	5.13	5.28	5.44	5.61	5.79	5.97	6.15	6.35	6.55	6.76
			10	2.27	2.35	2.43	2.52	2.61	2.71	2.81	2.91	3.02	3.13	3.25
			20	1.08	1.13	1.19	1.24	1.31	1.37	1.43	1.50	1.58	1.65	1.73

Table 2. Fair fee rate α_{ins} (%) based on the total value maximization strategy Γ^V.

Parameters				$\alpha_{\mathbf{m}}$										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	**0%**	**0.2%**	**0.4%**	**0.6%**	**0.8%**	**1%**	**1.2%**	**1.4%**	**1.6%**	**1.8%**	**2%**
1	10	10	5	3.08	3.47	3.96	4.57	5.36	6.57	8.55	12.12	17.66	23.55	29.93
			10	1.66	1.92	2.23	2.61	3.13	3.81	4.86	6.44	8.99	12.11	15.30
			20	0.82	0.96	1.09	1.21	1.33	1.44	1.50	1.59	1.65	1.70	1.81
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.38	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.12	3.80	4.84	6.56	9.19	12.17	15.31
			20	0.81	0.96	1.16	1.39	1.67	2.05	2.60	3.43	4.65	6.14	7.70
	30	10	5	15.05	15.92	16.91	18.00	19.25	20.66	22.32	24.34	26.79	29.80	33.58
			10	8.38	8.93	9.53	10.19	10.93	11.77	12.72	13.81	15.11	16.65	18.51
			20	4.32	4.63	4.94	5.27	5.59	5.90	6.20	6.49	6.76	6.99	7.16
		20	5	13.99	14.82	15.73	16.80	18.00	19.39	21.04	23.09	25.62	28.84	32.93
			10	7.85	8.38	8.96	9.62	10.35	11.20	12.19	13.34	14.70	16.38	18.37
			20	4.00	4.32	4.68	5.08	5.53	6.04	6.61	7.27	8.04	8.92	9.95
5	10	10	5	0.43	0.47	0.50	0.54	0.58	0.62	0.67	0.71	0.76	0.81	0.80
			10	0.18	0.19	0.21	0.22	0.23	0.22	0.17	0.07	−0.02	−0.14	−0.26
			20	0.08	0.09	0.08	0.04	−0.09	−0.23	−0.37	−0.51	−0.64	−0.79	−0.93
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.15	0.17	0.19	0.20	0.22	0.23	0.25	0.26
			20	0.03	0.04	0.04	0.04	0.03	0.02	−0.00	−0.10	−0.24	−0.38	−0.53
	30	10	5	5.33	5.48	5.64	5.79	5.95	6.11	6.27	6.43	6.60	6.75	6.94
			10	2.91	3.01	3.10	3.18	3.26	3.34	3.41	3.47	3.53	3.58	3.60
			20	1.58	1.64	1.68	1.72	1.74	1.75	1.75	1.74	1.74	1.73	1.71
		20	5	4.97	5.13	5.28	5.44	5.61	5.78	5.96	6.14	6.32	6.51	6.70
			10	2.27	2.35	2.43	2.51	2.59	2.67	2.74	2.82	2.88	2.94	3.00
			20	1.08	1.13	1.18	1.22	1.23	1.24	1.23	1.21	1.18	1.14	1.09

Table 3. Total value $V(0, \mathbf{X}(0); \Gamma^G)$ based on the guarantee value maximization strategy Γ^G.

Parameters				$\alpha_{\mathbf{m}}$										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	**0%**	**0.2%**	**0.4%**	**0.6%**	**0.8%**	**1%**	**1.2%**	**1.4%**	**1.6%**	**1.8%**	**2%**
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.94	0.92	0.92	0.91	0.90	0.90	0.90
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.97	0.96	0.95	0.94	0.93	0.93	0.92	0.91	0.91
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.90
			20	1.00	0.98	0.95	0.93	0.91	0.89	0.88	0.86	0.84	0.83	0.82
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.90	0.89
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.87	0.85	0.83	0.81	0.80
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.93
			20	1.00	0.98	0.97	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.91
			20	1.00	0.98	0.96	0.94	0.92	0.90	0.89	0.87	0.86	0.84	0.83

Table 4. Total value $V(0, \mathbf{X}(0); \Gamma^V)$ based on the policy value maximization strategy Γ^V.

Parameters				$\alpha_{\mathbf{m}}$										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	**0%**	**0.2%**	**0.4%**	**0.6%**	**0.8%**	**1%**	**1.2%**	**1.4%**	**1.6%**	**1.8%**	**2%**
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
			20	1.00	0.98	0.96	0.94	0.94	0.93	0.93	0.93	0.93	0.92	0.92
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.91	0.90
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.88	0.87	0.86	0.86	0.86
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.97	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.99	0.97	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.92
			20	1.00	0.98	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88	0.88

7. Conclusions

Determining accurate hedging costs of VA guarantees is a significant issue for VA providers. While the effect of management fees on policyholder's withdrawal behaviors is often not connected in the existing VA literature, it was demonstrated in this article that this effect on the pricing of GMWDB contract can be significant. As a form of market friction, management fees can affect policyholders' withdrawal behaviors, causing large deviations from those often assumed in the literature.

Two policyholder's withdrawal strategies were considered: total value maximization and guarantee value maximization, which differs from each other when management fees are present. We demonstrated that these two withdrawal strategies imply different fair insurance fee rates, where maximizing total value implies lower fair fees than those implied by maximizing guarantee value, which represents the maximal hedging costs from the insurer's perspective.

We identify the difference between the initial investment plus the value of the guarantee and the total value of the policyholder as the wealth manager's total value, which causes the discrepancy between the two withdrawal strategies. We further identify a number of factors that contribute to this discrepancy through a series of illustrating numerical experiments. Our findings identify the management fees as a potential cause of discrepancy between the fair fee rates implied by the guarantee value maximization strategy, often assumed from the insurer's perspective for VA pricing, and the prevailing market rates for VA contracts with GMWDB or similar riders.

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