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Asymptotic Ruin Probability of a Bidimensional Risk Model Based on Entrance Processes with Constant Interest Rate

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Received: 2 October 2018; Accepted: 15 November 2018; Published: 20 November 2018

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Abstract: In this paper, the risk model with constant interest based on an entrance process is investigated. Under the assumptions that the entrance process is a renewal process and the claims sizes satisfy a certain dependence structure, which belong to the different heavy-tailed distribution classes, the finite-time asymptotic estimate of the bidimensional risk model with constant interest force is obtained. Particularly, when inter-arrival times also satisfy a certain dependence structure, these formulas still hold.

Keywords: asymptotic; constant interest force; widely lower orthant dependence; ruin probability

1. Introduction

In this paper, we investigate a bidimensional risk model based on entrance processes, in which an insurance company operates two kinds of business. Suppose that the initial insurance fund for i-th class is x_i and $S_j^{(i)}$ is entry time of the j-th policy with $0 < S_1^{(i)} < S_2^{(i)} < \cdots$ and $S_j^{(i)} = \sum_{k=1}^{j} \theta_k^{(i)}$, $\theta_1^{(i)} = S_1^{(i)}$, $i = 1, 2$. The corresponding renewal process, up to the time t, is

$$N_i(t) = \sum_{j=1}^{\infty} I_{\{S_j^{(i)} \le t\}}.$$

where $I_{\{\cdot\}}$ is the indicator function.

Denote renewal function by $m_i(t) = E(N_i(t)) = \sum_{j=1}^{\infty} P(S_j^{(i)} \le t), t \ge 0$ and suppose the $m_i(t) < \infty$ for all $0 < t < \infty$ and $m_i(0) = 0, i = 1, 2$. Let the validity time of the j-th policy be $\{C_j^{(i)}, j = 1, 2, \cdots\}$ with probability $P(C_j^{(i)} = \alpha_\ell) = p_\ell$, $\ell = 1, 2, \cdots, K$, and $\alpha_1 < \alpha_2 < \cdots < \alpha_K$, where they are independent and identically distributed. The premium is $f_i(C_j^{(i)})$ and $f_i(\cdot)$ is a strictly increasing function. $D_j^{(i)}$ is claim time of the j-th policy and independent and identically distributed function $H_i(\cdot)$. $Y_j^{(i)}$ is the j-th claim size and identically distributed function $F_i(\cdot)$. Suppose that $D_j^{(i)}$, $Y_j^{(i)}$ and $C_j^{(i)}$ have the same distributions with random variables D_i, Y_i and C_i, respectively.

Assume that an insurance company invests risk free market with force of interest $\delta > 0$, then up to time t, the surplus process of the insurance company is written as:

$$\begin{aligned}\begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} &= \begin{pmatrix} x_1 e^{\delta t} \\ x_2 e^{\delta t} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{N_1(t)} f_1(C_j^{(1)}) e^{\delta(t-S_j^{(1)})} \\ \sum_{j=1}^{N_2(t)} f_2(C_j^{(2)}) e^{\delta(t-S_j^{(2)})} \end{pmatrix} \\ &- \begin{pmatrix} \sum_{j=1}^{N_1(t)} Y_j^{(1)} e^{\delta(t-S_j^{(1)}-D_j^{(1)})} I_{\{S_j^{(1)}+D_j^{(1)} \le t, D_j^{(1)} \le C_j^{(1)}\}} \\ \sum_{j=1}^{N_2(t)} Y_j^{(2)} e^{\delta(t-S_j^{(2)}-D_j^{(2)})} I_{\{S_j^{(2)}+D_j^{(2)} \le t, D_j^{(2)} \le C_j^{(2)}\}} \end{pmatrix}. \end{aligned} \tag{1}$$

Now the following two types of ruin times for a bidimensional risk model based on entrance processes are considered. we define the first time when both $R_1(t)$ and $R_2(t)$ become negative by

$$\tau_{\max}(x_1, x_2) = \inf\{t : \max\{R_1(t), R_2(t)\} < 0 | R_i(0) = x_i, i = 1, 2\},$$

the first time when both $R_1(t)$ or $R_2(t)$ become negative by

$$\tau_{\min}(x_1, x_2) = \inf\{t : \min\{R_1(t), R_2(t)\} < 0 | R_i(0) = x_i, i = 1, 2\}.$$

Then we define the corresponding ruin probabilities with the finite time $t > 0$ respectively by

$$\psi_{\max}(x_1, x_2, T) = P(\tau_{\max}(x_1, x_2) \le T) = P(\bigcap_{i=1}^{2} \{R_i(s) < 0\} \text{ for some } 0 \le s \le T), \tag{2}$$

and

$$\psi_{\min}(x_1, x_2, T) = P(\tau_{\min}(x_1, x_2) \le T) = P(\bigcup_{i=1}^{2} \{R_i(s) < 0\} \text{ for some } 0 \le s \le T). \tag{3}$$

