risks

Article

Longevity Modelling and Pricing under a Dependent Multi-Cohort Framework

Fadoua Zeddouk * and Pierre Devolder

Institute of Statistics, Biostatistics and Actuarial Sciences, Catholic University of Louvain, Voie du Roman Pays 20, 1348 Louvain-La-Neuve, Belgium; pierre.devolder@uclouvain.be
* Correspondence: fadoua.zeddouk@uclouvain.be

Received: 5 October 2020; Accepted: 13 November 2020; Published: 16 November 2020

Abstract: We propose a multi-cohort model that is able to capture the mortality correlation between different cohorts. The model is based on the Hull and White process to which we incorporate inter-generational risk factors, by modifying its stochastic part. We provide a pricing framework for a new survival forward contract under the Cost of Capital, risk-neutral and Sharpe approaches, allowing to cover the global multi-cohort longevity risk. We give numerical illustrations for Belgian cohorts, and we compute the price of the longevity derivative under the proposed methods, for different correlation levels.

Keywords: multi-cohort; longevity hedging; survival forward; Cost of Capital

1. Introduction

Insurance companies and pension funds are constantly exposed to mortality risk, and they are, therefore, becoming increasingly interested in longevity-linked securities to transfer this risk. However, only a few longevity derivatives have been launched for various reasons. One important cause is that no consensus has yet been reached regarding the best model for the mortality risk. Many continuous-time stochastic mortality models for a single generation have been proposed by a number of researchers, including Biffis (2005); Cairns et al. (2006a); Dahl (2004); Luciano and Vigna (2015); Milevsky and Promislow (2001); Schrager (2006).

However, the common trend in the evolution of the longevity of different populations is relevant, and it should be taken into account by entities seeking to hedge their exposures to mortality and/or longevity risk, as discussed in detail in Coughlan et al. (2011). In this way, these entities can assess the overall longevity risk, and reduce the basis risk between their own population and population associated with the hedging instruments.

Consequently, researchers have lately become more interested in developing multi-population models for the evolution of longevity rates (see, for instance, Chen et al. (2014); Enchev et al. (2017); Hunt and Villegas (2015); Ntamjokouen (2014)).

These multi-population mortality models are based on the assumption that the mortality experiences of the populations are linked together, and do not diverge in long term. According to Chen et al. (2014), this assumption could be justified by the long-term mortality evolution and, therefore, can be applicable to longevity risk modelling.

Some multi-population models that have been proposed in the literature are based on the generalization of the well-known Carter model Lee and Carter (1992). These models capture the mortality dependence by including an additional common factor between the multiple populations (see, for instance, the models that were presented by Danesi et al. (2015); Haberman et al. (2003); Li et al. (2015); Li and Lee (2005)). In addition, Jevtic et al. (2013) have proposed a model for the mortality intensity using common factors that affect all the cohorts, as well as specific factors that only affect specific cohorts.

In this paper, we consider a typical life insurer holding a portfolio of individuals with different ages, and who needs to hedge the longevity risk. We attempt to capture the eventual correlations across generations while using a multidimensional continuous-time mortality environment based on the Hull and White model. We first consider a portfolio of two different cohorts, where the correlation is based on the introduction of two risk factors modelled by independent Brownian motions; we then generalize our framework to n cohorts.

We assess the longevity risk related to these n generations through the pricing of a new longevity derivative that we call Global Survival forward contract (GS-forward). Therefore, we need to use a pricing approach in order to compute the price of this derivative. The different pricing approaches proposed in the literature are mainly based on the traditional pricing methods used in finance, such as the Sharpe ratio, the risk-neutral, and the Wang approaches, which have been adapted in the longevity context. However, these methods require the assessment of the risk premium, which is not easy due to lack of data in the longevity market. In the literature, some authors use life annuities or announced longevity bonds for calibration, or consider values usually used in finance. For instance, (Cairns et al. 2006b; Meyricke and Sherris 2014) have used the longevity bond announced by BNP/EIB in 2004 for calibration. Moreover, these classical pricing methods are not necessarily consistent with the directives of Solvency II, which is an important issue for insurance companies. The Cost of Capital method (COC) allows for avoiding these issues, since this method is consistent with Solvency II, and the Cost of Capital rate is fixed by the regulator. This approach has been used by (Levantesi and Menzietti 2017; Zeddouk and Devolder 2019) to price longevity derivatives, such as S-forward contracts, allowing to hedge the longevity risk for individuals belonging to one given cohort. The consistency between this Cost of Capital and the aforementioned classical methods have been discussed in detail in (Zeddouk and Devolder 2019).

Our aim is to determine, in closed form, the price of the GS-forward derivative under the Cost of Capital, risk-neutral (see Cairns et al. (2006b)), and Sharpe approaches (Biffis 2005; Milevsky et al. 2005) in the case of a multi-cohort portfolio with correlated mortality experiences, as well as to compare the price of a GS-forward with the corresponding individual S-forward contracts.

This paper is organized, as follows: in Section 2, we present the multi-dimensional model for two cohorts, we study the correlation between these two cohorts, and we provide the general pricing framework for the GS-forward under the COC, risk-neural, and Sharpe methods. Next, we generalize this framework for n cohorts in Section 3. In Section 4, we present a numerical illustration enabling the comparison of the different GS-forward prices and also between these GS-forward prices and individual S-forwards prices. Finally, in Section 5 we conclude.

2. Cohort-Based Longevity Model: Two Cohorts

In order to describe the force of mortality, we have used an affine model that allows for the valuation of longevity derivatives (Huang et al. (2019); Xu et al. (2020)). In particular, we have chosen the Hull and White process (HW), which is a cohort mortality model. Zeddouk and Devolder (2020) provided a comparison between different stochastic time-continuous models, and have shown that this model can accurately predict the mortality of the Belgian population. Moreover, the HW model was also used by Zeddouk and Devolder (2019) for mortality in order to price Survival-forwards and Survival-swaps. We first focus on the case of a portfolio of two different cohorts, then we generalize our study for n cohorts.

2.1. Correlation between Two Forces of Mortality

We consider two different cohorts Y and Z of individuals initially aged $x_0 = y$ and $x_0 = z$ at time $t = 0$.

The evolution of the mortality intensity of the two cohorts are assumed to be given by:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y dw_y(t) \tag{1}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z dw_z(t), \tag{2}$$

where A_y, B_y, b_y, σ_y and A_z, B_z, b_z, σ_z all positive constants, w_y and w_z are two dependent Brownian motions.

Using the Cholesky decomposition for correlated Brownian motions, Equations (1) and (2) can be alternatively written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y(\rho_y dw_1(t) + \sqrt{1-\rho_y^2}\, dw_2(t)) \tag{3}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z(\rho_z dw_1(t) + \sqrt{1-\rho_z^2}\, dw_2(t)), \tag{4}$$

with:

$$w_y = \rho_y w_1 + \sqrt{1-\rho_y^2}\, w_2, \tag{5}$$

and

$$w_z = \rho_z w_1 + \sqrt{1-\rho_z^2}\, w_2, \tag{6}$$

where w_1 and w_2 are two independent Brownian motions (independent risk factors), and ρ_y and ρ_z are two risk parameters that link the mortality intensity of the cohorts Y and Z to the two risk factors.

The forces of mortality that are given by Equations (3) and (4) are now based on two independent Brownian motions, generating risk factors that affect the mortality of the two cohorts, and allowing to introduce the inter-generational correlation with different levels.

The solutions of the SDEs (3) and (4) are given by:

$$\mu_y(t) = \mu_y(0)e^{-b_y t} + \frac{A_y}{b_y + B_y}(e^{B_y t} - e^{-b_y t}) + \sigma_y e^{-b_y t} \int_0^t e^{b_y u} dw_y(u) \tag{7}$$

$$\mu_z(t) = \mu_z(0)e^{-b_z t} + \frac{A_z}{b_z + B_z}(e^{B_z t} - e^{-b_z t}) + \sigma_z e^{-b_z t} \int_0^t e^{b_z u} dw_z(u), \tag{8}$$

w_y and w_z being given by (5) and (6).

In order to measure the correlation between the forces of mortality of the two cohorts at any time t, we compute the correlation $corr(\mu_y(t), \mu_z(t))$ that is induced by the model.

We first determine the covariance between the two forces of mortality:

$$\begin{aligned} Cov(\mu_y(t), \mu_z(t)) &= E_{\mathbb{P}}[(\mu_y(t) - E_{\mathbb{P}}(\mu_y(t))(\mu_z(t) - E_{\mathbb{P}}(\mu_z(t))] \\ &= \sigma_y \sigma_z e^{-(b_y+b_z)t} E_{\mathbb{P}}\left[\int_0^t e^{b_y u} dw_y(u) \cdot \int_0^t e^{b_z u} dw_z(u)\right]. \end{aligned} \tag{9}$$

Using the Ito multidimensional formula (see for instance Tristan (2017)), Equation (9) becomes:

$$\begin{aligned} Cov(\mu_y(t), \mu_z(t)) &= \sigma_y \sigma_z e^{-(b_y+b_z)t} \int_0^t e^{(b_y+b_z)u} \cdot \rho_{w_y,w_z} du \\ &= \frac{\sigma_y \sigma_z}{(b_y + b_z)}(1 - e^{-(b_y+b_z)t})\rho_{w_y,w_z}, \end{aligned} \tag{10}$$

where ρ_{w_y,w_z} is the correlation factor between w_y and w_z given by:

$$\rho_{w_y,w_z} = \rho_y \rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}. \tag{11}$$

