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How Much We Gain by Surplus-Dependent Premiums—Asymptotic Analysis of Ruin Probability

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Abstract: In this paper, we generate boundary value problems for ruin probabilities of surplus-dependent premium risk processes, under a renewal case scenario, Erlang (2) claim arrivals, and a hypoexponential claims scenario, Erlang (2) claim sizes. Applying the approximation theory of solutions of linear ordinary differential equations, we derive the asymptotics of the ruin probabilities when the initial reserve tends to infinity. When considering premiums that are *linearly* dependent on reserves, representing, for instance, returns on risk-free investments of the insurance capital, we firstly derive explicit solutions of the ordinary differential equations under considerations, in terms of special mathematical functions and integrals, from which we can further determine their asymptotics. This allows us to recover the ruin probabilities obtained for general premiums dependent on reserves. We compare them with the asymptotics of the equivalent ruin probabilities when the premium rate is fixed over time, to measure the gain generated by this additional mechanism of binding the premium rates with the amount of reserve owned by the insurance company.

Keywords: ruin probability; premiums dependent on reserves; risk process; Erlang distribution

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1. Introduction

Insurance companies maintain solvency via careful design of premium rates. The premium rates are primarily based on the claims history and carefully adjusted to evolving factors, such as the number of customers and/or the returns from the investments in the financial market. Collective risk models, introduced by Lundberg and Cramér, describe the evolution of the surplus of an insurance business when considering constant premium rates, for the simplicity of arguments. This model, a compound Poisson process with drift, is referred to in the actuarial mathematics literature as the Cramér-Lundberg model. In practical situations, risk models with surplus-dependent premiums capture the dynamics of the surplus of an insurance company better. The Reference Lin and Pavlova (2006) advised for a lower premium for higher surplus levels to improve competitiveness, whereas a higher premium is needed for lower surplus levels to reduce the probability of ruin.

Among surplus-dependent premiums, risk models with *risky* investments have been widely analyzed (see e.g., Albrecher et al. 2012; Frolova et al. 2002; Paulsen 1993; Paulsen and Gjessing 1997). See Paulsen (1998) and Paulsen (2008) for surveys on the topic. The special case of risk models with *linearly* dependent premiums can be interpreted as models with *riskless* investments, since the volatility of return on investments, or the proportion of the capital invested in the risky asset is zero. Under this scenario, exact expressions of the ruin probability are derived for compound Poisson risk models with interest on surplus and exponential-type upper bounds for renewal risk models with interest (see Cai and Dickson 2002, 2003). The Reference Cheung and Landriault (2012) investigated risk models with surplus-dependent premiums with dividend strategies and interest earning as a special case. The Reference Czarna et al. (2019) discussed the ruin probabilities with the scale function from the theory of the Lévy process for risk models when the claim arrival process is a spectrally negative Lévy process and the premium rate function is non-decreasing and locally Lipschitz-continuous.

Throughout this paper, we build on the method developed in Albrecher et al. (2013) to extend the derivation of ruin probabilities to surplus-dependent premium risk models with *Erlang* distributions (claim sizes or interarrival times). Recall from Albrecher et al. (2013), the risk model with surplus-dependent premiums described by

$$U(t) = u + \int_0^t p(U(s))ds - \sum_{k=1}^{N(t)} X_k, \tag{1}$$

where $U(t)$ denotes the surplus at time t and $p(U(t))$ describes the premium rate at time t, a positive function of the current surplus $U(t)$. When $p(.)$ is constant, this model reduces to the classical collective risk model, see Asmussen and Albrecher (2010). As in the classical collective risk theory, ruin defines the first time the surplus becomes negative. For T_u, the time of ruin, given by

$$T_u = \inf\{t \geqslant 0 | U(t) < 0\},$$

the probability of ruin with initial value u is defined as

$$\psi(u) = \mathbb{P}\{T_u < \infty | U(0) = u\}.$$

We focus on calculating the ruin probabilities under Erlang claims or Erlang arrivals scenarios. Previously, Willmot (2007) considered the mixed Erlang claim size class when examining various properties associated with the renewal risk processes with constant premium rates. Furthermore, Willmot and Woo (2007) employed the Erlang mixture to claim size distributions, when discussing the application of ruin-theoretic quantities. Various studies of ruin probabilities focus on the risk model with interclaim times, being Erlang(2) (see Dickson and Hipp 2001; Dickson and Li 2010; Tsai and Sun 2004) or distributed Erlang(n) (see Gerber and Shiu 2005; Li and Dickson 2006; Li and Garrido 2004).

We follow an algebraic approach to derive the equations satisfied by the ruin probabilities, similar to the one from Albrecher et al. (2010), and further perform an asymptotic analysis of their solutions. We even solve them explicitly in a few instances. To put it in perspective, recall that Albrecher et al. (2010) introduced an algebraic approach to study the Gerber–Shiu function and derived a linear ordinary differential equation (ODE) with constant coefficients for claims distribution with a rational Laplace transform. Later, in Albrecher et al. (2013), they extended this approach to an ODE with variable coefficients for surplus-dependent premium risk models. Using methods based on boundary value problems and Green's operators, Albrecher et al. (2010) derived explicit forms of the ruin probability in the classical model with exponential claim sizes. The Reference Albrecher et al. (2013) extended the method to surplus-dependent premium models with exponential arrivals, for which they derived exact and asymptotic results for a few premium functions, when the claims were exponentially distributed. Here we extend it to renewal models and Erlang claims.

The novelty of the paper consists of the explicit *asymptotic analysis* performed for reserve-dependent premiums with Erlang-distributed generic claim sizes or Erlang- distributed generic interarrival times. We separate the analysis between $p(\infty) = c$ and $p(\infty) = \infty$ and use the approximation theory of solutions of linear ordinary differential equations developed in Fedoryuk (1993) to conclude the asymptotics of the ruin probabilities when initial reserves tend to infinity. We compare those ruin probabilities for which the speed of going to zero is substantially different, skipping the analysis when only the constant is different. Let us note that these constants are available using the theory presented in Section 2.

Among the premium functions exploding at infinity, that is, $p(\infty) = \infty$, we consider the linear premium $p(u) = c + \varepsilon u$, in which ε can be interpreted as the interest rate on the available surplus. *Linear* premiums can be interpreted as returns from investments in bonds or risk-free assets. In this case, the risk model is the same as the surplus-generating process in Paulsen and Gjessing (1997) when the stochastic investment $\sigma = 0$. When considering premiums that are *linearly* dependent on reserves, we firstly derive *explicit formulas* for the ruin probabilities, using confluent geometric functions and their corresponding ODEs. From these explicit expressions, we can easily determine their asymptotics, only to match the ones obtained for general premiums dependent on reserves.

We show that when the investments are made on risk-free assets only, as bonds or treasury bills, the solvency is improved. We will look at the improvements on solvency when such investments are made, by analyzing the insurance risk models with or without investment returns, for claims and claim arrivals that are exponential or Erlang-distributed. We compare them with the asymptotics of the equivalent ruin probabilities when the premium rate is fixed over time, to measure the gain generated by this additional mechanism of binding the premium rates with the amount of reserve owned by the insurance company.