We know that Li et al. (2005) put forward into a new model (LIG model) based on an entrance process and discussed asymptotic normality of the risk process. Furthermore, Some scholars got some conclusions through the study of the LIG model. Li and Kong (2007) discussed the weak convergence properties of the model. Xiao et al. (2008) studied some limit properties of the model under constant interest. Xiao and Tang (2009) studied the infinite ruin probability with constant interest within Poisson process and class $\mathcal{R}_{-\alpha}$. Xiao et al. (2013) discussed the ruin probability of LIG model. It is clear that the above literatures are improved and investigated for one-dimensional risk model based on entrance processes. Recently, people have been interested in two-dimensional risk model, see, for example, Chan et al. (2003); Li et al. (2007); Zhang and Wang (2012) and so on. It is well known that these literatures are investigated in the classical model, and the risk model based on entrance processes is more important and actual. Therefore, on the basis of above literatures, we consider a bidimensional risk model based on entrance processes.

Because Theorem 1 of Xiao and Tang (2009) is obtained under the Poisson process and the regular variation class, we know that this is far from the actual. Hence, in this paper, we consider the LIG model and obtain the finite-time ruin probability under the class $\mathcal{L} \cap \mathcal{D}$ with constant interest when claim sizes satisfy a certain dependence under the renewal process. The conclusion also extend the above Theorem 1 and Theorem 3.1 of Xiao et al. (2013). At the same time, it indicates that tail characteristics of claim distribution determine the ruin probability of insurance company, which is of great significance to the safe operation and the risk assessment of insurance company.

This paper is organized as follows: The second Section introduces the preliminary knowledge. The third Section presents the main results of this paper. The fourth Section gives some lemmas. Finally, the fifth Section gives the proofs of main Theorems.

2. Some Preliminaries

Firstly, we give some markers. All limit relationships of this paper are for $x \to \infty$ unless stated otherwise. For the two positive function $f(\cdot)$ and $g(\cdot)$, if $\limsup \frac{f(x)}{g(x)} < \infty$, write $f(x) = O(g(x))$; if $\lim \frac{f(x)}{g(x)} = 0$, write $f(x) = o(g(x))$; if $\limsup \frac{f(x)}{g(x)} \leq 1$, write $f(x) \lesssim g(x)$; if $\liminf \frac{f(x)}{g(x)} \geq 1$, write $f(x) \gtrsim g(x)$; if $\lim \frac{f(x)}{g(x)} = 1$, write $f(x) \sim g(x)$; if $f(x) = O(g(x)), g(x) = O(f(x))$, write $f(x) \asymp g(x)$.

Here are some important concepts of heavy-tailed distributions.

Definition 1. *Say a distribution F belongs to the class $\mathcal{L}$, if F satisfies for any $y > 0$ (or equivalent for $y = 1$)*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

Say a distribution F belongs to the class $\mathcal{D}$, if F satisfies for any $0 < y < 1$ (or equivalent for $y = \frac{1}{2}$)

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Say a distribution F belongs to the class $\mathcal{C}$, if F satisfies

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

where their relationship is as follows:

$$\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}.$$

For more properties and applications of the heavy-tailed distribution, we can refer to Bingham et al. (1987) and Embrechts et al. (1997).

There are an important relationship between heavy-tailed distribution and Matuszewska index of the distribution, which is defined by

$$J_F^+ = -\lim_{x\to\infty} \frac{\log \overline{F}_*(x)}{\log x}, J_F^- = -\lim_{x\to\infty} \frac{\log \overline{F}^*(x)}{\log x},$$

where

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

Furthermore, other indices of the distribution F can be defined by

$$L_F = \lim_{y\downarrow 1} \overline{F}_*(y).$$

As for any $y > 0$, there is $\overline{F}^*(y) = 1/\overline{F}_*(y)$, hence

$$L_F = \lim_{y\downarrow 1} \overline{F}_*(y) = 1/\lim_{y\uparrow 1} \overline{F}^*(y).$$

Particularly, if $F \in \mathcal{C}$, then $L_F = 1$.

For more properties and applications of the heavy tailed distribution, we can refer to Tang and Tsitsiashvili (2003) and Yang and Wang (2010).

Here we introduce some concepts and properties of dependence.

Definition 2. *If there exists the finite real sequence* $\{g_U(n), n \geq 1\}$ *for* $x_i \in (-\infty, \infty), 1 \leq i \leq n$ *such that*

$$P(\bigcap_{i=1}^{n}\{\xi_i > x_i\}) \leq g_U(n)\prod_{i=1}^{n}P(\xi_i > x_i),$$

then we say random variable sequence $\{\xi_n, n \geq 1\}$ *are widely upper orthant dependent* $(WUOD)$.

If there exists the finite real sequence $\{g_L(n), n \geq 1\}$ *for* $x_i \in (-\infty, \infty), 1 \leq i \leq n$ *such that*

$$P(\bigcap_{i=1}^{n}\{\xi_i \leq x_i\}) \leq g_L(n)\prod_{i=1}^{n}P(\xi_i \leq x_i),$$

then we say random variable sequence $\{\xi_n, n \geq 1\}$ *are widely lower orthant dependent* $(WLOD)$.