Finally, the correlation $Corr(\mu_y(t), \mu_y(t))$ is:

$$
\begin{aligned}
Corr(\mu_y(t), \mu_z(t)) &= \frac{Cov(\mu_y(t), \mu_z(t))}{\sqrt{\mathrm{Var}_{\mathbb{P}}(\mu_y(t))}\sqrt{\mathrm{Var}_{\mathbb{P}}(\mu_z(t))}} \\
&= \rho_{w_y,w_z} \frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1-e^{-2b_y t}}\sqrt{1-e^{-2b_z t}}} \\
&= \left(\rho_y\rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}\right)\left(\frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1-e^{-2b_y t}}\sqrt{1-e^{-2b_z t}}}\right) \qquad (12) \\
&= \left(\rho_y\rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}\right)\varphi_{y,z}(t). \qquad (13)
\end{aligned}
$$

where $\varphi_{y,z}(t)$ is given by:

$$\varphi_{y,z}(t) = \left(\frac{2\sqrt{b_y b_z}}{b_y + b_z} \frac{1 - e^{-(b_y+b_z)t}}{\sqrt{1-e^{-2b_y t}}\sqrt{1-e^{-2b_z t}}}\right). \qquad (14)$$

Remark 1. *From Equation (12), we can see that, if $b_y = b_z$, which means that the reversion force of mortality is the same for the two noises, then $\varphi_{y,z}(t) = 1$, and $Corr(\mu_y(t), \mu_z(t))$ will be time-independent. In this case, we have also three possibilities:*

- *The two noises are independent, namely $\rho_y = 1$ and $\rho_z = 0$ or $\rho_y = 0$ and $\rho_z = 1$, then $Corr(\mu_y(t), \mu_z(t)) = 0$;*
- *The two noises are perfectly correlated, namely $\rho_y = \rho_z = 1$, or $\rho_y = \rho_z = 0$, then $Corr(\mu_y(t), \mu_z(t)) = 1$;*
- *The two noises are partially dependent, namely $1 < \rho_y\rho_z < 0$, then $Corr(\mu_y(t), \mu_z(t)) = \rho_y\rho_z + \sqrt{1-\rho_y^2}\sqrt{1-\rho_z^2}$;*

If $b_y \neq b_z$, then we have three possibilities:

- *The two noises are independent, namely $\rho_y = 1$ and $\rho_z = 0$ or $\rho_y = 0$ and $\rho_z = 1$, then $Corr(\mu_y(t), \mu_z(t)) = 0$;*
- *The two noises are perfectly correlated, namely $\rho_y = \rho_z = 1$, or $\rho_y = \rho_z = 0$, then $Corr(\mu_y(t), \mu_z(t)) = \varphi_{y,z}(t)$ (now not necessarily equal to 1);*
- *The two noises are partially dependent, then $Corr(\mu_y(t), \mu_z(t))$ becomes a function of time t (now not necessarily a constant).*

2.2. Correlation between Two Longevity Indexes

Now, let us study the correlation between two longevity indexes. The two longevity indexes related to the two cohorts Y and Z at time t are given by:

$$I_y(y+t, T-t) = e^{-\int_t^T \mu_y(u,\omega)du} \qquad (15)$$

$$I_z(z+t, T-t) = e^{-\int_t^T \mu_z(u,\omega)du}, \qquad (16)$$

where $I_y(y+t, T-t)$ is the longevity index of an individual of the cohort Y initially aged y, alive at time t, and surviving $T-t$ years more.

Let us compute the covariance of the two longevity indexes. This covariance is given by:

$$
\begin{aligned}
Cov(I_y(y+t,T-t), I_z(z+t,T-t)) &= E_{\mathbb{P}}[(I_y(y+t,T-t) - E_{\mathbb{P}}(I_y(y+t,T-t))) \\
&\quad \times (I_z(z+t,T-t) - E_{\mathbb{P}}(I_z(z+t,T-t)))] \\
&= E_{\mathbb{P}}(I_y(y+t,T-t) I_z(z+t,T-t)) - E_{\mathbb{P}}(I_y(y+t,T-t)) \\
&\quad \times E_{\mathbb{P}}(I_z(z+t,T-t)) \\
&= E_{\mathbb{P}}(e^{-\int_t^T (\mu_y(u,\omega)+\mu_z(u,\omega))du}) - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)} \\
&\quad \times e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)},
\end{aligned} \tag{17}
$$

where α_y, β_y and α_z, α_z are given by:

$$
\begin{cases}
\alpha_y(t,T) & = \frac{A_y}{b_y}[e^{-b_y T} \frac{e^{(B_y+b_y)T}-e^{(B_y+b_y)t}}{B_y+b_y} - \frac{e^{B_y T}-e^{B_y t}}{B_y}] - \frac{\sigma_y^2}{2b_y^2}[\frac{1}{b_y}(1-e^{-b_y(T-t)}) - T + t] \\
& \quad -\frac{\sigma_y^2}{4b_y^3}(1-e^{-b_y(T-t)})^2 \\
\beta_y(t,T) & = \frac{1}{b_y}(1-\exp(-b_y(T-t)))
\end{cases} \tag{18}
$$

and

$$
\begin{cases}
\alpha_z(t,T) & = \frac{A_z}{b_z}[e^{-b_z T} \frac{e^{(B_z+b_z)T}-e^{(B_z+b_z)t}}{B_z+b_z} - \frac{e^{B_z T}-e^{B_z t}}{B_z}] - \frac{\sigma_z^2}{2b_z^2}[\frac{1}{b_z}(1-e^{-b_z(T-t)}) - T + t] \\
& \quad -\frac{\sigma_z^2}{4b_z^3}(1-e^{-b_z(T-t)})^2 \\
\beta_z(t,T) & = \frac{1}{b_z}(1-\exp(-b_z(T-t))).
\end{cases} \tag{19}
$$

Let us now compute $E_{\mathbb{P}}(e^{-\int_t^T (\mu_y(u,\omega)+\mu_z(u,\omega))du})$.
For any given cohort of individuals aged x at time t, we have:

$$
I_x(x+t,T-t) = e^{-\int_t^T \mu_x^{\mathbb{P}}(u,\omega)du} = e^{X_x(t,T)}. \tag{20}
$$

Under the real-world risk measure $\mathbb{P}$, we have:

$$
X_x(t,T) \sim N(m_x(t,T),\, n_x(t,T)^2). \tag{21}
$$

Accordingly, the survival index I_x is log-normally distributed with X_x having a mean and a variance given by:

$$
\begin{cases}
m_x(t,T) = & \mu_x(t)\frac{(e^{-b(T-t)}-1)}{b} - \frac{Ae^{Bt}}{B(b+B)}(e^{B(T-t)}-1) - \frac{Ae^{Bt}}{b(b+B)}(e^{-b(T-t)}-1) \\
n_x^2(t,T) = & \frac{\sigma^2}{b^2}[T-t-\frac{1-e^{-b(T-t)}}{b} - \frac{(1-e^{-b(T-t)})^2}{2b}].
\end{cases} \tag{22}
$$

We have:

$$
\begin{aligned}
X_y(t,T) &\sim N(m_y(t,T),\, n_y(t,T)^2) \\
X_z(t,T) &\sim N(m_z(t,T),\, n_z(t,T)^2),
\end{aligned} \tag{23}
$$

where:

$$\begin{cases} m_y(t,T) = & \mu_y(t)\frac{(e^{-b_y(T-t)}-1)}{b_y} - \frac{A_y e^{B_y t}}{B_y(b_y+B_y)}(e^{B_y(T-t)}-1) - \frac{A_y e^{B_y t}}{b_y(b_y+B_y)}(e^{-b_y(T-t)}-1) \\ n_y^2(t,T) = & \frac{\sigma_y^2}{b_y^2}[T-t-\frac{1-e^{-b_y(T-t)}}{b_y} - \frac{(1-e^{-b_y(T-t)})^2}{2b_y}], \end{cases} \tag{24}$$

and

$$\begin{cases} m_z(t,T) = & \mu_z(t)\frac{(e^{-b_z(T-t)}-1)}{b_z} - \frac{A_z e^{B_z t}}{B_z(b_z+B_z)}(e^{B_z(T-t)}-1) - \frac{A_z e^{B_z t}}{b_z(b_z+B_z)}(e^{-b_z(T-t)}-1) \\ n_z^2(t,T) = & \frac{\sigma_z^2}{b_z^2}[T-t-\frac{1-e^{-b_z(T-t)}}{b_z} - \frac{(1-e^{-b_z(T-t)})^2}{2b_z}], \end{cases} \tag{25}$$

then:

$$X_y(t,T) + X_z(t,T) \sim N(m_{yz}(t,T),\, n_{yz}^2(t,T)), \tag{26}$$

where

$$\begin{cases} m_{yz}(t,T) = & m_y(t,T) + m_z(t,T) \\ n_{yz}^2(t,T) = & n_y^2(t,T) + n_z^2(t,T) + 2Cov(X_y(t,T), X_z(t,T)). \end{cases} \tag{27}$$

The covariance $Cov(X_y(t,T), X_z(t,T))$ is:

$$Cov(X_y(t,T), X_z(t,T)) = E_{\mathbb{P}}[X_y(t,T) - E_{\mathbb{P}}(X_z(t,T)) \cdot (X_y(t,T) - E_{\mathbb{P}}(X_z(t,T)))]. \tag{28}$$