After revisiting the model exhibiting $\text{Exp}(\lambda)$-distributed interarrival times with $\text{Exp}(\mu)$-distributed claim sizes, in this paper we consider

(i) $\text{Erlang}(2, \lambda)$-distributed interarrival times with $\text{Exp}(\mu)$-distributed claim sizes,
(ii) $\text{Exp}(\lambda)$-distributed interarrival times with $\text{Erlang}(2, \mu)$-distributed claim sizes.

We furthermore consider two premium functions:

P1. The premium function behaving like a constant at infinity

$$p(\infty) = c, \qquad p'(u) = O\left(\frac{1}{u^2}\right); \tag{2}$$

for $c > 0$, or

P2. The premium function exploding at infinity, $p(\infty) = \infty$ as

$$p(u) = c + \sum_{i=1}^{l} \epsilon_i u^i, \qquad \epsilon_i, c > 0. \tag{3}$$

The first case is satisfied by the rational and exponential premium functions. The second case is satisfied by the linear and quadratic premium functions.

The paper is organized as follows. In Section 2, we introduce the Gerber–Shiu function and present them as solutions of boundary value problems in models with premiums dependent on reserves and times and claims from distributions with rational Laplace transforms. In Sections 3 and 4, we perform the asymptotic analysis for the ruin probabilities for exponential and Erlang(2)-distributed claim sizes and interarrival times, alternatively, for models with premiums dependent on reserves. In each section, for linear premiums, the exact ruin probabilities are derived and the asymptotics confirmed to match those obtained for *general* premiums. Section 5 is dedicated to comparing the asymtotic results, highlighting the gain generated, as in higher solvency, when dynamically adjusting the premium rates to surplus. Conclusions are given in Section 6.

2. Ruin Probabilities—Method

Ruin probability is sometimes seen as a particular case of the Gerber–Shiu function $\Phi(u)$ defined in Gerber and Shiu (1998). $\Phi(u)$ is given by

$$\Phi(u) = \mathbb{E}\left[e^{-\delta T_u} \omega(U(T_u^-), |U(T_u)|) \mathbf{1}_{T_u < \infty} | U(0) = u\right], \tag{4}$$

where $e^{-\delta T_u}$ is the discount factor, ω is the penalty function of the surplus before ruin $U(T_u^-)$ and the deficit at ruin $U(T_u)$. Thus, the ruin probability $\psi(u)$ is a special case of the Gerber–Shiu function when $\delta = 0$ and $\omega = 1$.

Assuming that the distribution of the interclaim times $(\tau_k)_{k\geqslant 0}$ and the claim sizes $(X_k)_{k\geqslant 0}$ have rational Laplace transform, the density functions $f_\tau(t)$, $f_X(x)$ satisfy the linear ordinary differential equation

$$\mathcal{L}_\tau\left(\frac{d}{dt}\right)f_\tau(t)=0,\quad \mathcal{L}_X\left(\frac{d}{dy}\right)f_X(y)=0,$$

with initial conditions

$$\begin{aligned} f_\tau{}^{(k)}(0)&=0 \quad (k=0,1,\ldots,n-2),\quad f_\tau{}^{(n-1)}(0)=\alpha_0,\\ f_X{}^{(k)}(0)&=0 \quad (k=0,1,\ldots,m-2),\quad f_X{}^{(m-1)}(0)=\beta_0,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}_\tau\left(\frac{d}{dt}\right)&=\left(\frac{d}{dt}\right)^n+\alpha_{n-1}\left(\frac{d}{dt}\right)^{n-1}+\cdots+\alpha_0,\\ \mathcal{L}_X\left(\frac{d}{dx}\right)&=\left(\frac{d}{dx}\right)^m+\beta_{m-1}\left(\frac{d}{dx}\right)^{m-1}+\cdots+\beta_0,\end{aligned}$$

and α_i, β_i are the parameters of the rational Laplace transform related to derivatives of $f_\tau(t)$, $f_X(x)$.

For the risk models with surplus-dependent premiums, Albrecher et al. (2013) derived a compact integro-differential equation for $\Phi(u)$

$$\mathcal{L}_\tau\left(\delta-p(u)\frac{d}{du}\right)\Phi(u)=\alpha_0\left(\int_0^u\Phi(u-y)dF_X(y)+\omega(u)\right), \tag{5}$$

where $\omega(x)=\int_x^\infty\omega(x,y-x)dF_X(y)$. For a Gerber–Shiu function, the coefficients of ODE are variables (non-constant), and the boundary value problem developed by (Albrecher et al. 2013) is

$$\mathcal{L}_X\left(\frac{d}{du}\right)\mathcal{L}_\tau\left(\delta-p(u)\frac{d}{du}\right)\Phi(u)=\alpha_0\beta_0\Phi(u)+\alpha_0\mathcal{L}_X\left(\frac{d}{du}\right)\omega(u), \tag{6}$$

exhibiting one regularity condition

$$\Phi(\infty)=0$$

and m initial conditions

$$\Phi^{(k)}(0)=0\quad (k=0,\ldots,m-1).$$

The general solution of this boundary value problem has the form

$$\Phi(u)=\gamma_1 s_1(u)+\cdots+\gamma_m s_m(u)+Gg(u),$$

where $s_i(u)$, $i=1,\ldots,m$ are m stable solutions ($s_i(u)\to 0$ as $u\to\infty$), γ_i are constants determined by initial conditions, $g(u)=\alpha_0\mathcal{L}_X(\frac{d}{du})\omega(u)$, and $Gg(u)$ is the Green's operator for (6) (see Albrecher et al. 2013). Recall that the probability of ruin $\psi(u)$ is a special case of $\Phi(u)$ for $\delta=0$ and $\omega=1$. Furthermore, in this case we have $g(u)=0$. Thus, one has

$$\psi(u)=\gamma_1 s_1(u)+\cdots+\gamma_m s_m(u).$$

In the next sections we build on the above theory to analyse the case when either the generic interarrival time or the generic claim size follows an Erlang distribution. For the case when the inter-arrivals and claim sizes are exponentially distributed, the explicit and asymptotic results for ruin probability $\psi(u)$ derived by Albrecher et al. (2013); Asmussen and Albrecher (2010) are captured in the Appendix A.

3. Erlang$(2, \lambda)$ Distributed Interarrival Times with Exp(μ) Distributed Claim Sizes

Let the claim sizes $(X_k)_{k\geqslant 0}$ be exponentially distributed, with density

$$f_X(x) = e^{-\mu x},\ x \geqslant 0$$

and the interarrival times $(\tau_k)_{k\geqslant 0}$ be Erlang$(2, \lambda)$ distributed, with density

$$f_\tau(t) = \lambda^2 t e^{-\lambda t},\ t \geqslant 0.$$

We denote by $\psi_{l,2}(u)$ and $\Phi_{l,2}(u)$ the ruin probability and, respectively, the Gerber–Shiu function in this case.