Furthermore, if $\{\xi_n, n \geq 1\}$ *satisfy WUOD and WLOD at the same time, then we say random variables* $\{\xi_n, n \geq 1\}$ *are widely orthant dependent* (WOD).

For more detailed information, we can refer to Wang et al. (2013); Ghosh (1981) and Block et al. (1982).

Definition 3. *If real valued random variables* $X_i, i \geq 1$ *with distribution functions* $F_i, i \geq 1$ *satisfy for any* $i \neq j$

$$\lim_{x\to\infty} P(|X_i| \wedge X_j > x | X_i \vee X_j > x) = 0,$$

or, equivalently

$$\lim_{x\to\infty} \frac{P(X_i > x, X_j > x) + P(X_i < -x, X_j > x)}{\overline{F}_i(x) + \overline{F}_j(x)} = 0,$$

then we say random variable variables $X_i, i \geq 1$ *are pairwise quasi-asymptotically independent* $(PQAI)$.

If real valued random variables $X_i, i \geq 1$ *with distribution functions* $F_i, i \geq 1$ *satisfy for any* $i \neq j$

$$\lim_{x_i \wedge x_j\to\infty} P(|X_i| > x | X_j > x) = 0,$$

or, equivalently

$$\lim_{x_i \wedge x_j\to\infty} \frac{P(X_i > x, X_j > x) + P(X_i < -x, X_j > x)}{\overline{F}_j(x)} = 0,$$

then we say random variable variables $X_i, i \geq 1$ *are pairwise strong quasi-asymptotically independent* $(PSQAI)$.

Remark 1. *If random variables* $X_i, i \geq 1$ *are* $PSQAI$*, then they are* $PQAI$.

For more detailed information, we can refer to Li (2013) and Liu et al. (2012).

The first lemma comes from Theorem 2.1 of Li (2013).

Lemma 1. *Assume that* $\{X_j, 1 \leq j \leq n\}$ *are n real-valued random variables with functions of distribution* $F_j, 1 \leq j \leq n$. *Then*

$$P(\sum_{j=1}^{n} c_j X_j > x) \sim \sum_{j=1}^{n} P(c_j X_j > x).$$

holds if either (i) $\{X_j\}$ *are* $PSQAI$ *and* $F_j \in \mathcal{L} \cap \mathcal{D}$ *for* $1 \leq j \leq n$ *and* $(c_1, \cdots, c_n) \in [a, b]^n$, *or* (ii) $\{X_j\}$ *are* $PQAI$ *and* $F_j \in \mathcal{C}$ *for* $1 \leq j \leq n$ *and* $(c_1, \cdots, c_n) \in [a, b]^n$.

The following lemma comes from Proposition 2.2.1 of Bingham et al. (1987).

Lemma 2. *Let $F \in \mathcal{D}$. For any $0 < \alpha' < \alpha < \beta < \beta'$, there exist positive constants A_i and $B_i, i = 1,2$ satisfying the following inequality*

$$\frac{\overline{F}(y)}{\overline{F}(x)} \geq A_1(x/y)^{\alpha'}, \tag{4}$$

for any $x \geq y \geq B_1$, and the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq A_2(x/y)^{\beta'}. \tag{5}$$

for any $x \geq y \geq B_2$.

The following lemma comes from Theorem 3.3 of Cline and Samorodnitsky (1994), Lemma 3.4 of Liu and Wang (2016) and Lemma 3.5 of Tang and Tsitsiashvili (2003).

Lemma 3. *Let X be a random variable with distribution F and Y be a random variable independent of X. Suppose that H is the distribution of XY. If $EY^p < \infty$ for any $0 < J_F^+ \leq \beta < \beta'$ and some $p > J_F^+$, then there exist the following conclusions:*

(i) If $F \in \mathcal{D}$, then $\overline{F}(x) \asymp \overline{H}(x)$ and $x^{-\beta'} = o(\overline{F}(x))$;
(ii) If $F \in \mathcal{C}$, $H \in \mathcal{C}$.

The following lemma can be proved in Appendix A.

Lemma 4. (1) *Under the conditions of Theorem 1 (or Theorem 2), for all $T \geq T_0$, then we have*

$$\begin{aligned} &P(\textstyle\sum_{j=1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} I_{\{S_j^{(i)} \leq T\}} > x_i) \\ \sim &\textstyle\sum_{j=1}^{\infty} P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} I_{\{S_j^{(i)} \leq T\}} > x_i). \end{aligned} \tag{6}$$

(2) *Suppose the conditions of Theorem 2 are true and the inter-arrival times $\{\theta_j^i, j \geq 1\}$ are WLOD random variables satisfying (16) for some $\epsilon_{i0} > 0$, depending on F_i and $G_i, i = 1,2$. Then, for all $T \geq T_0$, the relation (6) still holds uniformly.*

3. Main Results

In this paper, we make the following assumptions:

A_1 Assume that the i-th class of random variables $\{C_j^{(i)}, j \geq 1\}, \{D_j^{(i)}, j \geq 1\}, \{Y_j^{(i)}, j \geq 1\}, \{S_j^{(i)}, j \geq 1\}, i = 1,2$, are independent mutually.