Using (7) and (8), we have:

$$\begin{aligned} Cov(X_y(t,T), X_z(t,T)) &= E_{\mathbb{P}}\Big[\Big(\int_t^T \sigma_y e^{-b_y t}\int_t^s e^{b_y v} dw_y(v)\Big)ds \cdot \Big(\int_t^T \sigma_z e^{-b_z t}\int_t^s e^{b_z v} dw_z(v)\Big)ds\Big] \\ &= E_{\mathbb{P}}\Big[\Big(\sigma_y \int_t^T \frac{1-e^{-b_y(T-v)}}{b_y} dw_y(v)\Big) \cdot \Big(\sigma_z \int_t^T \frac{1-e^{-b_z(T-v)}}{b_z} dw_z(v)\Big)\Big] \\ &= \sigma_y\sigma_z E_{\mathbb{P}}\Big[\Big(\int_t^T \frac{1-e^{-b_y(T-v)}}{b_y} dw_y(v)\Big) \cdot \Big(\int_t^T \frac{1-e^{-b_z(T-v)}}{b_z} dw_z(v)\Big)\Big]. \end{aligned}$$

Using the Ito multidimensional formula, we have:

$$\begin{aligned} Cov(X_y(t,T), X_z(t,T)) &= \sigma_y\sigma_z\rho_{w_y,w_z}\Big(\int_t^T \frac{1-e^{-b_y(T-v)}}{b_y} \cdot \frac{1-e^{-b_z(T-v)}}{b_z} dv\Big) \\ &= \sigma_y\sigma_z\rho_{w_y,w_z}\frac{1}{b_y b_z}\Big(\int_t^T (1-e^{-b_y(T-v)}) \cdot (1-e^{-b_z(T-v)}) dv\Big) \\ &= \sigma_y\sigma_z\rho_{w_y,w_z}\frac{1}{b_y b_z}\Big[(T-t) - \frac{1-e^{-b_z(T-t)}}{b_z} - \frac{1-e^{-b_y(T-t)}}{b_y} + \frac{1-e^{-(b_y+b_z)(T-t)}}{b_y+b_z}\Big] \\ &= \sigma_y\sigma_z\rho_{w_y,w_z}\Psi_{y,z}(t,T). \end{aligned} \tag{29}$$

Using (29), Equation (27) becomes:

$$\begin{cases} m_{yz}(t,T) = & m_y(t,T) + m_z(t,T) \\ n_{yz}^2(t,T) = & n_y^2(t,T) + n_z^2(t,T) + 2\sigma_y\sigma_z\rho_{w_y,w_z}\Psi_{y,z}(t,T). \end{cases} \tag{30}$$

Let:

$$\begin{aligned} K(t,T) &= e^{-\int_t^T(\mu_y(u,\omega)+\mu_z(u,\omega))du} \\ &= e^{X_y(t,T)+X_z(t,T)}. \end{aligned} \tag{31}$$

$E_{\mathbb{P}}(K(t,T))$ is then given by:

$$E_{\mathbb{P}}(K(t,T)) = e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2}}. \tag{32}$$

By replacing (32) in (17), we get:

$$Cov(I_y(y+t,T-t), I_z(z+t,T-t)) = e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2}} - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)} \cdot e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)}. \tag{33}$$

The correlation $Corr(I_y(y+t,T-t), I_z(z+t,T-t))$ is given by:

$$Corr(I_y(y+t,T-t), I_z(z+t,T-t)) = \frac{Cov(I_y(y+t,T-t), I_z(z+t,T-t))}{\bar{n}_y(t,T) \cdot \bar{n}_z(t,T)}, \tag{34}$$

where $\bar{n}_y(t,T)$ and $\bar{n}_z(t,T)$ are the standard deviations that are related to the two longevity indexes I_y and I_z.

Using (24) and (25), we can easily compute $\bar{n}_y(t,T)$ and $\bar{n}_z(t,T)$:

$$\bar{n}_y^2(t,T) = e^{2m_y(t,T)+n_y^2(t,T)} \cdot (e^{n_y^2(t,T)} - 1)$$
$$\bar{n}_z^2(t,T) = e^{2m_z(t,T)+n_z^2(t,T)} \cdot (e^{n_z^2(t,T)} - 1).$$

We finally get:

$$Corr(I_y(y+t,T-t), I_z(z+t,T-t)) = \frac{e^{m_{yz}(t,T)+\frac{n_{yz}^2(t,T)}{2}} - e^{\alpha_y(t,T)-\beta_y(t,T)\mu_y(t)} \cdot e^{\alpha_z(t,T)-\beta_z(t,T)\mu_z(t)}}{\sqrt{e^{2m_y(t,T)+n_y^2(t,T)}(e^{n_y^2(t,T)} - 1)}\sqrt{e^{2m_z(t,T)+n_z^2(t,T)}(e^{n_z^2(t,T)} - 1)}}. \tag{35}$$

Remark 2. *We remark that, if the two noises are independent, namely $\rho_y = 1$ and $\rho_z = 0$ or $\rho_y = 0$ and $\rho_z = 1$, then:*

- $Cov(X_y(t,T), X_z(t,T)) = 0$;
- *By elementary calculation, we find:*

$$m_{yz}(t,T) + \frac{n_{yz}^2(t,T)}{2} = \alpha_y(t,T) - \beta_y(t,T)\mu_y(t) + \alpha_z(t,T) - \beta_z(t,T)\mu_z(t),$$

which means that:

$$Corr(I_y(y+t,T-t), I_z(z+t,T-t)) = 0.$$

2.3. S-Forward Pricing

An S-forward (called individual S-forward in this paper) is a financial product exchanging at a fixed maturity T, the realized survival rate of a given population, in return for a fixed rate that was agreed at inception.

For a notional amount equal to one monetary unit, the payoff of this product at maturity becomes:

$$Payoff(T) = I(x,T) - {}_T\hat{p}_x, \tag{36}$$

where $I(x,T)$ is the realized survival rate at maturity, and ${}_T\hat{p}_x$ is a fixed survival rate that represents the probability to be alive be alive at age $x+T$ for an individual initially aged x.

This derivative allows to hedge the longevity risk that is related to an individual belonging to a given cohort. The pricing of this derivative under the COC approach as well as under the classical methods (risk-neutral, Sharpe, and Wang) was discussed in detail in Zeddouk and Devolder (2019).

2.4. GS-Forward Pricing

We consider the case of an insurer holding a portfolio of policyholders belonging to the cohort Y or Z. We assume that this insurer should pay one monetary unit to each individual alive at maturity T.

The payoff of this global S-forward (that we call GS-forward) at time t is given by:

$$Payoff(T) = I_y(y,T) + I_z(z,T) - ({}_T\hat{p}_y + {}_T\hat{p}_z).$$

where ${}_T\hat{p}_y$ and ${}_T\hat{p}_z$ are two fixed survival rates related to cohorts Y and Z, of an individual aged $y+T$ and $z+T$, respectively, at time 0, to be alive at age y and z, respectively, (F_0 measurable).

2.4.1. GS-Forward Pricing under the COC Approach

The price at time 0 under the COC approach is given by:

$$V_{COC}(0,T) = BE_0^{\mathbb{P}} + RM_0, \tag{37}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0,T)E_{\mathbb{P}}[I_y(y,T) + I_z(z,T) - ({}_T\hat{p}_y + {}_T\hat{p}_z)].$$

The risk margin is equal to the present value of the required returns on the future Solvency Capital Requirements (SCRs):

$$RM_0 = C\% \sum_{i=0}^{T-1} SCR_i\, P(0,i+1),$$

where SCR_i is the Solvency Capital Requirement that corresponds to the Value-at-Risk at a confidence level of 99.5% on a one-year period, and C is the Cost of Capital nowadays equal to 6% (Solvency II requirements).

The future SCRs being stochastic, we need to use an estimation of their future values at time 0. This estimation is denoted by: $S\hat{C}R_i = SCR_i \mid_0$.

The initial risk margin is then given by:

$$RM_0 = C\% \sum_{i=0}^{T-1} S\hat{C}R_i\, P(0,i+1).$$

Taking into account the Solvency II definition of the SCRs (Value-at-Risk at 99.5% on a one year), the expression of the SCR_i becomes:

$$SCR_i = \mathrm{VaR}_{99,5\%}[BE_{i+1}^{\mathbb{P}} P(i,i+1) - BE_i^{\mathbb{P}}],$$

where:

$$\begin{aligned} BE_{i+1}^{\mathbb{P}} = {} & (I_y(y,i+1)E_{\mathbb{P}}(\, I_y(y+i+1,T-i-1) + I_z(z,i+1)E_{\mathbb{P}}(\, I_z(z+i+1,T-i-1)) \\ & - ({}_T\hat{p}_y + {}_T\hat{p}_z))P(i+1,T) \end{aligned}$$

$$BE_i^{\mathbb{P}} = (I_y(y,i)E_{\mathbb{P}}(\, I_y(y+i,T-i) + I_z(z,i)E_{\mathbb{P}}(\, I_z(z+i,T-i) - ({}_T\hat{p}_y + {}_T\hat{p}_z))P(i,T).$$

The SCR_i is then given by:

$$
\begin{aligned}
SCR_i &= P(i,T)\,\mathrm{VaR}_{99,5\%}[I_y(y,i+1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) + I_z(z,i+1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1))\\
&\quad - I_y(y,i)E_{\mathbb{P}}(\,I_y(y+i,T-i)) - I_z(z,i)E_{\mathbb{P}}(\,I_z(z+i,T-i))]\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)I_y(y+i,1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) + I_z(z,i)I(z+i,1)\\
&\quad \times E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)) - I_y(y,i)E_{\mathbb{P}}(\,I_y(y+i,T-i)) - I_z(z,i)E_{\mathbb{P}}(\,I_z(z+i,T-i))]\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)I_y(y+i,1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - I_y(y,i)E_{\mathbb{P}}(\,I_y(y+i,T-i))\\
&\quad + I_z(z,i)I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)) - I_z(z,i)E_{\mathbb{P}}(\,I_z(z+i,T-i))]\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}[I_y(y,i)\big(I_y(y+i,1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_y(y+i,T-i))\\
&\quad + I_z(z,i)(I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_z(z+i,T-i))].
\end{aligned}
$$