3.1. General Premium

Based on the (Albrecher et al. 2010, 2013) technique, the boundary value problem (6) becomes

$$\left[\left(\frac{d}{du} + \mu\right)\left(\delta - p(u)\frac{d}{du} + \lambda\right)^2 - \lambda^2\mu\right]\Phi_{l,2}(u) = Gg(u), \quad u \geqslant 0.$$

For the special case $\delta = 0$ and $\omega = 1$, $g(u) = 0$, the ODE of the ruin probability $\psi_{l,2}(u)$ has the form

$$\left[\left(\frac{d}{du} + \mu\right)\left(-p(u)\frac{d}{du} + \lambda\right)^2 - \lambda^2\mu\right]\psi(u) = 0, \quad u \geqslant 0, \tag{7}$$

with

$$\psi_{l,2}(u) = \gamma_{21}s_{21}(u),$$

where $s_{21}(u)$ is a stable solution and γ_{21} is a constant to be determined by the initial conditions. Expanding ODE (7) leads to

$$p^2(u)\psi'''_{l,2}(u) + (2p'(u)p(u) - 2\lambda p(u) + \mu p^2(u))\psi''_{l,2}(u) + (\lambda^2 - 2\lambda p'(u) - 2\lambda\mu p(u))\psi'_{l,2}(u) = 0.$$

This is a third-order ODE with variable coefficients, that can be regarded as a second-order ODE in $h_{l,2}(u) = \psi'_{l,2}(u)$. Thus, one has

$$p^2(u)h''_{l,2}(u) + (2p'(u)p(u) - 2\lambda p(u) + \mu p^2(u))h'_{l,2}(u) + (\lambda^2 - 2\lambda p'(u) - 2\lambda\mu p(u))h_{l,2}(u) = 0. \tag{8}$$

In order to perform the asymptotic analysis described in (Fedoryuk 1993, p. 250), we consider the characteristic Equation of (8) when $p(u) = c$. Let $\hat{\rho}_1$ and $\hat{\rho}_2$ be the solutions of the square equation

$$\rho^2 + \frac{-2\lambda c + \mu c^2}{c^2}\rho + \frac{\lambda^2 - 2\lambda\mu c}{c^2} = 0,$$

that is, for $i = 1, 2$,

$$\hat{\rho}_i = \frac{2\lambda c - \mu c^2 \pm \sqrt{(2\lambda c - \mu c^2)^2 + 4\lambda c^2(2\mu c - \lambda)}}{2c^2}. \tag{9}$$

Moreover, let

$$\rho_1(u) = \frac{1}{2}\left(-q_1(u) - \sqrt{q_1^2(u) - 4q_0(u)}\right) \tag{10}$$

and

$$\rho_2(u) = \frac{1}{2}\left(-q_1(u) + \sqrt{q_1^2(u) - 4q_0(u)}\right) \tag{11}$$

be the solutions of the characteristic equation $\rho^2 + q_1(u)\rho + q_0(u) = 0$, where

$$q_1(u) = \frac{2p'(u)p(u) - 2\lambda p(u) + \mu p^2(u)}{p^2(u)}$$

and

$$q_0(u) = \frac{\lambda^2 - 2\lambda p'(u) - 2\lambda\mu p(u)}{p^2(u)}.$$

Further, as in (Fedoryuk 1993), denote

$$\rho_i^{(1)}(u) = \frac{-\rho_i'(u)}{2\rho_i(u) + q_1(u)}, \qquad i = 1, 2. \tag{12}$$

Theorem 1. *Let C_i ($i = 1, 2, 3$) be some constants. If* (2) *holds with*

$$\frac{2c}{\lambda} > \frac{1}{\mu}, \tag{13}$$

then

$$\psi_{l,2}(u) \sim -\frac{C_1}{\hat{\rho}_1} e^{\hat{\rho}_1 u} \tag{14}$$

where $\hat{\rho}_1 < 0$. If (3) holds, then

$$\psi_{l,2}(u) \sim C_3 \int_u^\infty \exp\left\{ \int_0^y (\rho_1(z) + \rho_1^{(1)}(z)dz \right\} dy. \tag{15}$$

Remark 1. *Condition* (13) *is derived from $\hat{\rho}_i$ in Equation* (9)*, which is also consistent with the safety loading condition for the classical risk model when the premium is constant, that is, $\frac{2c}{\lambda} - \frac{1}{\mu} > 0$. Under the assumption complimentary to* (13)*,*

$$\frac{2c}{\lambda} < \frac{1}{\mu},$$

we have $\hat{\rho}_{1,2} > 0$, and hence, both asymptotic particular solutions are unstable. Their difference might still tend to zero, but the Fedoryuk (1993) theory is not refined enough to recover the finer asymptotics in this case.

Proof. Note that in the case of premium function (2), we have

$$q_1(u) = \frac{-2\lambda c + \mu c^2}{c^2} + O\left(\frac{1}{u^2}\right)$$

and

$$q_0(u) = \frac{\lambda^2 - 2\lambda\mu c}{c^2} + O\left(\frac{1}{u^2}\right).$$

Further, under assumption (13), $\hat{\rho}_1 < 0$ and $\hat{\rho}_2 > 0$, for $\hat{\rho}_{1,2}$ defined in (9). Then Conditions (1) and (2) of (Fedoryuk 1993, p. 250) are satisfied, and hence, choosing the stable solution (tending to zero as u tends to infinity), we have

$$h_{l,2}(u) \sim e^{\hat{\rho}_1 u}, \tag{16}$$

whenever (13) is satisfied. Thus, asymptotics (14) hold true.

In the second case, of premium function (3), we observe that the solutions of the characteristic equation $\rho^2 + q_1(u)\rho + q_0(u) = 0$ satisfy:

$$\rho_1(u) = \frac{1}{2}\left(-q_1(u) - \sqrt{q_1^2(u) - 4q_0(u)}\right) \sim -\mu + \frac{q_0(u)}{\mu} \sim -\mu - \frac{2\lambda}{\epsilon_l} u^{-l} \tag{17}$$

and

$$\rho_2(u) = \frac{1}{2}\left(-q_1(u) + \sqrt{q_1^2(u) - 4q_0(u)}\right) \sim \frac{2\lambda}{\epsilon_l} u^{-l}. \tag{18}$$

Moreover, in this case, $q_0(u) \sim \frac{-2\lambda\mu}{\epsilon_l} u^{-l}$ and $q_1(u) \sim \mu$. Thus, Conditions (1), (2), and (19) of (Fedoryuk 1993, p. 254) are satisfied, and we can conclude (15). Note that from (18)

$$\int_u^{\infty} \exp\left\{\int_0^y (\rho_2(z) + \rho_2^{(1)}(z)dz\right\}dy$$

equals infinity for all u. Hence, by (17), only

$$\int_u^{\infty} \exp\left\{\int_0^y (\rho_1(z) + \rho_1^{(1)}(z)dz\right\}dy$$

can produce the stable asymptotics (15) in a sense that it tends to zero as u tends to infinity. □

Observe that, indeed in all considered cases, $\psi_{l,2}(u) \to 0$ as $u \to +\infty$, that is, we choose the asymptotics of the stable solutions.