A_2 Assume that $E(N_i(t))^{p+1} < \infty$ for any fixed $t > 0$ and some $p > J_{F_i}^+, i = 1,2$.

Theorem 1. *Consider the bidimensional risk model (1) under the assumptions $A_1 - A_2$. Let $p_0 = P(S_1^{(i)} < T_0) > 0$ for some $0 < T_0 < \infty$. Assume that claim sizes, $\{Y_j^{(i)}, j \geq 1\}$ be PSQAI random variables with common distribution $F_i \in \mathcal{L} \cap \mathcal{D}, i = 1,2$ such that $J_{F_i}^- > 0$, respectively. Then for $T \geq T_0$, we have*

$$\psi_{\max}(x_1, x_2, T) \sim \prod_{i=1}^{2} (\sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s)). \tag{7}$$

and

$$\psi_{\min}(x_1, x_2, T) \sim \sum_{i=1}^{2} \sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s). \tag{8}$$

where $a \wedge b = \min\{a, b\}$.

Proof. (i) Firstly, we deal with the relation (7). Write $U_i(t) = \sum_{j=1}^{N_i(t)} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \le t, D_j^{(i)} \le C_j^{(i)}\}}$. Due to (1) and (2), we know that for all $T \ge T_0$,

$$\psi_{\max}(x_1, x_2, T) = P(\bigcap_{i=1}^{2} \{U_i(s) - \sum_{j=1}^{N_i(t)} f_i(C_j^{(i)}) e^{-\delta S_j^{(i)}} > x_i\} \text{ for some } 0 \le s \le T). \tag{9}$$

By (9), we know for all $T \ge T_0$,

$$P(\bigcap_{i=1}^{2} \{U_i(T) - \sum_{j=1}^{N_i(T)} f_i(C_j^{(i)}) e^{-\delta S_j^{(i)}} > x_i\}) \le \psi_{\max}(x_1, x_2, T) \le P(\bigcap_{i=1}^{2} \{U_i(T) > x_i\}). \tag{10}$$

Because Lemma 4 (1) and random variables $\{C_j^{(i)}, j \ge 1\}$, $\{D_j^{(i)}, j \ge 1\}$, $\{Y_j^{(i)}, j \ge 1\}$, $\{S_j^{(i)}, j \ge 1\}$, $i = 1, 2$, are independent mutually, it is clear that

$$\begin{aligned} \psi_{\max}(x_1, x_2, T) &\le \textstyle\prod_{i=1}^{2} P(U_i(T) > x_i) \\ &\sim \textstyle\prod_{i=1}^{2} P(\sum_{j=1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \le T, D_j^{(i)} \le C_j^{(i)}\}} I_{\{S_j^{(i)} \le T\}} > x_i) \\ &\sim \textstyle\prod_{i=1}^{2} (\sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s)). \end{aligned} \tag{11}$$

holds uniformly for all $T \ge T_0$.

Because $f_i(C_j^{(i)}), j = 1, \cdots, K$ are bounded and $\{Y_j^{(i)}, j \ge 1\}$ are $PSQAI$ random variables with distributions $F_i \in \mathcal{L} \cap \mathcal{D}, i = 1, 2$, respectively, it is easy to prove that $\{Y_j^{(i)} + f_i(C_j^{(i)}), j \ge 1\}$ are $PSQAI$ random variables, whose distributions belong to the class $\mathcal{L} \cap \mathcal{D}, i = 1, 2$, respectively. Hence, by Lemma 4 (1), we know

$$\begin{aligned} \psi_{\max}(x_1, x_2, T) &\ge P(\textstyle\bigcap_{i=1}^{2} \{U_i(T) - \sum_{j=1}^{N_i(T)} f_i(C_j^{(i)}) e^{-\delta S_j^{(i)}} > x_i\}) \\ &= \textstyle\prod_{i=1}^{2} P(\{U_i(T) - \sum_{j=1}^{N_i(T)} f_i(C_j^{(i)}) e^{-\delta S_j^{(i)}} > x_i\}) \\ &= \textstyle\prod_{i=1}^{2} P(\sum_{j=1}^{N_i(t)} (Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \le t, D_j^{(i)} \le C_j^{(i)}\}} - f_i(C_j^{(i)}) e^{-\delta S_j^{(i)}}) > x_i) \\ &\sim \textstyle\prod_{i=1}^{2} P(\sum_{j=1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \le t, D_j^{(i)} \le C_j^{(i)}\}} I_{\{S_j^{(i)} \le t\}} > x_i) \\ &\sim \textstyle\prod_{i=1}^{2} (\sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s)). \end{aligned} \tag{12}$$

Combining (11) with (12), we obtain that (7) holds uniformly for all $T \ge T_0$.