Therefore, the estimation of $SCR_i\mid_0$ can be given by:

$$
\begin{aligned}
SCR_i\mid_0 &= P(i,T)[\mathrm{VaR}_{99,5\%}[E_{\mathbb{P}}(I_y(y,i))\big((I_y(y+i,1)E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_y(y+i,T-i)))\\
&\quad + E_{\mathbb{P}}(I_z(z,i))((I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_z(z+i,T-i)))]\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}[E_{\mathbb{P}}(I_y(y,i))\big((I_y(y+i,1)\\
&\quad \times E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)) - E_{\mathbb{P}}(\,I_y(y+i,1))E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1)))\\
&\quad + E_{\mathbb{P}}(I_z(z,i))((I_z(z+i,1)E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1))\\
&\quad - E_{\mathbb{P}}(\,I_z(z+i,1))E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)))]\\
&= P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y+i,1) - E_{\mathbb{P}}(\,I_y(y+i,1)))\cdot\theta + \xi\cdot(I_z(z+i,1) - E_{\mathbb{P}}(\,I_z(z+i,1)))\cdot\eta)],
\end{aligned}
$$

where ν, θ, ξ, and η are constants given by:

$$
\begin{aligned}
\nu &= E_{\mathbb{P}}(I_y(y,i))\\
\theta &= E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1))\\
\xi &= E_{\mathbb{P}}(I_z(z,i))\\
\eta &= E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1)).
\end{aligned}
$$

The risk margin at time 0 is then equal to:

$$
RM_0 = 6\%\sum_{i=0}^{T-1} P(0,i+1)P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y+i,1) - E_{\mathbb{P}}(\,I_y(y+i,1)))\cdot\theta + \xi\cdot(I_z(z+i,1) - E_{\mathbb{P}}(\,I_z(z+i,1)))\cdot\eta)].
$$

The price of the GS-forward under the COC approach is finally given by:

$$
\begin{aligned}
V^{\,y,z}_{\,COC}(0,T) &= P(0,T)(E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z))\\
&\quad + 6\%\sum_{i=0}^{T-1} P(0,i+1)P(i,T)[\mathrm{VaR}_{99,5\%}(\nu\cdot(I_y(y+i,1) - E_{\mathbb{P}}(\,I_y(y+i,1)))\cdot\theta + \xi\cdot(I_z(z+i,1)\\
&\quad - E_{\mathbb{P}}(\,I_z(z+i,1)))\cdot\eta)].
\end{aligned} \tag{38}
$$

2.4.2. GS-Forward Pricing under the Risk-Neutral Approach

The price of a GS-forward at time t under the risk-neutral probability measure $\mathbb{Q}$, is given by:

$$
V^{y,z}_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_y(y,T) + I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z)\mid\mathcal{F}_0], \tag{39}
$$

where λ and λ' are the two market prices of longevity risk that are linked to the two independent risk factors, modelled by the two Brownian motions w_1 and w_2 (Equations (5) and (6)).

We assume that the market price of risk is constant ($\lambda(t) = \lambda$ and $\lambda'(t) = \lambda'$).

The two SDEs of the forces of mortality under the real-word measure $\mathbb{P}$ are given by Equations (3) and (4). Under the risk-neutral measure, these formulas become:

$$d\mu_y^{\mathbb{Q}_{\lambda\lambda'}}(t) = (A_y e^{B_y t} - b_y \mu_y(t) + \tau\sigma_y)dt + \sigma_y(\rho_y dw_1^*(t) + \sqrt{1-\rho_y^2}dw_2^*(t)) \tag{40}$$

$$d\mu_z^{\mathbb{Q}_{\lambda\lambda'}}(t) = (A_z e^{B_z t} - b_z \mu_z(t) + \tau'\sigma_z)dt + \sigma_z(\rho_z dw_1^*(t) + \sqrt{1-\rho_z^2}dw_2^*(t)), \tag{41}$$

where:

- $w_1^*(t) = w_1(t) - \lambda t$, and $w_2^*(t) = w_2(t) - \lambda' t$;
- $\tau = \lambda\rho_y + \lambda'\sqrt{1-\rho_y^2}$ and $\tau' = \lambda\rho_z + \lambda'\sqrt{1-\rho_z^2}$.

We have:

$$E_{\mathbb{Q}_{\lambda\lambda'}}(I(y,T)) = e^{\alpha_y^{\mathbb{Q}_{\lambda\lambda'}}(0,T) - \beta_y^{\mathbb{Q}_{\lambda\lambda'}}(0,T)\mu_y^{\mathbb{Q}_{\lambda\lambda'}}(0)}$$
$$E_{\mathbb{Q}_{\lambda\lambda'}}(I(z,T)) = e^{\alpha_z^{\mathbb{Q}_{\lambda\lambda'}}(0,T) - \beta_z^{\mathbb{Q}_{\lambda\lambda'}}(0,T)\mu_z^{\mathbb{Q}_{\lambda\lambda'}}(0)},$$

where $\alpha_y^{\mathbb{Q}_{\lambda\lambda'}}, \beta_y^{\mathbb{Q}_{\lambda\lambda'}}$ and $\alpha_z^{\mathbb{Q}_{\lambda\lambda'}}, \beta_z^{\mathbb{Q}_{\lambda\lambda'}}$ are given by:

$$\begin{cases} \alpha_y^{\mathbb{Q}_{\lambda\lambda'}}(t,T) & = \frac{A_y}{b_y}[e^{-b_y T}\frac{e^{(B_y+b_y)T} - e^{(B_y+b_y)t}}{B_y+b_y} - \frac{e^{B_y T} - e^{B_y t}}{B_y}] - \frac{\sigma_y^2}{2b_y^2}[\frac{1}{b_y}(1-e^{-b_y(T-t)}) - T + t] \\ & -\frac{\sigma_y^2}{4b_y^3}(1-e^{-b_y(T-t)})^2 - \frac{\sigma_y\tau}{b_y}(1-e^{-b_y(T-t)}) \\ \beta_y^{\mathbb{Q}_{\lambda\lambda'}}(t,T) & = \frac{1}{b_y}(1-e^{-b_y(T-t)}) \end{cases}$$

and

$$\begin{cases} \alpha_z^{\mathbb{Q}_{\lambda\lambda'}}(t,T) & = \frac{A_z}{b_z}[e^{-b_z T}\frac{e^{(B_z+b_z)T} - e^{(B_z+b_z)t}}{B_z+b_z} - \frac{e^{B_z T} - e^{B_z t}}{B_z}] - \frac{\sigma_z^2}{2b_z^2}[\frac{1}{b_z}(1-e^{-b_z(T-t)}) - T + t] \\ & -\frac{\sigma_z^2}{4b_z^3}(1-e^{-b_z(T-t)})^2 - \frac{\sigma_z\tau'}{b_z}(1-e^{-b_z(T-t)}) \\ \beta_z^{\mathbb{Q}_{\lambda\lambda'}}(t,T) & = \frac{1}{b_z}(1-e^{-b_z(T-t)}). \end{cases}$$

2.4.3. GS-Forward Pricing under the Sharpe Approach

Let us now compute the GS-forward price under the Sharpe approach. The price at time 0 is:

$$V_{Sharpe}^{y,z}(0,T) = P(0,T)\left[E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z) + S\sqrt{\mathrm{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T))}\right], \tag{42}$$

where S represents the Sharpe ratio, and $\mathrm{Var}_{\mathbb{P}}(I_y(y,T))$ is the variance of the survival index.

Because $I_y(y,T)$ and $I_z(z,T)$ are dependent, we have:

$$\mathrm{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T)) = \mathrm{Var}_{\mathbb{P}}(I_y(y,T)) + \mathrm{Var}_{\mathbb{P}}(I_z(z,T)) + 2Cov(I_y(y,T), I_z(z,T)). \tag{43}$$

The price under the Sharpe approach is finally given by:

$$\begin{aligned} V_{Sharpe}^{y,z}(0,T) = P(0,T)\Big[&E_{\mathbb{P}}(I_y(y,T)) + E_{\mathbb{P}}(I_z(z,T)) - ({}_T\hat{p}_y + {}_T\hat{p}_z) \\ &+ S\sqrt{\mathrm{Var}_{\mathbb{P}}(I_y(y,T)) + \mathrm{Var}_{\mathbb{P}}(I_z(z,T)) + 2Cov(I_y(y,T), I_z(z,T))}\Big]. \end{aligned} \tag{44}$$

From (23), $\text{Var}_{\mathbb{P}}(I_y(y,T))$ and $\text{Var}_{\mathbb{P}}(I_z(z,T))$ are given by:

$$\text{Var}_{\mathbb{P}}(I_y(y,T)) = e^{(2m_y(0,T)+n_y^2(0,T))} \cdot (e^{n_y^2(0,T)} - 1)$$
$$\text{Var}_{\mathbb{P}}(I_z(z,T)) = e^{(2m_z(0,T)+n_z^2(0,T))} \cdot (e^{n_z^2(0,T)} - 1).$$

We have the expression of the covariance $Cov(I_y(y,T), I_z(z,T))$ given by formula (33). The expression of the variance $\text{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T))$ is given by:

$$\begin{aligned}\text{Var}_{\mathbb{P}}(I_y(y,T) + I_z(z,T)) &= e^{(2m_y(0,T)+n_y^2(0,T))} \cdot (e^{n_y^2(0,T)} - 1) + e^{(2m_z(0,T)+n_z^2(0,T))}(e^{n_z^2(0,T)} - 1) \\ &+ 2e^{m_{yz}(0,T)+\frac{n^2(0,T)}{2})} - e^{\alpha_y(0,T)-\beta_y(0,T)\mu_x^y(t)} \cdot e^{\alpha_z(0,T)-\beta_z(0,T)\mu_x^z(0)}.\end{aligned}$$

2.5. Consistency between S-Forward and GS-Forward Pricing Methods

Let us now derive the price of the individual S-forward contract from the GS-forward's price formulas given in Section 2.4.