3.2. Linear Premium

Now we perform the asymptotic analysis of the special case of the linear premium rate which corresponds to investments of reserves into bonds with the interest rate $\varepsilon > 0$. Substituting $p(u) = c + \varepsilon u$ into ODE (8), we have

$$\begin{aligned} &(c+\varepsilon u)^2 h''_{l,2}(u) + (2\varepsilon(c+\varepsilon u) - 2\lambda(c+\varepsilon u) + \mu(c+\varepsilon u)^2)h'_{l,2}(u) \\ &+ (\lambda^2 - 2\lambda\varepsilon - 2\lambda\mu(c+\varepsilon u))h_{l,2}(u) = 0. \end{aligned} \tag{19}$$

Before we solve this equation and perform the asymptotic analysis, we will show how the asymptotics of $\psi_{l,2}$ can be derived from Theorem 1. In this case, we have

$$q_1(u) = \frac{2\varepsilon - 2\lambda}{c + \varepsilon u} + \mu \text{ and } q_0(u) = \frac{\lambda^2 - 2\lambda\varepsilon}{(c+\varepsilon u)^2} - \frac{2\lambda\mu}{c+\varepsilon u}.$$

The discriminant is

$$q_1^2(u) - 4q_0(u) = \frac{4\varepsilon^2}{(c+\varepsilon u)^2} + \frac{4\mu\varepsilon + 4\lambda\mu}{c+\varepsilon u} + \mu^2,$$

and therefore,

$$\begin{aligned} \rho_1(u) &= \frac{1}{2}\left(-q_1(u) - \sqrt{q_1^2(u) - 4q_0(u)}\right) \\ &= -\frac{1}{2}\mu - \frac{\varepsilon - \lambda}{c+\varepsilon u} - \frac{1}{2}\sqrt{\mu^2 + \frac{4\mu(\varepsilon+\lambda)}{c+\varepsilon u} + \frac{4\varepsilon^2}{(c+\varepsilon u)^2}}. \end{aligned}$$

Applying Taylor expansion, we can conclude that

$$\sqrt{q_1^2(u) - 4q_0(u)} \sim \mu + \frac{2(\varepsilon+\lambda)}{c+\varepsilon u}, \quad \text{as } u \to \infty.$$

Additionally, observe that

$$q_1'(u) = -\frac{\varepsilon(2\varepsilon - 2\lambda)}{(c+\varepsilon u)^2} \text{ and } \left(\sqrt{q_1^2(u) - 4q_0(u)}\right)' = \frac{-\frac{4\varepsilon^3}{(c+\varepsilon u)^3} - \frac{\varepsilon(2\varepsilon\mu + 2\lambda\mu)}{(c+\varepsilon u)^2}}{\sqrt{\frac{4\varepsilon^2}{(c+\varepsilon u)^2} + \frac{4\mu\varepsilon + 4\lambda\mu}{c+\varepsilon u} + \mu^2}}.$$

This gives

$$
\begin{aligned}
\frac{\rho_1'(u)}{\sqrt{q_1^2(u)-4q_0(u)}} &= \frac{\frac{1}{2}\left(-q_1(u)-\sqrt{q_1^2(u)-4q_0(u)}\right)'}{\sqrt{\frac{4\varepsilon^2}{(c+\varepsilon u)^2}+\frac{4\mu\varepsilon+4\lambda\mu}{c+\varepsilon u}+\mu^2}} \\
&= \frac{\frac{\varepsilon(\varepsilon-\lambda)}{(c+\varepsilon u)^2}}{\sqrt{\frac{4\varepsilon^2}{(c+\varepsilon u)^2}+\frac{4\mu\varepsilon+4\lambda\mu}{c+\varepsilon u}+\mu^2}} + \frac{\frac{2\varepsilon^3}{(c+\varepsilon u)^3}+\frac{\varepsilon(\varepsilon\mu+\lambda\mu)}{(c+\varepsilon u)^2}}{\frac{4\varepsilon^2}{(c+\varepsilon u)^2}+\frac{4\mu\varepsilon+4\lambda\mu}{c+\varepsilon u}+\mu^2}.
\end{aligned}
$$

Using (12), we finally derive

$$
\rho_1(u)+\rho_1^{(1)}(u) = \rho_1(u) + \frac{\rho_1'(u)}{\sqrt{q_1^2(u)-4q_0(u)}} \sim -\mu - \frac{2\varepsilon}{c+\varepsilon u}, \quad \text{as } u\to\infty.
$$

Thus, for $u \to \infty$,

$$
\exp\left\{\int_0^y (\rho_1(z)+\rho_1^{(1)}(z)dz\right\} \sim e^{-\mu y}\left(\frac{c+\varepsilon y}{c}\right)^{-2}
$$

and by (15)

$$
\psi_{l,2}(u) \sim C_3 \int_u^\infty e^{-\mu y}\left(\frac{c+\varepsilon y}{c}\right)^{-2} dy, \tag{20}
$$

for some constant C_3. The same asymptotics can be derived by solving (19) explicitly. Note that (19) is the general confluent equation 13.1.35 in Abramowitz and Stegun (1965, p. 505), which has the form

$$
\begin{aligned}
& w''(z) + \left[\frac{2A}{Z} + 2f'(z) + \frac{bh'(z)}{h(z)} - h'(z) - \frac{h''(z)}{h'(z)}\right] w'(z) \\
& \quad + \left[\left(\frac{bh'(z)}{h(z)} - h'(z) - \frac{h''(z)}{h'(z)}\right)\left(\frac{A}{Z}+f'(z)\right) + \frac{A(A-1)}{Z^2} + \frac{2Af'(z)}{Z}\right. \\
& \quad \left. + f''(z) + f'^2(z) - \frac{ah'^2(z)}{h(z)}\right] w(z) = 0.
\end{aligned}
$$

For our ODE (19), let

$$
\begin{aligned}
& Z = \frac{c+\varepsilon u}{\varepsilon},\ f(Z) = h(Z) = \mu Z, \\
& A = \frac{1}{2} - \frac{\lambda}{\varepsilon} - \frac{1}{2}\sqrt{1+\frac{4\lambda}{\varepsilon}},\ a = 1 + \frac{\varepsilon + 2\lambda + \varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon},\ b = 1+\sqrt{1+\frac{4\lambda}{\varepsilon}},
\end{aligned}
$$

where the corresponding solutions are

$$
\begin{aligned}
h_{l,21}(u) =& C_{21} e^{-\mu u}(c+\varepsilon u)^{-\frac{1}{2}+\frac{\lambda}{\varepsilon}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{\varepsilon}}} \\
& \cdot \mathrm{M}\left(1+\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}, 1+\sqrt{1+\frac{4\lambda}{\varepsilon}}, \frac{\mu(c+\varepsilon u)}{\varepsilon}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
h_{l,22}(u) =& C_{22} e^{-\mu u}(c+\varepsilon u)^{-\frac{1}{2}+\frac{\lambda}{\varepsilon}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{\varepsilon}}} \\
& \cdot \mathrm{U}\left(1+\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}, 1+\sqrt{1+\frac{4\lambda}{\varepsilon}}, \frac{\mu(c+\varepsilon u)}{\varepsilon}\right),
\end{aligned}
$$

where C_{21} and C_{22} are constants. From (Abramowitz and Stegun 1965, p. 504), we know that

$$\mathrm{M}(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \text{ and } \mathrm{U}(a,b,z) \sim z^{-a}, \text{ as } z \to \infty.$$