Next, we handle (8). By (3), we know that

$$\psi_{\min}(x_1, x_2, T) = P(\tau_1(x_1 \le T)) + P(\tau_2(x_2 \le T)) - \psi_{\max}(x_1, x_2, T). \tag{13}$$

where $(\tau_i(x_i) = \inf\{t : R_i(t) < 0 | R_i(0) = x_i\}, i = 1, 2$. Then by (7) with its one-dimensional case, it is clear that

$$P(\tau_i(x_i) \le T) \sim \sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s), \quad i = 1, 2. \tag{14}$$

holds uniformly for all $T \geq T_0$.

Again by (7) and $F_i \in \mathcal{L} \cap \mathcal{D} \subset \mathcal{D}, i = 1, 2$, we have that

$$\begin{aligned} &\limsup_{x_1 \wedge x_2 \to \infty} \sup_{T \in \Lambda \cap [T_0, \infty)} \frac{\psi_{\max}(x_1, x_2, T)}{\sum_{i=1}^{2} \sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}_i(e^{\delta(s+y)} x_i) dH_i(y) dm_i s} \\ &\leq \limsup_{x_1 \wedge x_2 \to \infty} \sup_{T \in \Lambda \cap [T_0, \infty)} \frac{\overline{F}_1(x_1) m_1(T) \overline{F}_2(x_2) m_2(T) (\sum_{\ell=1}^{K} p_\ell H_i(\alpha_\ell \wedge (T-s)))^2}{\overline{F}_1(x_1 e^{\delta T}) m_1(T_0) (\sum_{\ell=1}^{K} p_\ell H_i(\alpha_\ell \wedge (T_0 - s)))} \\ &= 0. \end{aligned} \tag{15}$$

Hence, by (13)–(15), we prove that (8) holds uniformly for all $T \geq T_0$. □

Theorem 2. *Consider the bidimensional risk model* (1) *under the assumption* $A_1 - A_2$. *Let* $p_0 = P(S_1^{(i)} < T_0) > 0$ *for some* $0 < T_0 < \infty$. *Assume that claim sizes,* $\{Y_i, i \geq 1\}$ *be PQAI random variables with common distribution* $F_i \in \mathcal{C}, i = 1, 2$ *such that* $J_{F_i}^- > 0$, *respectively. Then the relations* (7) *and* (8) *hold uniformly for* $T \geq T_0$.

Proof. Similarly, when $f_i(C_j^{(i)}), j = 1, \cdots, K$ are bounded and $\{Y_j^{(i)}, j \geq 1\}$ are $PQAI$ random variables with distributions $F_i \in \mathcal{C}, i = 1, 2$, respectively, it is easy to prove that $\{Y_j^{(i)} + f_i(C_j^{(i)}), j \geq 1\}$ are $PQAI$ random variables, whose distributions belong to the class $\mathcal{C}, i = 1, 2$, respectively. Hence, applying the same method of proof of Theorem 1, we know that (7) and (8) still hold uniformly for all $T \geq T_0$. □

Theorem 3. *Under the conditions of Theorem* 2, *suppose that entry inter-arrival times,* $\{\theta_k^{(i)}, k \geq 1\}$ *are* $WLOD$ *random variables with common distribution* G_i *satisfying*

$$\lim_{n \to \infty} g_{L_i}(n) e^{-\epsilon_{i0} n} = 0. \tag{16}$$

holds for some $\epsilon_{i0} > 0$, *depending on* F_i *and* $G_i, i = 1, 2$. *Then, the relations* (7) *and* (8) *still hold uniformly for all* $T \geq T_0$.

Proof. By the Lemma 4 (2), Theorem 2 and similar proof of Theorem 1, the relations (7) and (8) still hold uniformly for all $T \geq T_0$. □

Corollary 1. *Consider one-dimensional risk model satisfying the same conditions as those in Theorem* 1, *and denote ruin time by* $\tau(x) = \inf\{t : R(t) < 0 | R(0) = x\}$, *where*

$$R(t) = x e^{\delta t} + \sum_{j=1}^{N(t)} f(C_j) e^{\delta(t - S_j)} - \sum_{j=1}^{N(t)} Y_j e^{\delta(t - S_j - D_j)} I_{\{S_j + D_j \leq t, D_j \leq C_j\}},$$

then, we have

$$\psi(x, T) = P(\tau(x) \leq T) \sim \sum_{\ell=1}^{K} p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}(e^{\delta(s+y)} x) dH(y) dm(s).$$

It is easy to prove Corollary 1 from the proof of relation (7) of Theorem 1.

Remark 2. *Corollary* 1 *is a partial extension for the results of Theorem 3.1 of Xiao et al. (2013), Theorem 1 of Xiao and Tang (2009), and Theorem 3.1 of Xiao and Xie (2018).*

4. Conclusions

In summary, this paper studies the two-dimensional independent risk model based on entrance processes with constant interest rate. Under the assumptions that the entry process of policies of two kinds of business of insurance companies have different renewal processes, the claims sizes of two kinds of business are independent of each other, and the claims sizes of the same kind of business are pairwise strong quasi-asymptotically independent, which belong to the class $\mathcal{L} \cap \mathcal{D}$, the maximum finite-time ruin probability and the minimum finite-time ruin probability are obtained, respectively. If intervals of entry time of the policy satisfy the wide lower quadrant dependence, The finite-time maximum ruin probability and the finite-time minimum ruin probability are also obtained.