Under the real-world measure $\mathbb{P}$, the mortality intensity Equations (3) and (4) can be written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y d\hat{w}_y(t) \tag{45}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z d\hat{w}_z(t), \tag{46}$$

where $\hat{w}_y(t) = \rho_y w_1(t) + \sqrt{1-\rho_y^2}\, w_2(t)$ and $\hat{w}_z(t) = \rho_z w_1(t) + \sqrt{1-\rho_z^2}\, w_2(t)$.

For each of the three pricing methods, we can determine the price of the individual S-forward contract directly from the GS-forward price formula, by considering just one cohort.

- Cost of Capital approach:

The prices of the individual S-forwards under the Cost of Capital approach for cohorts Y and Z are given by:

$$\begin{aligned}V^y_{COC}(0,T) &= P(0,T)\,(E_{\mathbb{P}}[\,I_y(y,T) - {}_T\hat{p}_y]) + 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I_y(y,i))[\text{VaR}_{99,5\%}(I_y(y+i,1)) \\ &- E_{\mathbb{P}}(\,I_y(y+i,1))]E_{\mathbb{P}}(\,I_y(y+i+1,T-i-1))P(0,i+1)P(i,T)\end{aligned} \tag{47}$$

$$\begin{aligned}V^z_{COC}(0,T) &= P(0,T)\,(E_{\mathbb{P}}[\,I_z(z,T) - {}_T\hat{p}_z]) + 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I_z(z,i))[\text{VaR}_{99,5\%}(I_z(z+i,1)) \\ &- E_{\mathbb{P}}(\,I_z(z+i,1))]E_{\mathbb{P}}(\,I_z(z+i+1,T-i-1))P(0,i+1)P(i,T)\end{aligned} \tag{48}$$

- Sharpe approach:

The prices of the individual S-forwards under the Sharpe approach for cohorts Y and Z are given by:

$$V^y_{Sharpe}(0,T) = P(0,T)\left[E_{\mathbb{P}}(I_y(y,T)) - {}_T\hat{p}_y + S\sqrt{\text{Var}_{\mathbb{P}}(I_y(y,T))}\right]$$
$$V^z_{Sharpe}(0,T) = P(0,T)\left[E_{\mathbb{P}}(I_z(z,T)) - {}_T\hat{p}_z + S\sqrt{\text{Var}_{\mathbb{P}}(I_z(z,T))}\right]$$

- Risk-neutral

The prices of the individual S-forwards under the risk-neutral approach for cohorts Y and Z are given by:

$$V^y_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_y(y,T)) - {}_T\hat{p}_y) \mid \mathcal{F}_0]$$
$$V^z_{\mathbb{Q}_{\lambda\lambda'}}(0,T) = P(0,T)[E_{\mathbb{Q}_{\lambda\lambda'}}(I_z(z,T)) - {}_T\hat{p}_z) \mid \mathcal{F}_0].$$

Under the risk-neutral approach, Equations (45) and (46) can be written as:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t) + \tau\sigma_y)dt + \sigma_y d\overline{w}_y(t) \tag{49}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t) + \tau'\sigma_z)dt + \sigma_z d\overline{w}_z(t), \tag{50}$$

where $\overline{w}_y(t) = \rho_y w_1^*(t) + \sqrt{1-\rho_y^2}\, w_2^*(t)$, and $\overline{w}_z(t) = \rho_z w_1^*(t) + \sqrt{1-\rho_z^2}\, w_2^*(t)$.
We can see that under the risk-neutral measure $\mathbb{Q}_{\lambda\lambda'}$, the drifts have been changed by adding $\tau\sigma_y$ and $\tau'\sigma_z$ where $\tau = \lambda\rho_y + \lambda'\sqrt{1-\rho_y^2}$ and $\tau' = \lambda\rho_z + \lambda'\sqrt{1-\rho_z^2}$.

If we consider the special case where $\rho_y = 1$, and only one risk parameter $\rho_z = \rho$, τ and τ' become:

$$\tau = \lambda$$
$$\tau' = \lambda\rho + \lambda'\sqrt{1-\rho^2}.$$

We define the market price of risk λ^* for the two cohorts, as follows:

$$\lambda = \lambda^*$$
$$\tau' = \lambda\rho + \lambda'\sqrt{1-\rho^2} = \lambda^*.$$

We find that:

$$\lambda = \lambda^*$$
$$\lambda' = \lambda^*\sqrt{\frac{1-\rho}{1+\rho}}.$$

For illustration, if we consider $\rho = 0.95$ and $\lambda = -20\%$, we find that $\lambda' = -3.2\%$. We can see that the values of λ and λ' are very different, but the market prices of risk τ and τ' on which the final price depends are equal.

We conclude that COC and Sharpe methods are coherent between global and individual models; however, in the risk-neutral approach, we can also obtain coherency between the models, but we have to impose conditions on the values of λ and λ'.

3. Cohort-Based Longevity Model: *n* Cohorts

Let us now generalize our study to n cohorts. We consider n different cohorts of individuals initially aged x_1, $x_2 .... x_n$ at time $t = 0$, and we use the HW model for describing the evolution of mortality of each cohort.

The mortality intensities that are related to these cohorts are given by:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{B_1 t} - b_1\mu_{x_1}(t) \\ A_2 e^{B_2 t} - b_2\mu_{x_2}(t) \\ . \\ . \\ . \\ A_n e^{B_n t} - b_n\mu_{x_n}(t) \end{pmatrix} \text{dt+} \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} & . & . & \rho_{1,n} \\ \rho_{2,1} & 1 & \rho_{2,3} & . & . & \rho_{2,n} \\ . & \rho_{3,2} & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & . & . \\ \rho_{n,1} & \rho_{n,2} & \rho_{n,3} & . & . & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix}. \tag{51}$$

where w_1, $w_2 ... w_n$ are independent.

$$\text{We denote } M = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} & . & . & \rho_{1,n} \\ \rho_{2,1} & 1 & \rho_{2,3} & . & . & \rho_{2,n} \\ . & \rho_{3,2} & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & . & . \\ \rho_{n,1} & \rho_{n,2} & \rho_{n,3} & . & . & 1 \end{pmatrix}$$

M is a symmetric $n \times n$ Matrix that represents the impact of the risk factors between two cohorts $\rho_{i,j}$.

Equation (51) is the general case, since it includes n independent risk factors that affect each cohort. We consider three particular cases in order to limit the number of risk factors, and also to look at extreme situations. We study three possibilities that express how the cohorts are correlated:

1. One common risk factor: the cohorts are equally dependent, which means that the forces of mortality of these cohorts are given by:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{B_1 t} - b_1 \mu_{x_1}(t) \\ A_2 e^{B_2 t} - b_2 \mu_{x_2}(t) \\ . \\ . \\ . \\ A_n e^{B_n t} - b_n \mu_{x_n}(t) \end{pmatrix} \text{dt+} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_1(t) \\ . \\ . \\ . \\ \sigma_n dw_1(t) \end{pmatrix}.$$

In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = \varphi_{x_k,x_l}(t),$$

where $\varphi_{x_k,x_l}(t)$ is given by:

$$\varphi_{x_k,x_l}(t) = \Big(\frac{2\sqrt{b_{x_k} b_{x_l}}}{b_{x_k} + b_{x_l}} \frac{1 - e^{-(b_{x_k}+b_{x_l})t}}{\sqrt{1 - e^{-2b_{x_k}t}}\sqrt{1 - e^{-2b_{x_l}t}}}\Big).$$

2. Two common risk factors: the cohorts are dependent with different degrees:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{B_1 t} - b_1 \mu_{x_1}(t) \\ A_2 e^{B_2 t} - b_2 \mu_{x_2}(t) \\ . \\ . \\ . \\ A_n e^{B_n t} - b_n \mu_{x_n}(t) \end{pmatrix} dt + \begin{pmatrix} \rho_1 & \sqrt{1-\rho_1^2} & 0 & . & . & 0 \\ \rho_2 & \sqrt{1-\rho_2^2} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \rho_n & \sqrt{1-\rho_n^2} & 0 & . & . & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix},$$

where w_1, w_2 ... w_n are n independent Brownian motions. In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = \Big(\rho_k \rho_l + \sqrt{1-\rho_k^2}\sqrt{1-\rho_l^2}\Big)\varphi_{x_k,x_l}(t),$$

3. One risk factor by cohort: the cohorts are completely independent:

$$\begin{pmatrix} d\mu_{x_1}(t) \\ d\mu_{x_2}(t) \\ . \\ . \\ . \\ d\mu_{x_n}(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{B_1 t} - b_1 \mu_{x_1}(t) \\ A_2 e^{B_2 t} - b_2 \mu_{x_2}(t) \\ . \\ . \\ . \\ A_n e^{B_n t} - b_n \mu_{x_n}(t) \end{pmatrix} \text{dt+} \begin{pmatrix} \sigma_1 dw_1(t) \\ \sigma_2 dw_2(t) \\ . \\ . \\ . \\ \sigma_n dw_n(t) \end{pmatrix},$$

where w_1, w_2 ... w_n are n-independent Brownian motions. In this case, the correlation is given by:

$$Corr(\mu_{x_k}(t), \mu_{x_l}(t)) = 0.$$

3.1. GS-Forward Pricing

We consider the case of an insurer holding a portfolio of policyholders belonging to the cohorts X_1, X_2 , X_3 .... X_n. We assume that this insurer should pay one monetary unit to each individual alive at maturity T.