Thus, we have $h_{l,21}(u) \to \infty$ and $h_{l,22}(u) \to 0$ for $u \to \infty$. Since $s_2(u)$ is the stable solution, we have

$$s_2(u) = C \int_u^\infty e^{-\mu v}(c+\varepsilon v)^{-\frac{1}{2}+\frac{\lambda}{\varepsilon}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{\varepsilon}}} \cdot \mathrm{U}\left(1+\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}, 1+\sqrt{1+\frac{4\lambda}{\varepsilon}}, \frac{\mu(c+\varepsilon v)}{\varepsilon}\right) dv.$$

Thus, as $u \to \infty$, the ruin probability has the following asymptotics:

$$\psi_{l,2}(u) \sim C \int_u^\infty e^{-\mu y} \cdot (c+\varepsilon y)^{-\frac{1}{2}+\frac{\lambda}{\varepsilon}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{\varepsilon}}} \cdot \left(\frac{\mu(c+\varepsilon y)}{\varepsilon}\right)^{-1-\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}} dy,$$

equivalent to

$$\psi_{l,2}(u) \sim C\left(\frac{\mu}{\varepsilon}\right)^{-1-\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}} \int_u^\infty e^{-\mu y} \cdot (c+\varepsilon y)^{-2} dy, \quad \text{as } u \to \infty.$$

By (20), this asymptotic behaviour is the same as the one derived using Theorem 1. Furthermore, one can simplify the above asymptotics by applying the integration-by-parts formula,

$$\begin{aligned}\psi_{l,2}(u) &\sim C \cdot \left(\frac{\mu}{\varepsilon}\right)^{-1-\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}} \cdot \frac{1}{\varepsilon}\left(\frac{e^{-\mu u}}{c+\varepsilon u} - \mu \int_u^\infty \frac{e^{-\mu v}}{c+\varepsilon v} dv\right) \\ &= C \cdot \left(\frac{\mu}{\varepsilon}\right)^{-1-\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}} \cdot \frac{1}{\varepsilon}\left(\frac{e^{-\mu u}}{c+\varepsilon u} - \frac{\mu}{\varepsilon} e^{\frac{\mu}{\varepsilon}c} \Gamma\left(\frac{\mu}{\varepsilon}(c+\varepsilon u), 0\right)\right), \quad \text{as } u \to \infty. \end{aligned} \tag{21}$$

4. Exp(λ)-Distributed Interarrival Times with Erlang($2,\mu$)-Distributed Claim Sizes

Let the claim sizes $(X_k)_{k\geqslant 0}$ be Erlang$(2,\mu)$-distributed and the interarrival times $(\tau_k)_{k\geqslant 0}$ be Exp(λ)-distributed, that is,

$$f_X(x) = \mu^2 x e^{-\mu x},\ x \geqslant 0 \quad \text{and} \quad f_\tau(t) = \lambda e^{-\lambda t},\ t \geqslant 0.$$

We denote by $\psi_{l,3}(u)$ and $\Phi_{l,3}(u)$ the ruin probability and the Gerber–Shiu function in this case.

4.1. General Premium

Applying the same technique as in (Albrecher et al. 2010, 2013), the boundary value problem (6) becomes

$$\left[\left(\frac{d}{du}+\mu\right)^2\left(\delta - p(u)\frac{d}{du}+\lambda\right) - \lambda\mu^2\right]\Phi(u)_{l,3} = Gg(u), \quad u \geqslant 0.$$

For $\delta = 0$ and $\omega = 1$, $g(u) = 0$ and the ODE of $\psi_{l,3}(u)$ has the following form:

$$\left[\left(\frac{d}{du}+\mu\right)^2\left(-p(u)\frac{d}{du}+\lambda\right) - \lambda\mu^2\right]\psi_{l,3}(u) = 0, \quad u \geqslant 0, \tag{22}$$

equivalent to

$$\left(\frac{d^2}{du^2}+2\mu\frac{d}{du}+\mu^2\right)(-p(u)\psi'_{l,3}(u)+\lambda\psi_{l,3}(u))=\lambda\mu^2\psi_{l,3}(u). \tag{23}$$

Hence, one can rewrite it as

$$p(u)\psi'''_{l,3}(u)+(2p'(u)+2\mu p(u)-\lambda)\psi''_{l,3}(u)+(p''(u)+2\mu p'(u)+\mu^2 p(u)-2\mu\lambda)\psi'_{l,3}(u)=0.$$

Denoting $h_{l,3}(u)=\psi'_{l,3}(u)$, we have the following equation in $h_{l,3}(u)$

$$p(u)h''_{l,3}(u)+(2p'(u)+2\mu p(u)-\lambda)h'_{l,3}(u)+(p''(u)+2\mu p'(u)+\mu^2 p(u)-2\mu\lambda)h_{l,3}(u)=0. \tag{24}$$

We first analyze the general premium rate. We denote now, as in (Fedoryuk 1993),

$$\tilde{q}_1(u)=\frac{2p'(u)+2\mu p(u)-\lambda}{p(u)}$$

and

$$\tilde{q}_0(u)=\frac{p''(u)+2\mu p'(u)+\mu^2 p(u)-2\mu\lambda}{p(u)}.$$

By $\tilde{\rho}_{1,2}$ we denote the roots of the quadratic equation

$$\rho^2+\frac{2\mu c-\lambda}{c}\rho+\frac{\mu^2 c-2\mu\lambda}{c}=0,$$

that is, for $i=1,2$,

$$\tilde{\rho}_i=\frac{\lambda-2\mu c\pm\sqrt{(\lambda-2\mu c)^2+4\mu c(2\lambda-\mu c)}}{2c}. \tag{25}$$

Further, let

$$\tilde{\rho}_i^{(1)}(u)=\frac{-\tilde{\rho}_i'(u)}{2\tilde{\rho}_i(u)+q_1(u)}, \qquad i=1,2$$

with

$$\tilde{\rho}_1(u)=\frac{1}{2}\left(-\tilde{q}_1(u)-\sqrt{\tilde{q}_1^2(u)-4\tilde{q}_0(u)}\right) \tag{26}$$

and

$$\tilde{\rho}_2(u)=\frac{1}{2}\left(-\tilde{q}_1(u)+\sqrt{\tilde{q}_1^2(u)-4\tilde{q}_0(u)}\right) \tag{27}$$

be solutions of the characteristic equation $\rho^2+\tilde{q}_1(u)\rho+\tilde{q}_0(u)=0$.