Author Contributions: L.X. completed the article preliminarily, and the Professor H.X. perfected it.

Funding: This research was funded by Natural Scientific Funds of China (71261023).

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Proof of Lemma 4

Proof. (1) Firstly, we deal with the upper bound of (6). Along with the method of proof of Theorem 2.1 of Hao and Tang (2008), for any positive integer N, we have

$$\begin{aligned}
&P(\textstyle\sum_{j=1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\
&= P(\textstyle\sum_{j=1}^{N_i(T)} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i) \\
&= (\textstyle\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}) P(\sum_{j=1}^{n} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i, N_i(T) = n) \\
&= I_1(x_i, t, N) + I_2(x_i, t, N).
\end{aligned} \tag{A1}$$

For $I_2(x_i, t, N)$. Because of the Lemma 2, there exist positive constants β, A_1 and B_1 for any $h > 0$ and $\frac{x_i}{n} \ge B_1$. Hence, we have

$$\begin{aligned}
I_2(x_i, t, N) \le& \sum_{n=N+1}^{\infty} P(\sum_{j=1}^{n} Y_j^{(i)} e^{-\delta(S_1^{(i)}+D_j^{(i)})} I_{\{S_1^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i, S_n^{(i)} \le T \le S_{n+1}^{(i)}) \\
=& \sum_{n=N+1}^{\infty} \int_0^T P(\sum_{j=1}^{n} Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i, S_n^{(i)} - S_1^{(i)} \le T - s \le S_{n+1}^{(i)} - S_1^{(i)}) \\
&\cdot P(S_1^{(i)} \in ds) \\
\le& \sum_{n=N+1}^{\infty} \int_0^T P(\sum_{j=1}^{n+1} Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i) \cdot P(N_i(T-s) = n) P(S_1^{(i)} \in ds) \\
=& \sum_{N\le n\le x_i/B_1} \int_0^T P(\sum_{j=1}^{n+1} Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i) P(N_i(T-s) = n) dm_i(s) \\
&+ \sum_{n> x_i/B_1} \int_0^T P(\sum_{j=1}^{n+1} Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i) P(N_i(T-s) = n) dm_i(s) \\
\le& \sum_{N\le n\le x_i/B_1} \int_0^T (n+1) P(Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} > x_i/(n+1)) P(N_i(T-s) = n) \\
&\cdot dm_i(s) + \sum_{n> x_i/B_1} \int_0^T P(N_i(T-s) = n) dm_i(s) \\
\le& \sum_{N\le n\le x_i/B_1} \int_0^T C_1 P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i\le T, D_i\le C_i\}} > x_i)(n+1)^{\beta+1} P(N_i(T-s) = n) dm_i(s) \\
&+ \int_0^T P(N_i(T-s) > x_i/B_1) dm_i(s)
\end{aligned}$$

$$\leq A_1 \int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i \leq T, D_i \leq C_i\}} > x_i) E(N_i(T-s)+1)^{\beta+1} I_{\{N \leq N_0(T-s) \leq x_i/B_1\}} dm_i(s)$$
$$+ P(N_i(T-s) > x_i/B_1) m_i(T)$$
$$\leq A_1 E(N_i(T-s)+1)^{\beta+1} I_{\{N \leq N_0(T-s) \leq x_i/B_1\}} \int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i \leq T, D_i \leq C_i\}} > x_i) dm_i(s)$$
$$+ e^{-h(x_i/B_1)} E(e^{hN_i(T)}) m_i(T).$$

If $N \to \infty$, then we have

$$E(N_i(T-s)+1)^{\beta+1} I_{\{N \leq N_i(T-s) \leq x_i/B_1\}} \to 0,$$

Because of the Lemma 3.2 of Hao and Tang (2008), we know

$$e^{-h(x_i/B_1)} E(e^{hN_i(T)}) \to 0.$$

Hence, for $x_i > 0$, we have

$$\begin{aligned} &\lim_{N\to\infty} \sup_{t\in(0,T]} \frac{I_2(x_i,t,N)}{\int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i \leq T, D_i \leq C_i\}} > x_i) dm_i(s)} \\ &= \lim_{N\to\infty} \sup_{t\in(0,T]} \frac{I_2(x_i,t,N)}{\sum_{\ell=1}^K p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s)} \\ &= 0. \end{aligned} \tag{A2}$$

For $I_1(x_i,t,N)$. Because of the Lemma 1, for $t \in (0,T]$, we have

$$\begin{aligned} I_1(x_i,t,N) \quad &\sim \textstyle\sum_{n=1}^N \sum_{i=1}^n P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} > x_i, N_i(T)=n) \\ &\sim (\textstyle\sum_{n=1}^\infty \sum_{j=1}^n - \sum_{n=N+1}^\infty \sum_{j=1}^n) P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} > x_i, N_i(T)=n) \\ &= J_1(x_i,t,N) + J_2(x_i,t,N). \end{aligned} \tag{A3}$$