The payoff of this GS-forward at time T that is evaluated at time 0 is given by:

$$Payoff(T) = \sum_{k=1}^{n} \left(I_{x_k}(x_k, T) - {}_T\hat{p}_{x_k} \right),$$

where ${}_T\hat{p}_{x_k}$ is the fixed survival rate related to cohort X_k, and $I_{x_k}(x_k, T)$ the survival index of an individual aged x_k at time 0, to be alive at age $x_k + T$.

3.1.1. GS-Forward Pricing under the COC Approach

The price at time 0 under the COC approach is given by:

$$V^{G}_{COC}(0, T) = BE_0^{\mathbb{P}} + RM_0 \tag{52}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0, T)E_{\mathbb{P}}\big[\sum_{j=1}^{n} (I_{x_j}(x_j, T) - {}_T\hat{p}_{x_j}) \big].$$

The risk margin at time 0 is:

$$RM_0 = C\% \sum_{i=0}^{T-1} S\hat{C}R_i \, P(0, i + 1).$$

Subsequently, the expression of the SCR_i is given by:

$$SCR_i = \mathrm{VaR}_{99,5\%} \left[BE_{i+1}^{\mathbb{P}} P(i, i + 1) - BE_i^{\mathbb{P}} \right],$$

where:

$$BE_{i+1}^{\mathbb{P}} = (\sum_{k=1}^{n} I_{x_k}(x_k, i + 1)E_{\mathbb{P}}(I_{x_k}(x_k + i + 1, T - i - 1) - {}_T\hat{p}_{x_k})P(i + 1, T)$$
$$BE_i^{\mathbb{P}} = (\sum_{k=1}^{n} I_{x_k}(x_k, i)E_{\mathbb{P}}(I_{x_k}(x_k + i, T - i) - {}_T\hat{p}_{x_k})P(i, T).$$

The estimation of $SCR_i \mid_0$ can be given by:

$$\begin{aligned} SCR_i \mid_0 \quad &= \quad P(i, T)\Big[\mathrm{VaR}_{99,5\%} \big[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k, i))((I_{x_k}(x_k + i, 1) \\ &\times \quad E_{\mathbb{P}}(I_{x_k}(x_k + i + 1, T - i - 1)) - E_{\mathbb{P}}(I_{x_k}(x_k + i, T - i))\big]\Big]. \end{aligned}$$

RM_0 is then equal to:

$$\begin{aligned} RM_0 &= 6\% \sum_{i=0}^{T-1} P(0,i+1)P(i,T)\Big[\,\mathrm{VaR}_{99,5\%}\,[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k,i))((I_{x_k}(x_k+i,1) \\ &\times\ E_{\mathbb{P}}(I_{x_k}(x_k+i+1,T-i-1)) - E_{\mathbb{P}}(I_{x_k}(x_k+i,T-i))]\Big]. \end{aligned}$$

Finally, the price of the GS-forward with the COC approach is:

$$\begin{aligned} V^{G}_{COC}(0,T) &= P(0,T)E_{\mathbb{P}}[\sum_{k=1}^{n}(I_{x_k}(x_k,T) - {}_T\hat{p}_{x_k})] + 6\% \sum_{i=0}^{T-1} P(0,i+1)P(i,T) \\ &\times\ \Big[\,\mathrm{VaR}_{99,5\%}\,[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k,i))((I_{x_k}(x_k+i,1) \\ &\times\ E_{\mathbb{P}}(I_{x_k}(x_k+i+1,T-i-1)) - E_{\mathbb{P}}(I_{x_k}(x_k+i,T-i))]\Big]. \end{aligned} \tag{53}$$

3.1.2. GS-Forward Pricing under the Risk-Neutral Approach

The price of the GS-forward at time 0 is given by:

$$V^{G}_{\mathbb{Q}_{\lambda_k}}(0,T) = P(0,T)\Big[\sum_{k=1}^{n} E_{\mathbb{Q}_{\lambda_k}}(I_{x_k}(x_k,T) - {}_T\hat{p}_{x_k} \mid \mathcal{F}_0\Big], \tag{54}$$

where λ_k is the market price of risk associated to the cohort x_k.

3.1.3. GS-Forward Pricing under the Sharpe Approach

The price of the GS-forward at time 0 under the Sharpe approach is given by:

$$V^{G}_{Sharpe}(0,T) = P(0,T)\left[\sum_{k=1}^{n}(E_{\mathbb{P}}(I_{x_k}(x_k,T)) - {}_T\hat{p}_{x_k} + S\sqrt{\mathrm{Var}_{\mathbb{P}}(\sum_{k=1}^{n} I_{x_k}(x_k,T))}\right], \tag{55}$$

where S is the Sharpe ratio, and ${}_T\hat{p}_{x_k}$ is the fixed rate that is related to the cohort X_k.

Because $I_{x_k}(x_k,T)$ are dependent, we have:

$$\mathrm{Var}_{\mathbb{P}}\,\Big(\sum_{k=1}^{n} I_{x_k}(x_k,T)\Big) = \sum_{k=1}^{n}\mathrm{Var}_{\mathbb{P}}\,\big(I_{x_k}(x_k,T)\big) + 2\sum_{k=1}^{n}\sum_{\substack{l\neq k\\ l=1}}^{n} Cov\big(I_{x_k}(x_k,T), I_{x_l}(x_l,T)\big). \tag{56}$$

The price at time 0 under the Sharpe approach is finally given by:

$$V^{G}_{Sharpe}(0,T) = P(0,T)\left[\sum_{k=1}^{n} E_{\mathbb{P}}(I_{x_k}(x_k,T)) - {}_T\hat{p}_{x_k} + S\sqrt{\sum_{k=1}^{n}\mathrm{Var}_{\mathbb{P}}(I_{x_k}(x_k,T)) + 2\sum_{k=1}^{n}\sum_{\substack{l\neq k\\ l=1}}^{n} Cov(I_{x_k}(x_k,T), I_{x_l}(x_l,T))}\right], \tag{57}$$

where $Cov(I_{x_k}(x_k,T), I_{x_l}(x_l,T))$ is given by (33).

3.2. Application to a Portfolio of Annuities

Life and pension annuities are typical products that allow retirees to obtain lifelong incomes. For fixed lifetime annuities, the longevity risk is completely supported by the insurer or the pension fund. Therefore, it is crucial for the annuity provider to hedge this risk, which is not easy given the limited solutions available on the market.

Our pricing framework has the advantage to be flexible, as it can also be generalized to hedge a portfolio of annuities: we consider a pension fund holding a portfolio of policyholders that belong to the cohorts $X_1, X_2, \ldots X_n$, to whom he provides fixed annuities of one monetary unit at each date $t_1, t_2, \ldots, t_m$ if the policyholder is alive.

For each cohort X_k, the annuity provider can mitigate the corresponding longevity risk by entering a survival swap (S-swap).

This S-swap is considered to be an aggregation of S-forward contracts (see for instance Zeddouk and Devolder (2019)).

We denote, by $P^{x_k}(0, t_1, t_2, ..., t_m)$, the price at time 0 of an S-swap for the individuals belonging to the cohort X_k, with $t_1, t_2, ..., t_m$ as exchange dates $(0 < t_1 < t_2 < ... < t_m)$. Without arbitrage opportunities, the price of this S-swap is equal to the sum of the prices of the corresponding individual S-forward contracts (see, for instance, (Zeddouk and Devolder 2019)):

$$P^{x_k}(0, t_1, t_2, ..., t_m) = \sum_{i=1}^{m} V^{x_k}(0, t_i), \tag{58}$$

where $V^{x_k}(0, t_i)$ is the price at time 0 of the individual S-forward contract related to a given cohort X_k for a maturity t_i.

Therefore, if we do not take into account the eventual correlation between cohorts, the price of the S-swap for individuals that belong to the different cohorts $X_1, X_2, ..X_n$ is given by:

$$\bar{P}^{x_k}(0, t_1, t_2, ..., t_m) = \sum_{k=1}^{n} \sum_{i=1}^{m} V^{x_k}(0, t_i) \tag{59}$$

Alternatively, if such product exists, the annuity provider can cover the longevity risk for all individuals that belong to the different cohorts $X_1, X_2, ..X_n$, by entering one derivative instead of a derivative by cohort, while taking the eventual correlation between cohorts into account. The price at time 0 of this Global S-swap (GS-swap) is given by:

$$P^{G}(0, t_1, t_2, ..., t_m) = \sum_{i=1}^{m} V^{G}(0, t_i) \tag{60}$$

where $V^G(0, t_i)$ is the price of the GS-forward contract.

4. Numerical Simulation

In this section, we compute the price of different GS-forward contracts based on the Belgian population data, and using the different pricing methods.

4.1. Assumptions

To price the GS-forwards, we consider these assumptions:

- An insurer with a portfolio of pure endowment contracts paying 1€ at maturity T;
- Two maturities: $T = 5$ and $T = 10$;
- N_0=10 000 initial policyholders for each cohort;
- We consider two examples: individuals initially aged $y = 55$ and $z = 60$, or individuals initially aged $y = 60$ and $z = 65$, old in 2015;
- According to the literature, the values of ρ are around 0.98 (Jevtic et al. 2013). However, to have a better idea on the price evolution, we consider the following values for ρ: 0.95, 0.98, 1 (in view of illustration, we also consider the extreme value $\rho = 0$);
- Data from the Belgian IA|BE unisex mortality table[1];
- The risk-less interest rate denoted r, is considered constant, $r = 1\%$;

---

[1] IA|BE mortality projection for the Belgian population (2015) available at www.iabe.be.