Theorem 2. *Let C_i ($i=1,2,3,4$) be some constants. If (3) holds, then*

$$\psi_{l,2}(u)\sim-\frac{C_1}{\tilde{\rho}_1}e^{\tilde{\rho}_1 u}, \tag{28}$$

where $\tilde{\rho}_1<0$ for

$$\lambda>\frac{\mu c}{2} \tag{29}$$

and

$$\psi_{l,2}(u)\sim-\frac{C_1}{\tilde{\rho}_1}e^{\tilde{\rho}_1 u}-\frac{C_2}{\tilde{\rho}_2}e^{\tilde{\rho}_2 u} \tag{30}$$

where $\tilde{\rho}_{1,2} < 0$ *for*

$$\lambda < \frac{\mu c}{2}. \tag{31}$$

Moreover, if (3) *holds, then*

$$\psi_{l,2}(u) \sim C_3 \int_u^\infty \exp\left\{\int_0^y (\tilde{\rho}_1(z) + \tilde{\rho}_1^{(1)}(z)dz\right\}dy + C_4 \int_u^\infty \exp\left\{\int_0^y (\tilde{\rho}_2(z) + \tilde{\rho}_2^{(1)}(z)dz\right\}dy. \tag{32}$$

Remark 2. *Again, the conditions* (29) *and* (31) *are derived from* $\tilde{\rho}_i$ *in Equation* (25)*. In this case, the safety loading condition for the classical risk model with constant premium is* $\frac{c}{\lambda} - \frac{2}{\mu} > 0$.

Proof. Note that in the first case of premium function (2), we have

$$\tilde{q}_1(u) = \frac{2\mu c - \lambda}{c} + O\left(\frac{1}{u^2}\right)$$

and

$$\tilde{q}_0(u) = \frac{\mu^2 c - 2\mu\lambda}{c} + O\left(\frac{1}{u^2}\right).$$

Here, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are different. Then, Conditions (1) and (2) of (Fedoryuk 1993, p. 250) are satisfied, and hence, choosing the stable solution (tending to zero as u tends to infinity)

$$h_{l,2}(u) \sim e^{\tilde{\rho}_1 u} \tag{33}$$

for $\tilde{\rho}_2 > 0$ or

$$h_{l,2}(u) \sim e^{\tilde{\rho}_1 u} + e^{\tilde{\rho}_2 u} \tag{34}$$

for $\tilde{\rho}_2 < 0$. Similarly like in the proof of Theorem 1, this observation completes the proof of (28) and (30).

In the case of premium function (3), observe that the solutions of the characteristic equation $\rho^2 + q_1(u)\rho + q_0(u) = 0$ converge for $u \to \infty$ to

$$\tilde{\rho}_1(u) = \frac{1}{2}\left(-\tilde{q}_1(u) - \sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}\right) \to -\mu \tag{35}$$

and

$$\tilde{\rho}_2(u) = \frac{1}{2}\left(-\tilde{q}_1(u) + \sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}\right) \to -\mu, \tag{36}$$

since in this case, $q_0(u) \to \mu^2$ and $q_1(u) \to 2\mu$. Although we are not in the set-up of asymptotically simple roots Equation (9) from (Fedoryuk 1993, p. 251), (Fedoryuk 1993, (9)) still holds true. Observe now that for large u,

$$\tilde{\rho}_1(u) - \tilde{\rho}_2(u) + \tilde{\rho}_1^{(1)}(u) - \tilde{\rho}_2^{(1)}(u)$$

does not change signs. Indeed, note that

$$\tilde{\rho}_1(u) - \tilde{\rho}_2(u) = -\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)} < 0.$$

Moreover,

$$\tilde{\rho}_1^{(1)}(u) = \frac{1}{2}\frac{1}{|\tilde{q}_1^2(u) - 4\tilde{q}_0(u)|}\left(\tilde{q}_1'(u)\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)} + \tilde{q}_1(u)\tilde{q}_1'(u) - 2\tilde{q}_0'(u)\right)$$

and

$$\tilde{\rho}_2^{(1)}(u) = \frac{1}{2}\frac{1}{|\tilde{q}_1^2(u) - 4\tilde{q}_0(u)|}\left(-\tilde{q}_1'(u)\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)} + \tilde{q}_1(u)\tilde{q}_1'(u) - 2\tilde{q}_0'(u)\right).$$

Therefore,

$$\tilde{\rho}_1^{(1)}(u) - \tilde{\rho}_2^{(1)}(u) = \frac{1}{2}\frac{1}{\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}}\tilde{q}_1'(u),$$

which is negative for large values of u because

$$q_1'(u) \sim \frac{-2l}{u^2} < 0.$$

Finally, $q_1''(u) \sim \frac{4l}{u^3}$ and

$$q_0'(u) \sim \frac{-2\mu l}{u^2}, \qquad q_0''(u) \sim \frac{4\mu l}{u^3}.$$

Thus, Conditions (1), (2), and (9) of Fedoryuk (1993, pp. 251–52) are satisfied and, similarly with the proof of Theorem 1, we conclude the proof of (32). □

4.2. Linear Premium

Using the same method as in the previous case, and considering a linear premium $p(u) = c + \varepsilon u$, one has

$$(c+\varepsilon u)h_{l,3}''(u) + (2\varepsilon + 2\mu(c+\varepsilon u) - \lambda)h_{l,3}'(u) + (2\mu\varepsilon + \mu^2(c+\varepsilon u) - 2\mu\lambda)h_{l,3}(u) = 0. \quad (37)$$

As in the previous section, before we solve this equation explicitly, and then using its solution to perform the asymptotic analysis, we will first show how the asymptotic behaviour of $\psi_{l,3}$ can be derived from Theorem 2. Note that, in this case, we consider

$$\tilde{q}_1(u) = \frac{2\varepsilon - \lambda}{c+\varepsilon u} + 2\mu \text{ and } \tilde{q}_0(u) = \frac{2\mu\varepsilon - 2\mu\lambda}{c+\varepsilon u} + \mu^2.$$

Further, we have the discriminant

$$\tilde{q}_1^2(u) - 4\tilde{q}_0(u) = \frac{(2\varepsilon-\lambda)^2}{(c+\varepsilon u)^2} + \frac{4\lambda\mu}{c+\varepsilon u},$$

and hence for $i = 1, 2$,

$$\begin{aligned}\tilde{\rho}_i(u) &= \frac{1}{2}\left(-\tilde{q}_1(u) \pm \sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}\right)\\ &= -\mu - \frac{2\varepsilon-\lambda}{2(c+\varepsilon u)} \pm \frac{1}{2}\sqrt{\frac{(2\varepsilon-\lambda)^2}{(c+\varepsilon u)^2} + \frac{4\lambda\mu}{c+\varepsilon u}}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\tilde{\rho}_i'(u)}{\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}} &= \frac{\frac{1}{2}\left(-\tilde{q}_1(u) \pm \sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}\right)'}{\sqrt{\frac{(2\varepsilon-\lambda)^2}{(c+\varepsilon u)^2} + \frac{4\lambda\mu}{c+\varepsilon u}}}\\ &= \frac{\varepsilon(2\varepsilon-\lambda)}{2(c+\varepsilon u)\sqrt{(2\varepsilon-\lambda)^2 + 4\lambda\mu(c+\varepsilon u)}} \pm \frac{-\frac{\varepsilon(2\varepsilon-\lambda)}{c+\varepsilon u} - 2\varepsilon\lambda\mu}{2(2\varepsilon-\lambda)^2 + 8\lambda\mu(c+\varepsilon u)}\\ &\sim \mp\frac{\varepsilon}{4(c+\varepsilon u)}, \quad \text{as } u \to \infty.\end{aligned}$$