For $J_1(x_i,t,N)$, we obtain

$$\begin{aligned} J_1(x_i,t,N) \quad &= \textstyle\sum_{j=1}^\infty P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} > x_i, N_i(T) \geq j) \\ &= \textstyle\int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i \leq T, D_i \leq C_i\}} > x_i) dm_i(s) \\ &= \textstyle\sum_{\ell=1}^K p_\ell \int_0^T \int_0^{\alpha_\ell \wedge (T-s)} \overline{F}(e^{\delta(s+y)} x_i) dH_i(y) dm_i(s). \end{aligned} \tag{A4}$$

For $J_2(x_i,t,N)$. Applying the similar method of dealing with $I_2(x_i,t,N)$, we have

$$\begin{aligned} J_2(x_i,t,N) &\leq \sum_{n=N+1}^\infty \sum_{j=1}^n P(Y_j^{(i)} e^{-\delta(S_1^{(i)}+D_j^{(i)})} I_{\{S_1^{(i)}+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} > x_i, N_i(T)=n) \\ &\leq \sum_{n=N}^\infty \sum_{j=1}^{n+1} P(Y_j^{(i)} e^{-\delta(s+D_j^{(i)})} I_{\{s+D_j^{(i)} \leq T, D_j^{(i)} \leq C_j^{(i)}\}} > x_i) \cdot P(N_i(T-s)=n) dm_i(s) \\ &\leq \int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i \leq T, D_i \leq C_i\}} > x_i) dm_i(s) \cdot \sum_{n=N}^\infty (n+1) P(N_i(T) \geq n). \end{aligned}$$

Hence, for $x_i > 0$,

$$\begin{aligned}
&\lim_{N\to\infty}\sup_{t\in(0,T]} \frac{J_2(x_i,t,N)}{\int_0^T P(Y_i e^{-\delta(s+D_i)} I_{\{s+D_i\le T, D_i\le C_i\}} > x_i)dm_i(s)} \\
&= \lim_{N\to\infty}\sup_{t\in(0,T]} \frac{J_2(x_i,t,N)}{\sum_{\ell=1}^K p_\ell \int_0^T \int_0^{\alpha_\ell\wedge(T-s)} \overline{F}_i(e^{\delta(s+y)}x_i)dH_i(y)dm_i(s)} \\
&= 0.
\end{aligned} \tag{A5}$$

By the relations (A1)–(A5), we obtain

$$\begin{aligned}
&P(\textstyle\sum_{j=1}^\infty Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\
&\lesssim \textstyle\sum_{\ell=1}^K p_\ell \int_0^T \int_0^{\alpha_\ell\wedge(T-s)} \overline{F}_i(e^{\delta(s+y)}x_i)dH_i(y)dm_i(s).
\end{aligned} \tag{A6}$$

Next, we cope with the lower bound of (6).

For $m = 0,1,2,\cdots$, we write $\Delta_m = \sum_{j=1}^\infty Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}}$. According to the similar method of Tang and Tsitsiashvili (2004), for all integer m such that $\sum_{j=m+1}^\infty j^{-2} < 1$, we have

$$\begin{aligned}
P(\Delta_m > x_i) &\le P(\textstyle\sum_{j=m+1}^\infty Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > \sum_{j=m+1}^\infty \frac{x_i}{n^2}) \\
&\le P(\textstyle\bigcup_{j=m+1}^\infty (Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > \frac{x_i}{n^2})) \\
&\le \textstyle\sum_{j=m+1}^\infty P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} > \frac{x_i}{n^2}).
\end{aligned} \tag{A7}$$

Because of the Lemma 2, there exist $0 < \alpha < J_F^- \le J_F^+ < \infty, A_i$ and $B_i, i = 1,2$ satisfying the relations (4) and (5). The events are written as $A_1(j,x_i) = (j^{-2}e^{\delta(S_j^{(i)}+D_j^{(i)})} \le B_2/x_i)$, $A_2(j,x_i) = (B_2/x_i < j^{-2}e^{\delta(S_j^{(i)}+D_j^{(i)})} \le 1)$ and $A_3(j,x_i) = (j^{-2}e^{\delta(S_j^{(i)}+D_j^{(i)})} > 1)$. Hence, the relation (A7) is written as

$$\begin{aligned}
P(\Delta_m > x_i) &\le \sum_{j=m+1}^\infty P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} > \frac{x_i}{n^2}) \\
&= \sum_{k=1}^3 \sum_{j=m+1}^\infty P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} > \frac{x_i}{n^2}, A_k(j,x_i)) \\
&= \sum_{k=1}^3 I_k(m,x_i).
\end{aligned}$$

By Chebyshev's inequality and the Lemma 3 (i), we have

$$\begin{aligned}
I_1(m,x_i) &\le \textstyle\sum_{j=m+1}^\infty P(A_1(j,x_i)) \le (\frac{x_i}{B_2})^{-\beta'} \sum_{j=1}^\infty j^{2\beta'}(E(e^{-\delta\beta' S_1^{(i)}}))^j \\
&= o(\overline{F}(x_i)).
\end{aligned} \tag{A8}$$