- The optimal parameters for the HW model are given by Table 1:

Table 1. Optimal parameters value for the different cohorts, HW model.

Age in 2015	$\mu_x(0)$	A	B	b	σ
55	0.00466531	0.00042258	0.11428187	0.11669113	0.00200113
60	0.00722197	0.00089226	0.11571836	0.15355787	0.00166015
65	0.01056770	0.00231775	0.11562220	0.25062948	0.01770006

- The fixed rates are reported in Table 2:

Table 2. The fixed survival rates ${}_T\hat{p}_x$ for different ages and maturities.

T/Age	55	60	65
$T = 5$	0.9737899	0.9605744	0.9419321
$T = 10$	0.9395278	0.9107331	0.8658090

- For simplification purpose, we chose $\rho_y = 1$, and we consider only one risk parameter $\rho_z = \rho$

  The two mortality intensities for cohorts Y and Z given by (3) and (4) now become:

$$d\mu_y(t) = (A_y e^{B_y t} - b_y \mu_y(t))dt + \sigma_y dw_1(t) \tag{61}$$

$$d\mu_z(t) = (A_z e^{B_z t} - b_z \mu_z(t))dt + \sigma_z(\rho dw_1(t) + \sqrt{1-\rho^2} dw_2(t)) \tag{62}$$

ρ_{w_y,w_z} and the correlation given by Equation (13) become:

$$\rho_{w_y,w_z} = \rho \tag{63}$$

$$Corr(\mu_y(t), \mu_z(t)) = \rho \varphi_{y,z}(t)$$

4.2. Correlation between Cohorts

Before computing the prices of the GS-forward contracts, let us determine the correlation between the cohorts. To do so, we first compute $\varphi_{y,z}(t)$ for the two portfolios considered:

- x_0= 55 and x_0= 60 in 2015 (denoted $\varphi_{55,60}(t)$); and,
- x_0= 60 and x_0= 65 in 2015 (denoted $\varphi_{60,65}(t)$).

We use Belgian data from IA|BE and compute $\varphi_{55,60}(t)$ and $\varphi_{60,65}(t)$ for $t = 1$ to $t = 10$. Table 3 presents the results:

Table 3. Some values of $\varphi_{y,z}(t)$.

	$\varphi_{55,60}(t)$	$\varphi_{60,65}(t)$
$t = 1$	0.9999436	0.9995160
$t = 2$	0.9997768	0.9981148
$t = 3$	0.9995068	0.9959374
$t = 4$	0.9991450	0.9931869
$t = 5$	0.9987058	0.9900917
$t = 6$	0.9982057	0.9868707
$t = 7$	0.9976623	0.9837081
$t = 8$	0.9970927	0.9807412
$t = 9$	0.9965130	0.9780582
$t = 10$	0.9959375	0.9757037

Now that we have the values of $\varphi_{y,z}(t)$, we can compute the correlation between the forces of mortality, and between the longevity indexes.

For ease of notation, we write: $Corr(\mu_y(t), \mu_z(t)) = Corr(\mu_{y,z}(t))$

The correlation between the forces of mortality are reported in Table 4:

Table 4. Values of $Corr(\mu_{y,z}(T))$.

	$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$	$Corr(\mu_{55,60}(T))$	$Corr(\mu_{60,65}(T))$
$T = 5$	0.9487705	0.9405871	0.9787317	0.9702899	0.9987058	0.9900917
$T = 10$	0.9461406	0.9269185	0.9760188	0.9561896	0.9959375	0.9757037

The correlation between longevity indexes is given by (35). For ease of notation, we write: $Corr(I_y(y, T), I_z(z, T)) = Corr(I_{y,z}(T))$. Table 5 reports the results:

Table 5. Values of $Corr(I_{y,z}(T))$.

	$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$	$Corr(I_{55,60}(T))$	$Corr(I_{60,65}(T))$
$T = 5$	0.9498743	0.9490736	0.9798716	0.9790453	0.9998698	0.9990265
$T = 10$	0.9495996	0.9475659	0.9795936	0.9774933	0.9995898	0.9974450

4.3. Pricing GS-Forwards

Let us now compute the prices of the different GS-forward contracts under the different methods.

4.3.1. Pricing GS-Forwards under the Risk-Neutral Approach

Because the risk-neutral method is based on the expectation of the survival index, the price of the GS-forward under this approach does not depend on ρ. This price is equal to the sum of the prices of the individual S-forwards that are related to the corresponding cohorts:

$$\begin{aligned} V^{y}(0,T)+V^{z}(0,T) &= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T)) - {}_T\hat{p}_y) \mid \mathcal{F}_0] + P(0,T)[E_{\mathbb{Q}_{\lambda'}}(I_z(z,T)) - {}_T\hat{p}_z) \mid \mathcal{F}_0] \\ &= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T)) + E_{\mathbb{Q}_{\lambda'}}(I_z(z,T)) - {}_T\hat{p}_y - {}_T\,\hat{p}_z) \mid \mathcal{F}_0] \\ &= P(0,T)[E_{\mathbb{Q}_\lambda}(I_y(y,T) + I_z(z,T)) - ({}_T\hat{p}_y + {}_T\,\hat{p}_z) \mid \mathcal{F}_0] \\ &= V^{y,z}(0,T) \end{aligned} \quad (64)$$

Thus, we will only focus on the COC and Sharpe approaches.

4.3.2. Pricing GS-Forwards under the COC Approach

Let us now compute the price of the GS-forward while using the HW model under the COC approach.

Table 6 reports the prices of the different GS-forwards $V^{y,z}_{COC}(0,T)$ for different ages, maturities, and correlation levels:

Table 6. The prices of the different GS-forwards for different ages, maturities, and correlations, Cost of Capital (COC) approach.

	$\rho = 0$			$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	BE	***RM***	***Price***	***RM***	***Price***	***RM***	***Price***	***RM***	***Price***
$V^{55,60}_{COC}(0,5)$	71.0608	25.5553	96.6161	31.3349	102.3958	31.5374	102.5983	31.6411	102.7020
$V^{55,60}_{COC}(0,10)$	193.7744	63.4303	257.2047	76.1547	269.9291	76.5697	270.3441	76.7914	270.5658
$V^{60,65}_{COC}(0,5)$	84.1478	22.0132	106.1611	27.4256	111.5735	27.5616	111.7095	27.6549	111.8028
$V^{60,65}_{COC}(0,10)$	169.7714	18.2327	188.0041	27.1015	196.8729	27.5122	197.2836	27.6512	197.4226

4.3.3. Pricing GS-Forwards under the Sharpe Approach

Let us now compute the price of the GS-forward under the Sharpe approach, while using $S = 10\%$, which is consistent with the values found in the literature (see for instance (Barrieu and Veraart 2016), and (Zeddouk and Devolder 2019)). Table 7 reports the prices of the different GS-forwards $V^{y,z}_{Sharpe}(0,T)$, as well as the best estimates and the premiums P for different ages, maturities, and correlation levels:

Table 7. The prices of the different GS-forwards for different ages, maturities, and correlations, Sharpe method.

	$\rho = 0$			$\rho = 0.95$		$\rho = 0.98$		$\rho = 1$	
	BE	***P***	***Price***	***P***	***Price***	***P***	***Price***	***P***	***Price***
$V^{55,60}_{Sharpe}(0,5)$	71.0608	12.3187	83.3796	17.0625	88.1234	17.1910	88.2519	17.2761	88.3370
$V^{55,60}_{Sharpe}(0,10)$	193.7744	26.1532	219.9276	36.0616	229.8361	36.3305	230.1050	36.5088	230.2832
$V^{60,65}_{Sharpe}(0,5)$	84.1479	10.2784	94.4263	14.3410	98.4889	14.4507	98.5986	14.5234	98.6713
$V^{60,65}_{Sharpe}(0,10)$	169.7714	19.6761	189.4476	27.3093	197.0808	27.5159	197.2874	27.6528	197.4243

4.4. Comparison between Individual and GS-Forward Prices

In our framework, we consider an insurer with a portfolio of individuals that belong to two different cohorts. Hence, to hedge the longevity risk, this insurer needs to buy a GS-forward based on these two cohorts. Such a product can be available as an OTC derivative, but if it is not the case, the insurer should enter two different S-forwards that correspond to the two different cohorts. It could be interesting to check if these two situations are equivalent. To do so, we compare the prices of the individual S-forward contracts to the prices of the GS-forward. We denote, by $V^y(0,T)$ and $V^z(0,T)$, the prices of these S-forward contracts at time 0 for a maturity T, simultaneously for cohorts Y and Z.

In order to measure the difference between buying a GS-forward for the whole portfolio, and buying an individual S-forward by cohort, let us define $\Delta_\rho^{y,z}$ by the following:

$$\Delta_\rho^{y,z}(0,T) = \frac{V^y(0,T) + V^z(0,T) - V^{y,z}(0,T)}{V^{y,z}(0,T)}.$$

We compute $\Delta\rho^{y,z}(0,T)$ for the Cost of Capital and Sharpe approaches.

4.4.1. Cost of Capital Approach

Under the COC approach, the prices of the individual S-forward contracts for each cohort Y and Z are reported in Table 8:

Table 8. The prices of the different individual S-forward contracts under the COC approach.

	Prices
$V_{COC}^{55}(0,5)$	50.3640
$V_{COC}^{55}(0,10)$	143.2363
$V_{COC}^{60}(0,5)$	52.3390
$V_{COC}^{60}(0,10)$	127.3379
$V_{COC}^{65}(0,5)$	59.4699
$V_{COC}^{65}(0,10)$	108.2631

We report the values of $\Delta_\rho^{y,z}$ in Table 9:

Table 9. $\Delta_\rho^{y,z}(0,T)$ values for different ages and maturities, COC approach.