Additionally,

$$\begin{aligned}\tilde{\rho}_i(u) + \tilde{\rho}_i^{(1)}(u) &= \tilde{\rho}_i(u) \mp \frac{\tilde{\rho}_i'(u)}{\sqrt{\tilde{q}_1^2(u) - 4\tilde{q}_0(u)}} \\ &\sim -\mu + \left(-\frac{3\varepsilon}{4} + \frac{\lambda}{2}\right)\frac{1}{c+\varepsilon u} \pm \sqrt{\frac{\lambda\mu}{c+\varepsilon u}}, \quad \text{as } u \to \infty.\end{aligned}$$

Thus, we conclude that

$$\exp\left\{\int_0^y (\tilde{\rho}_i(z) + \tilde{\rho}_i^{(1)}(z)dz\right\} = e^{-\mu y \pm \frac{2}{\varepsilon}\left(\sqrt{\lambda\mu(c+\varepsilon y)} - \sqrt{\lambda\mu c}\right)}\left(\frac{c+\varepsilon y}{c}\right)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}}$$

and by Theorem 2, we have that

$$\begin{aligned}\psi_{l,3}(u) \sim &C_3 \int_u^\infty e^{-\mu y + \frac{2}{\varepsilon}\left(\sqrt{\lambda\mu(c+\varepsilon y)} - \sqrt{\lambda\mu c}\right)}\left(\frac{c+\varepsilon y}{c}\right)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy \\ &+ C_4 \int_u^\infty e^{-\mu y - \frac{2}{\varepsilon}\left(\sqrt{\lambda\mu(c+\varepsilon y)} - \sqrt{\lambda\mu c}\right)}\left(\frac{c+\varepsilon y}{c}\right)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy, \end{aligned} \tag{38}$$

for some constants C_3 and C_4 as $u \to \infty$. The same asymptotic behaviour can be observed by first solving ODE (37) explicitly. Recall that a general Bessel equation (Sherwood and Reed 1939, p. 211)

$$x^2\frac{d^2y}{dx^2} + \left[(1-2m)x - 2\alpha x^2\right]\frac{dy}{dx} + \left[p^2a^2x^{2p} + \alpha^2x^2 + \alpha(2m-1)x + m^2 - p^2n^2\right] = 0,$$

exhibits solutions involving Bessel functions (see Sherwood and Reed (1939, p. 211) and Logan (2012, p. 460) for details). In our ODE (37), we let

$$x = c + \varepsilon u,\ m = -\frac{1}{2} + \frac{\lambda}{2\varepsilon},\ \alpha = -\frac{\mu}{\varepsilon},\ p = \frac{1}{2},\ p^2a^2 = -\frac{\lambda\mu}{\varepsilon^2},\ n = -1 + \frac{\lambda}{\varepsilon},$$

and employing the property $\mathrm{K}_{-v}(z) = \mathrm{K}_v(z)$ (see Abramowitz and Stegun (1965, p. 375)), it can be verified that the corresponding solution is

$$\begin{aligned}h_{l,3}(u) = &C_{31}e^{-\frac{\mu}{\varepsilon}(c+\varepsilon v)} \cdot (c+\varepsilon v)^{-\frac{1}{2}+\frac{\lambda}{2\varepsilon}} \cdot \mathrm{BesselI}\left[-1+\frac{\lambda}{\varepsilon}, 2\sqrt{\frac{(v+\frac{c}{\varepsilon})\lambda\mu}{\varepsilon}}\right] \\ &+ C_{32}e^{-\frac{\mu}{\varepsilon}(c+\varepsilon v)} \cdot (c+\varepsilon v)^{-\frac{1}{2}+\frac{\lambda}{2\varepsilon}} \cdot \mathrm{BesselK}\left[-1+\frac{\lambda}{\varepsilon}, 2\sqrt{\frac{(u+\frac{c}{\varepsilon})\lambda\mu}{\varepsilon}}\right],\end{aligned}$$

where C_{31} and C_{32} are some real constants and BesselI and BesselK are modified Bessel functions. The solution is the same as the one in Example 2.4. in Paulsen and Gjessing (1997), which is solved by the changing of variables $u = z - c/\varepsilon$. In this case, $n = -1 + \frac{\lambda}{\varepsilon}$ has to be restricted to an integer. This yields

$$s_{31}(u) = C_{31}\int_u^\infty e^{-\frac{\mu}{\varepsilon}(c+\varepsilon v)} \cdot (c+\varepsilon v)^{-\frac{1}{2}+\frac{\lambda}{2\varepsilon}} \cdot \mathrm{BesselI}\left[-1+\frac{\lambda}{\varepsilon}, 2\sqrt{\frac{(v+\frac{c}{\varepsilon})\lambda\mu}{\varepsilon}}\right]dv,$$

$$s_{32}(u) = C_{32}\int_u^\infty e^{-\frac{\mu}{\varepsilon}(c+\varepsilon v)} \cdot (c+\varepsilon v)^{-\frac{1}{2}+\frac{\lambda}{2\varepsilon}} \cdot \mathrm{BesselK}\left[-1+\frac{\lambda}{\varepsilon}, 2\sqrt{\frac{(v+\frac{c}{\varepsilon})\lambda\mu}{\varepsilon}}\right]dv.$$

Since

$$\mathrm{I}_v(z) \sim \frac{e^z}{\sqrt{2\pi z}} \text{ and } \mathrm{K}_v(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \quad \text{as } z \to \infty$$

(see Abramowitz and Stegun (1965, p. 377)), we have that, for $u \to \infty$,

$$\psi_{l,3}(u) \sim \frac{C_{31}}{\sqrt{2\pi\sqrt{\lambda\mu}}} e^{-\frac{\mu c}{\varepsilon}} \int_u^\infty e^{-\mu y + \frac{2}{\varepsilon}\sqrt{\lambda\mu(c+\varepsilon y)}} \cdot (c+\varepsilon y)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy$$
$$+ C_{32}\sqrt{\frac{\pi}{2\sqrt{\lambda\mu}}} e^{-\frac{\mu c}{\varepsilon}} \int_u^\infty e^{-\mu y - \frac{2}{\varepsilon}\sqrt{\lambda\mu(c+\varepsilon y)}} \cdot (c+\varepsilon y)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy,$$

which is consistent with (38), and hence with Theorem 2.