Because of the relations (4) and (5), for all $x_i \ge \max\{B_1, B_2\}$, we can obtain

$$\begin{aligned}
I_2(m,x_i) &\le A_2\overline{F}(x_i) \textstyle\sum_{j=m+1}^\infty E(j^{2\beta'} e^{-\delta\beta' S_j^{(i)}} I_{A_2(j,x_i)}) \\
&\le A_2\overline{F}(x_i) \textstyle\sum_{j=m+1}^\infty j^{2\beta'}(E(e^{-\delta\beta' S_1^{(i)}}))^j.
\end{aligned} \tag{A9}$$

and

$$I_3(m,x_i) \le \frac{\overline{F}(x_i)}{A_1}\sum_{j=m+1}^{\infty} E(j^{2\alpha'} e^{-\delta\alpha' S_j^{(i)}} I_{A_3(j,x_i)}) \le \frac{\overline{F}(x_i)}{A_1}\sum_{j=m+1}^{\infty} j^{2\alpha'}(E(e^{-\delta\alpha' S_1^{(i)}}))^j. \quad \text{(A10)}$$

Because of $F \in \mathcal{D}$ and $p_0 > 0$, there exists a positive number $M > 0$ satisfying for $T \ge T_0$,

$$\begin{aligned} & Y_1^{(i)} e^{-\delta(S_1^{(i)}+D_1^{(i)})} I_{\{S_1^{(i)}+D_1^{(i)}\le T, D_1^{(i)}\le C_1^{(i)}\}} I_{\{S_1^{(i)}\le T\}} \\ & \ge P(Y_1^{(i)} e^{-\delta(S_1^{(i)}+D_1^{(i)})} I_{\{S_1^{(i)}+D_1^{(i)}\le T, D_1^{(i)}\le C_1^{(i)}\}} I_{\{S_1^{(i)}\le T\}}, S_1^{(i)} \le T_0) \\ & \ge p_0 P(Y_1^{(i)} e^{-\delta(T_0+D_1^{(i)})} I_{\{S_1^{(i)}+D_1^{(i)}\le T, D_1^{(i)}\le C_1^{(i)}\}} I_{\{S_1^{(i)}\le T\}}) \\ & \ge M\overline{F}(x_i). \end{aligned} \quad \text{(A11)}$$

By the relations (A8)–(A10), we have

$$\begin{aligned} & \lim_{m\to\infty}\limsup_{x_i\to\infty} \frac{P(\Delta_m > x_i)}{\overline{F}(x_i)} \\ & = \lim_{m\to\infty}\limsup_{x_i\to\infty} \sum_{j=m+1}^{\infty} \frac{P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} > \frac{x_i}{n^2})}{\overline{F}(x_i)} = 0. \end{aligned} \quad \text{(A12)}$$

For $\varepsilon > 0$, by the relations (A11) and (A12), there exist integer $n_0 > 0$ and enough large number x_i. Then we have

$$\begin{aligned} & P(\sum_{j=n_0+1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\ & \le \sum_{j=n_0+1}^{\infty} P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} > \frac{x_i}{n^2}) \\ & \le \varepsilon\overline{F}(x_i) \le \varepsilon P(Y_1^{(i)} e^{-\delta(S_1^{(i)}+D_1^{(i)})} I_{\{S_1^{(i)}+D_1^{(i)}\le T, D_1^{(i)}\le C_1^{(i)}\}} I_{\{S_1^{(i)}\le T\}}). \end{aligned} \quad \text{(A13)}$$

Let the above number n_0 be fixed. Because of the Lemma 1, the relation (A13), we know

$$\begin{aligned} & P(\sum_{j=1}^{\infty} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\ & \ge P(\sum_{j=1}^{n_0} Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\ & \sim (\sum_{j=1}^{\infty} - \sum_{j=n_0+1}^{\infty}) P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i) \\ & \ge (1-\varepsilon)\sum_{j=1}^{\infty} P(Y_j^{(i)} e^{-\delta(S_j^{(i)}+D_j^{(i)})} I_{\{S_j^{(i)}+D_j^{(i)}\le T, D_j^{(i)}\le C_j^{(i)}\}} I_{\{S_j^{(i)}\le T\}} > x_i). \end{aligned} \quad \text{(A14)}$$

Combining (A6) with (A14) and arbitrariness of ε, we prove the Lemma 4 (1).

(2) Because the inter-arrival times $\{\theta_j^{(i)}, j \ge 1\}$ are $WLOD$ random variables satisfying (9) for some $\epsilon_{i0} > 0$, depending on F_i and G_i, $i = 1, 2$, by (11) and (12) of Block et al. (1982), it is clear that

$$Ee^{-\delta\beta' S_n^{(i)}} \le (Ee^{-\delta\beta' S_1^{(i)}})^n.$$

holds for any $\beta' > 0$ and $n \ge 1$. Hence, we know the relations (A8)–(A10) still hold. Along with the similar proof of Lemma 4 (1), it is easy to prove Lemma 4 (2). □

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