	$\boldsymbol{\rho = 0}$	$\boldsymbol{\rho = 0.95}$	$\boldsymbol{\rho = 0.98}$	$\boldsymbol{\rho = 1}$
$\Delta_\rho^{55,60}(0,5)$	6.300%	0.300%	0.102%	0.00098%
$\Delta_\rho^{55,60}(0,10)$	5.198%	0.231%	0.085%	0.00310%
$\Delta_\rho^{60,65}(0,5)$	5.320%	0.211%	0.089%	0.00550%
$\Delta_\rho^{60,65}(0,10)$	5.020%	0.289%	0.080%	0.00978%

4.4.2. Sharpe Approach

Under the Sharpe approach, the prices of these S-forward contracts for each cohort Y and Z without the correlation effect are reported in Table 10:

Table 10. The prices of the different individual S-forward contracts under Sharpe approach.

	Prices
$V^{55}_{Sharpe}(0,5)$	42.5466
$V^{55}_{Sharpe}(0,10)$	121.7403
$V^{60}_{Sharpe}(0,5)$	45.7909
$V^{60}_{Sharpe}(0,10)$	108.5467
$V^{65}_{Sharpe}(0,5)$	52.8839
$V^{65}_{Sharpe}(0,10)$	88.8951

We report the values of $\Delta^{y,z}_{\rho}$ in Table 11.

Table 11. $\Delta^{y,z}_{\rho}(0,T)$ values for different ages and maturities, Sharpe approach.

	$\boldsymbol{\rho = 0}$	$\boldsymbol{\rho = 0.95}$	$\boldsymbol{\rho = 0.98}$	$\boldsymbol{\rho = 1}$
$\Delta^{55,60}_{\rho}(0,5)$	5.940%	0.240%	0.097%	0.00062%
$\Delta^{55,60}_{\rho}(0,10)$	4.710%	0.196%	0.079%	0.00160%
$\Delta^{60,65}_{\rho}(0,5)$	4.490%	0.188%	0.077%	0.00350%
$\Delta^{60,65}_{\rho}(0,10)$	4.220%	0.183%	0.078%	0.00880%

For the two approaches, we can see that:

- if $\rho = 0$, we observe a strong non-additive effect for the two random variables;
- if $0 < \rho < 1$, the prices reflect the non-additive effect as well as the dependence effect between cohorts;
- if $\rho = 1$, the values of $\Delta^{y,z}_{\rho}(0,T)$ are almost equal to 0, which means that COC and Sharpe approaches are additive when the two cohorts are completely correlated; and,
- when ρ decreases, the GS-forward contract becomes less expensive than individual S-forwards.

5. Conclusions

In this paper, we have explored the correlation between the mortality of different cohorts, and its impact in the pricing of the GS-forward contract under three pricing approaches. To describe mortality, we have used the Hull and White model, into which we incorporate risk factors, which allow for the introduction of inter-generational correlations with different levels. We have provided the GS-forward price in closed form under the Cost of Capital, risk-neutral and Sharpe methods for different dependence levels. In addition, to measure the impact of including this dependence, we have compared both GS-forward and individual S-forward prices. For the risk-neutral approach, the correlation between generations does not have an effect on the GS-forward price, whereas, for the Cost of Capital and Sharpe approaches, we have observed a significant correlation effect on the prices. For these two pricing methods, if the cohorts are not perfectly correlated, the GS-forward becomes less expensive than the sum of the respective individual S-forward contracts.

Author Contributions: Conceptualization, F.Z. and P.D.; methodology, F.Z. and P.D.; software, F.Z. and P.D.; validation, F.Z. and P.D.; formal analysis, F.Z. and P.D.; investigation, F.Z. and P.D.; resources, F.Z. and P.D.; data curation, F.Z. and P.D.; writing—original draft preparation, F.Z. and P.D.; writing—review and editing, F.Z. and P.D.; visualization, F.Z. and P.D.; supervision, F.Z. and P.D.; project administration, F.Z. and P.D.; funding acquisition, F.Z. and P.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

Barrieu, Pauline M., and Luitgard A. M. Veraart. 2016. Pricing q -forward contracts: An evaluation of estimation window and pricing method under different mortality models. *Scandinavian Actuarial Journal* 2: 146–66. [CrossRef]

Biffis, Enrico. 2005. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics and Economics* 37: 443–68. [CrossRef]

Cairns, Andrew J. G., David Blake, and Kevin Dowd. 2006a. Pricing deaths: Frameworks for the valuation and Securitization of Mortality Risk. *Astin Bulletin* 36: 79–120. [CrossRef]

Cairns, Andrew J. G., David Blake, and Kevin Dowd. 2006b. A two-factor model for stochastic mortality with parameter uncertainty: Theory and calibration. *Journal of Risk and Insurance* 73: 687–718. [CrossRef]

Chen, Hua, Richard MacMinn, and Tao Sun. 2014. Multi-Population Mortality Models: A Factor Copula Approach. *SSRN Electronic Journal* 63: 135–46. [CrossRef]

Coughlan, Guy D., Marwa Khalaf-Allah, Yijing Ye, Sumit Kumar, Andrew J. G. Cairns, David Blake, and Kevin Dowd. 2011. Longevity Hedging 101: A Framework for Longevity Basis Risk Analysis and Hedge Effectiveness. *North American Actuarial Journal* 15: 150–76. [CrossRef]

Dahl, Mikkel. 2004. Stochastic mortality in life insurance: Market reserves and mortality-linked insurance contracts. *Insurance: Mathematics and Economics* 35: 113–36. [CrossRef]

Danesi, Ivan Luciano, Steven Haberman, and Pietro Millossovich. 2015. Forecasting mortality in subpopulations using Lee Carter type models: A comparison. *Insurance: Mathematics and Economics* 62: 151–61. [CrossRef]

Enchev, Vasil, Torsten Kleinow, and Andrew J. G. Cairns. 2017. Multi-population mortality models: Fitting, forecasting and comparisons. *Scandinavian Actuarial Journal* 2017: 319–42. [CrossRef]

Tristan, Guillaume. 2017. A Useful Result on the Covariance Between Ito Integrals. *Journal of Advances in Mathematics and Computer Science* 25: 1–12.

Haberman, Steven, Vladimir Kaishev, Pietro Millossovich, Andres Villegas, Steven Baxter, Andrew Gaches, Sveinn Gunnlaugsson, and Mario Sison. 2003. *Longevity Basis Risk: A Methodology for Assessing Basis Risk*. Cass Business School and Hymans Robertson LLP. London: Institute and Faculty of Actuaries (IFoA) and Life and Longevity Markets Association (LLMA).

Huang, Zhiping, Michael Sherris, Andres Villegas, and Jonathan Ziveyi. 2019. *The Application of Affine Processes in Cohort Mortality Risk Models*. Research paper. Kensington: UNSW Business School.

Hunt, Andrew, and Andrés M. Villegas. 2015. Robustness and convergence in the Lee-Carter model with cohort effects. *Insurance: Mathematics and Economics* 64: 186–202. [CrossRef]

Jevtic, Petar, Elisa Luciano, and Elena Vigna. 2013. Mortality Surface by Means of Continuous Time Cohort Models. *Insurance: Mathematics and Economics* 53: 122–33. [CrossRef]

Lee, Ronald D., and Lawrence R. Carter. 1992. Modeling and forecasting u. s. mortality. *Journal of the American Statistical Association* 87: 659–71.

Levantesi, Susanna, and Massimiliano Menzietti. 2017. Maximum Market Price of Longevity Risk under Solvency Regimes: The Case of Solvency II. *Insurance: Mathematics and Economics* 5: 1–21. [CrossRef]

Li, Johnny Siu-Hang, Rui Zhou, and Mary Hardy. 2015. A step-by-step guide to building two population stochastic mortality models. *Insurance: Mathematics and Economics* 63: 121–34. [CrossRef]

Li, Nan, and Ronald Lee. 2005. Coherent Mortality Forecasts for a Group of Population: An Extension of the Lee Carter Method. *Demography* 42: 575–94. [CrossRef] [PubMed]

Luciano, Elisa, and Elena Vigna. 2015. Non mean reverting affine processes for stochastic mortality. *ICER Working Papers*. doi:10.2139/ssrn.724706. [CrossRef]

Meyricke, Ramona, and Michael Sherris. 2014. Longevity risk, cost of capital and hedging for life insurers under Solvency II. *Insurance: Mathematics and Economics* 55: 147–55. [CrossRef]

Milevsky, Moshe A., and S. David Promislow. 2001. Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics* 29: 299–318. [CrossRef]

Milevsky, Moshe A., S. David Promislow, and Virginia R. Young. 2005. *Financial Valuation of Mortality Risk via the Instantaneous Sharpe Ratio: Application to Pricing Pure Endowments*. Working Paper. Ann Arbor: Department of Mathematics, University of Michigan.

Ntamjokouen, Achille. 2014. Modeling Life Expectancy at Birth for Multi-Population: A Cointegration Approach. *SSRN Electronic Journal*. [CrossRef]

Schrager, David F. 2006. Affine stochastic mortality. *Insurance: Mathematics and Economics 38*: 81–97. [CrossRef]

Xu, Yajing, Michael Sherris, and Jonathan Ziveyi. 2020. Continuous-time multi-cohort mortality modelling with affine processes. *Scandinavian Actuarial Journal* 526–52. [CrossRef]

Zeddouk, Fadoua, and Pierre Devolder. 2019. Pricing of Longevity Derivatives and Cost of Capital. *Risks 7*: 41. [CrossRef]

Zeddouk, Fadoua, and Pierre Devolder. 2020. Mean reversion in stochastic mortality: Why and how? *European Actuarial Journal*. [CrossRef]