5. Asymptotic Analysis—Comparison Results

5.1. *Exp*(λ)*-Distributed Interarrival Times with Exp*(μ)*-Distributed Claim Sizes*

In this case, for the linear premium, we have the asymptotic result (A3), that is,

$$\psi(u) \sim \frac{\mu}{\lambda c^{\lambda/\varepsilon}} C e^{-\mu u}(c+\varepsilon u)^{\frac{\lambda}{\varepsilon}-1},$$

where C is some constant. For a constant premium c, the result for the ruin probability is

$$\psi_{c,1}(u) = \frac{\lambda}{c\mu} e^{-(\mu - \frac{\lambda}{c})u}, \quad \text{for any } u \geqslant 0.$$

Thus, we have

$$\psi_{l,1}(u) \sim C \cdot \psi_{c,2}(u) \cdot e^{-\frac{\lambda}{c}u} \cdot (c+\varepsilon u)^{\frac{\lambda}{\varepsilon}-1}, \quad \text{as } u \to \infty.$$

5.2. *Erlang*$(2,\lambda)$*-Distributed Interarrival Times with Exp*(μ)*-Distributed Claim Sizes*

Recall the asymptotic result (21) for risk models with linear premiums, in this case,

$$\psi_{l,2}(u) \sim C_1 \cdot \left(\frac{\mu}{\varepsilon}\right)^{-1-\frac{\varepsilon+2\lambda+\varepsilon\sqrt{1+\frac{4\lambda}{\varepsilon}}}{2\varepsilon}} \cdot \frac{1}{\varepsilon}\left(\frac{e^{-\mu u}}{c+\varepsilon u} - \frac{\mu}{\varepsilon} e^{\frac{\mu}{\varepsilon}c}\Gamma\left(\frac{\mu}{\varepsilon}(c+\varepsilon u), 0\right)\right), \quad \text{as } u \to \infty,$$

and the explicit result for risk models with constant premiums gives

$$\psi_{c,2}(u) = C_2 e^{-\frac{c\mu - 2\lambda + \sqrt{c^2\mu^2 + 4c\lambda\mu}}{2c}u}, \quad u \geqslant 0,$$

see (Dickson and Hipp 1998, 2001). Taking the limit and applying L'Hôpital's rule, the ratio between $\psi_{l,2}(u)$ and $\psi_{c,2}(u)$ behaves asymptotically as

$$\frac{\psi_{l,2}(u)}{\psi_{c,2}(u)} \sim C_3 e^{-\frac{c\mu + 2\lambda - \sqrt{c^2\mu^2 + 4c\lambda\mu}}{2c}u}(c+\varepsilon v)^{-2}, \quad u \to \infty,$$

where C_3 is some constant. Hence, $\frac{\psi_{l,2}(u)}{\psi_{c,2}(u)}$ tends to zero as u tends to infinity.

This means that as the initial surplus u increases, one has more premium income for risk models with linear premiums, thus the ruin probability $\psi_{l,2}(u)$ for risk models with linear premiums decays to zero exponentially faster than the ruin probability $\psi_{c,2}(u)$ for constant premium risk models. As expected, this means that risk models with linear premiums are less risky than the constant premium ones.

5.3. Exp(λ)-Distributed Interarrival Times with Erlang$(2,\mu)$-Distributed Claim Sizes

We start by recalling the asymptotic result (38) for risk models with linear premiums

$$\psi_{l,3}(u) \sim C_1 \int_u^\infty e^{-\mu y + \frac{2}{\varepsilon}\left(\sqrt{\lambda\mu(c+\varepsilon y)} - \sqrt{\lambda\mu c}\right)} \left(\frac{c+\varepsilon y}{c}\right)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy + C_2 \int_u^\infty e^{-\mu y - \frac{2}{\varepsilon}\left(\sqrt{\lambda\mu(c+\varepsilon y)} - \sqrt{\lambda\mu c}\right)} \left(\frac{c+\varepsilon y}{c}\right)^{-\frac{3}{4}+\frac{\lambda}{2\varepsilon}} dy, \quad u \to \infty,$$

where C_1 and C_2 are constants. The explicit result for the constant premiums case can also be derived from an ordinary differential equation, as

$$\psi_{c,3}(u) = C_3 e^{\sigma_{31} u} + C_4 e^{\sigma_{32} u}, \quad u \geqslant 0,$$

where $\sigma_{31,32} = -\frac{2c\mu - \lambda \pm \sqrt{\lambda^2 + 4c\lambda\mu}}{2c} < 0$ and C_3 and C_4 are some constants, see Bergel and Egidio dos Reis (2015); Li and Garrido (2004) for details. Taking the limit and applying L'Hôpital's rule, we can conclude that

$$\lim_{u\to\infty} \frac{\psi_{l,3}(u)}{\psi_{c,3}(u)} = 0.$$

Thus, as the initial surplus u increases, the ruin probability $\psi_{l,3}(u)$ for risk models with linear premiums decreases to zero faster than the ruin probability $\psi_{c,3}(u)$ for constant premiums. Again, this means that risk models with constant premiums are more risky than linear premium ones, as expected; thus, there is a gain in terms of solvency when binding premiums to reserves.

6. Conclusions

It is much easier to calculate the ruin probabilities for risk models with constant premiums, and explicit results for constant cases abound in the risk theory literature; however, the risk models with surplus-dependent premiums are more applicable in real life. For these complex cases, we have results in terms of the confluent hypergeometric function and modified Bessel function at most, or only asymptotic results, from which one can make inferences.

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Appendix A

We recall the results for ruin probabilities, in models with premiums dependent on reserves, general and linear premiums, when both inter-arrivals and claim sizes are exponentially distributed. For a classical compound Poisson process with exponential claims, the following explicit and asymptotic results for ruin probability $\psi(u)$ can be found in Albrecher et al. (2013); Asmussen and Albrecher (2010).

General premium. The ruin probability $\psi(u)$ has the following explicit expression

$$\psi(u) = \frac{\lambda \int_u^{\infty} e^{-\mu v + \int_0^v \frac{\lambda}{p(y)} dy} \frac{1}{p(v)} dv}{1 + \lambda \int_0^{\infty} e^{-\mu v + \int_0^v \frac{\lambda}{p(y)} dy} \frac{1}{p(v)} dv}. \quad \text{(A1)}$$

The asymptotic estimate of ruin probability for $p(\infty) = c$ is

$$\psi(u) \sim \frac{\mu}{\lambda} C e^{-\mu u + \lambda \int_0^u \frac{dw}{p(w)}}, \quad u \to \infty$$

and for $p(\infty) = \infty$ is

$$\psi(u) \sim \frac{\mu}{\lambda} C \frac{1}{p(u)} e^{-\mu u + \lambda \int_0^u \frac{dw}{p(w)}}, \quad u \to \infty$$

where C is a constant. We write $f(u) \sim g(u)$ for some functions f and g when $\lim_{u \to +\infty} f(u)/g(u) = 1$.

Linear premium. The explicit form of ruin probability $\psi(u)$ is

$$\psi_{l,1}(u) = \frac{\lambda \varepsilon^{\lambda/\varepsilon - 1}}{\mu^{\lambda/\varepsilon} c^{\lambda/\varepsilon} e^{-\mu c/\varepsilon} + \lambda \varepsilon^{\lambda/\varepsilon - 1} \Gamma(\frac{\lambda}{\varepsilon}, \frac{\mu c}{\varepsilon})} \Gamma(\frac{\lambda}{\varepsilon}, \frac{\mu(c + \varepsilon u)}{\varepsilon}), \quad \text{(A2)}$$

where $\Gamma(\eta, x)$ is the incomplete gamma function defined as

$$\Gamma(\eta, x) = \int_x^{\infty} t^{\eta - 1} e^{-t} dt.$$

Moreover, when $p(u) = c + \varepsilon u$, we have

$$\psi(u) \sim \frac{\mu}{\lambda c^{\lambda/\varepsilon}} C e^{-\mu u} (c + \varepsilon u)^{\frac{\lambda}{\varepsilon} - 1}, \quad \text{as } u \to \infty. \quad \text{(A3)}$$

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